NOTES ON MODAL LOGICS

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1. These notes are a sequel to [7]. Familiarity with that paper will be assumed. The same notations will be used, except that a single arrow ($\rightarrow$) will denote material implication, while L and M are respectively necessity and possibility.

The paper [7] needed an emendation, for the alleged proof of Theorem 2.3 does not hold water. The Theorem itself is true and will follow from the results of section 2 below.

In sections 2 and 3 sufficient conditions for thesishood in S_a and in vS_a will be proved; they are named «completeness theorems» for reasons which will become apparent in section 4, where possible extensions of the relational semantic of Kripke to some of the systems of [7] will be examined.

Sections 5 and 6 are devoted to various topics related to the comparisons between the weak modal systems defined in [7].

2. *A completeness theorem in* S_a.

Convention: In this paper a «*tautology*» (or a «*PC-thesis*») will denote any substitution instance of a thesis of the classical propositional calculus proper (without connective L).

Definition 2.1. The system S_a^o is defined by the following axiom schemes and rules: P_1, P_2, P_3, D, vP_1, vP_2, υP_3, υD.

Definition 2.2. An *extended assignment* of values is a unary function from modal formulas (i.e. S_a^o-formulas) to the classical set of values $\{t,f\}$, which is constructed as in PC, except that we have adjoined to PC a denumerable set of supplementary propositional variables, $(q_i)_{i\in N}$ in one-one correspondence with all the S_a^o-formulas of the form Lx_i, and that, for an extended assignment α

$\alpha(Lx_i) = t$ if x_i is a tautology;
$\alpha(Lx_i) = \alpha(q_i)$ if x_i is not a tautology.

It must be remarked that when the value of a formula like $p_1 \to L(p_2 \to L(p_3 \to p_3))$ is computed for a given assignment α, the values $\alpha(p_3)$, $\alpha(L(p_3 \to p_3))$ and $\alpha(p_2)$ are not used; we have simply

$$\alpha(p_1 \to L(p_2 \to L(p_3 \to p_3))) = \alpha(p_1) \to^* \alpha(q_k)$$

where q_k is the supplementary variable corresponding to $L(p_2 \to L(p_3 \to p_3))$ – using the properties

$$\alpha(\neg x) = \neg^* \alpha(x)$$

and $\alpha(x \to y) = \alpha(x) \to^* \alpha(y)$

where $\neg^*$ and $\to^*$ are the representations of $\neg$ and $\to$ in the classical two-valued matrix for PC.

Definition 2.3. A formula is *E-valid* if it takes the value t for every extended assignement.

Lemma 2.4. Every E-valid formula is an S_a^o-thesis. – This is the fundamental result.

Proof: The axioms are E-valid. Then:

Let us suppose that there is a thesis, x, which is not E-valid. There should be a formal deduction of x in S_a^o, and in that deduction there should be a first non-valid formula, z. It will be shown that z should be preceded by another non-valid formula; whence a contradiction.

If z was obtained by D

$$\begin{array}{l} u \\ u \to z \\ \hline z \end{array}$$

there should be an extended assignment α such that $\alpha(z) = f$; whence for the same assignment either $\alpha(u) = f$ or $\alpha(u \to z) = \alpha(u) \to \alpha(z) = f$, and either u or $u \to z$ should not be E-valid.

If $z = Ly$ was obtained by D

$$\begin{array}{l} Lu \\ L(u \to y) \\ \hline Ly \end{array}$$

y should not be a tautology, then either u or $u \to y$ should not be a tautology. In either case we could find an extended assignement α' such that $\alpha'(Lu) = f$ or another, α'' such that $\alpha''(L(u \to y)) = f$.

Lemma 2.5 If $\vdash_{S_a^o} Lx$, then $\vdash_{PC} x$. For, if not, Lx would not be E-valid.

Lemma 2.6 Rule W is admissible in S_a^o; from Lemma 2.5 and the fact all PC-theses are S_a^o-theses.

Lemma 2.7. S_a^o and S_a have the same theses. For S_a is obtained from S_a^o by postulating rule W, which is admissible in S_a^o.

It follows:

Theorem 2.7. Every E-valid formula is an S_a-thesis.

Corollary 2.8. $\vdash_{Sa}$ Lx if and only if $\vdash_{PC}$ x. From Lemmas 2.4 and 2.7 and Theorem 2.1 of [7]. – But this is Theorem 2.3 of [7], whose proof needed an amendment.

3. *A completeness theorem in* νS_a.

Definition 3.1. T being the particular tautology $p \rightarrow p$, the *T-reduction* of a modal formula is the transformation which consists in replacing by T every maximal subformula which either is a tautology or has the form Lz where z is a tautology.

Definition 3.2. The *T-reduct* of a formula x is the formula obtained by iterating the T-reduction as many times as it is possible.

If s is the operation of T-reducing, s^k the k-th iteration of the same, and $\bar{s}$ the operation of forming the T-reduct, then there is a number n such that $\bar{s} = s^n$.

Definition 3.3. A modal formula is *T-valid* if its T-reduct is T.

Theorem 3.4. Every T-valid formula is a νS_a-thesis.

Proof: Every axiom is T-valid. Then:

Let us suppose that there is a νS_a-thesis, x, which is not T-valid. We will argue as in the proof of Lemma 2.4: there should be, in the formal deduction of x, a first formula z which should not be T-valid.

As $\bar{s}(x) = T$ iff $\bar{s}(Lx) = T$, z can be obtained neither by W nor by I.

If z was obtained by D

$$\left|\begin{array}{l} u \\ u \rightarrow z \\ z \end{array}\right.\downarrow$$

we would have

$$\bar{s}(u) = s^n(u) = T$$

$$\bar{s}(u \rightarrow z) = s^m(u \rightarrow z) = s^m(u) \rightarrow s^m(z) = T$$

and, with $k = \max(n,m) + 1$

$$s^k(u) = T$$

$$s^k(u) \rightarrow s^k(z) = T \leftrightarrow (T \rightarrow T)$$

whence $s^k(z) = T$ -contrary to hypothesis.

If z = Ly is obtained by νD

$$\begin{array}{|l} Lu \\ L(u \to y) \\ Ly \end{array}$$

we would have $\bar{s}(Lu) = T$ and $\bar{s}(L(u \to y)) = T$, whence

$\bar{s}(u) = (u \to y) = T$ as in the preceding case, again contradicting the hypothesis.

4. *Variations about Kripke's semantics.*

The twelve systems defined in [7] had been constructed in 1955-58, before Kripke (and also Kanger, Hintikka, C.A. Meredith and Prior) had created the possible worlds semantics. Is it possible to interpret them in a similar way?

As they are all «non-normal» (in the sense of Kripke [4] or as well in the sense of Lemmon [5]), except $\nu\varrho S_a = T$ (of Feys) and $\nu\varrho\nu S_a = S4$, it will be necessary to use more complex semantics than the frames of [3] or of [5].

For $\varrho Sa = S0.5$, the problem has been solved by Cresswell ([1], see also Hughes and Cresswell [2], pp. 286-288), using non-normal worlds different from those of Kripke [4].

For $\varrho\nu\varrho S_a$ the problem has been solved by myself. The result, presented to the Logic Symposion of Patras (August 1980) will be published in full later: [9]. It can be proved that $\varrho\nu\varrho S_a$ is complete for the model structure consisting of a triple ⟨G, K, R⟩, where K is the set of «possible worlds», R is a binary relation between worlds (accessibility), and $G \in K$ is «the real world» (or «actual world»). Every world is accessible to itself and every world is accessible to the real world (this results into a kind of «restricted transitivity»: for all worlds w_1, w_2, if GRw_1 and w_1Rw_2 then GRw_2). It is not possible to suppress the mention of distinguished «real world»: the logic is incomplete on any class of frames with non-normal worlds, these non-normal worlds being defined in the way of Kripke [4], of Cresswell [1], or of the «semi-normal worlds» defined below.

For S_a, νS_a, $\varrho\nu S_a$, and another logic (defined below) I have found reasonable conjectures, which will presently be expounded.

Definition 4.1. Non-normal worlds are defined as in Cresswell [1], except that accessibility is not restricted: Lx true in w is compatible with x false in w′ accessible to w, while Lx false in w is compatible with x true in every world accessible to w.

Definition 4.2. A *Semi-normal world* is one in which Lx can be true in w even if x is false in a w′ accessible to w, but Lx can be false in w only if there is a w′ accessible to w in which x is false (the semi-normal worlds are the «non-normal worlds» of [8], section 5).

Conjecture (I): S_a is complete for the model structure ⟨G, K, R⟩ where G (the real world) is semi-normal (or normal), while the other worlds are non-normal (or semi-normal, or normal).

Conjecture (II): νS_a is complete for the model structure ⟨G, K, R⟩ where all the worlds are semi-normal (or normal) – It is then useless to single out a world as being the real one.

Conjecture (III): $\varrho\nu S_a$ is complete for the model structure ⟨G, K, R⟩ where the real world is normal, while all the other worlds are semi-normal (or normal), the relation R being reflexive and transitive.

Conjecture (IV): If the model structure is like in Conjecture (III), except that R is not bound to be transitive, it determines a logic which can be axiomatized by

$$\nu P_1,\ \nu P_2,\ \nu P_3,\ D,\ \varrho\nu D,\ \varrho W,\ I$$

This system could be represented as $\varrho\nu S_a \cap \nu\varrho S_a$, or as $\varrho S_a \cup \nu S_a$.

It is easy to prove that all these classes of model structures are sound for the corresponding logics, i.e. that every thesis is valid (see [8], section 5, for Conjectures (III) and (IV)). It remains to prove that every valid formula is a thesis of the corresponding logic...

Now, the E-validity of Definition 2.3 is but a disguize of the relational semantic notion contained in Conjecture (I), while the same can be said for the T-validity of Definition 3.3 compared to the relational semantic notion of Conjecture (II). – Then, from Theorems 2.7 and 3.1, it follows:

Theorem 4.3 Conjecture (I) is true.

Theorem 4.4 Conjecture (II) is true.

Conjectures (III) and (IV) remain open problems. I have not even a conjecture for the other systems defined in [7].

5. *The role of rule W.*

It is well known that Kripke's semantics with normal worlds is particularly simple for the normal systems in which ϱW is not a schema of theses (see Lemmon [5]).

Things seem to be different for the comparison of weak modal logics studied in [7]. The simplicity of the discussion rests on Theorem 5.1 : $\nu S = \nu\nu S$, which is proved by the derivability of rule W.

A weaker but similar result could be obtained if W was admissible in a system S, and the admissibility was conserved by the operations ϱ and ν. Indeed W is admissible in S_a^o and remains admissible in ϱS_a^o (Lemma 2.6 above) and $\nu\varrho S_a^o$ (which is the system K of Lemmon [5]). – But the way to extend such results is not apparent...

6. *A central result.*

It appears that the Theorem 7.3 of [7] plays a somewhat central role in the comparison of weak modal systems, as appears from its similarity with result of Lewis and Langford ([6], p. 499) used by Simons [10], and with a result of Yonemitsu [12] used by Hughes and Cresswell ([2] pp. 227-228). It may be interesting to give it the most general form.

On one hand, in the formulation of Theorem 7.3 ([7], p. 16), the tautology t may be replaced by an arbitrary formula, u, as proved below:

(1)	$u \rightarrow (t \rightarrow u)$	– is a tautology
(2)	$\vdash L(u \rightarrow (t \rightarrow u))$	– by S_a
(3)	$Lu \vdash L(t \rightarrow u)$	– by (2), νD
(4)	$u \rightarrow t$	– is a tautology
(5)	$\vdash L(u \rightarrow t)$	– by S_a
(6)	$Lu \vdash L(u \rightarrow t)$	– by (3), (5), S_a
(7)	$Lu \vdash L(LLu \leftrightarrow LLt)$	– by (6) and C (twice)
(8)	$Lu \vdash LLu \leftrightarrow LLt$	– by (7) and W
(9)	$Lu, LLu \vdash LLt$	– by (8) and PC
(10)	$LLu \vdash Lu$	– by W
(11)	$LLu \vdash LLt$	– by (9), (10)

and the argument goes on as in [7], p. 16, last line, and p. 17.

On the other hand, it was understood in [7] that when a rule «holds» in a system, that means that it is *derivable* in it, and not only *admissible* in the system (see for instance Wang [11]). But it is apparent, from the proof of Theorem 7.3 of [7] completed by the foregoing argument, that the theorem remains true if *all* instances of the word «holds» are understood as meaning «is admissible». Then we have two generalized results:

Theorem 6.1. When the rule

$$C: \; L(x \leftrightarrow y) \; / \; L(Lx \leftrightarrow Ly)$$

is derivable in a system at least as strong as S_a, the rule

$$I: \; Lx \; / \; LLx$$

is derivable iff $\vdash$LLu for a formula u.

Theorem 6.2. When the rule C is admissible in a system at least as strong as S_a, the rule I is admissible iff $\vdash$LLu for a formula u.

Corollary 6.3. When C is derivable in a canonical system, the schema I: $\vdash Lx \rightarrow LLx$
holds iff $\vdash$LLu for a formula u.

Corollary 6.4. Iff we add any axiom of the form $\vdash$LLu to a system at least as strong as ϱS_c, we get a system at least as strong as$\varrho v \varrho S_a$.

Simons applied Corollary 6.4 to S3; Yonemitsu and Hughes-Cresswell applied Theorem 6.1 to S1 (for Lewis and Langford postulated C under the name of «rule of replacement of strict equivalents»).

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