DIVISIONS OF INTENSIONAL UNIVERSES

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1. Introduction

In [4] (see bibliography) a special treatment for a language with intensional problems as to individual expressions (oblique use of individual names) had been developed, with the understanding that it is clear from the context of the problem when intensions are considered different and when not. This treatment consisted in using what are called «intensional universes», i.e., universes of discourse which may contain more than one individual for each individual of the usual universes. More precisely, if *according to a given context* two individual expressions are considered as having different intensions (even if they have the same extension), they will be treated as denoting different individuals; only if they have also the same intension, will they denote the same individual. Thus «evening star» and «morning star» may denote different individuals, while «evening star» and «the star seen specially in the evening» may denote the same individual. (1)

The identity used to define these enlarged universes corresponds to the sameness of intension (according to the given context) and will be called «intensional identity». Due to the fact that the intensional universes have more individuals than the corresponding extensional ones, they are also richer in propositional functions. Thus, besides the intensional identity (symbolized by «$=$»), we may have another two-place relation, the extensional identity («$=_e$»), which corresponds to the identity in the usual universes (evening star $\neq$ morning star; evening star $=_e$ morning star). If we form the equivalence classes of the extensionally identical individuals, then the universe of discourse constructed on these equivalence classes is in a one-one relation to the corresponding universe in the usual treatment.

In spite of using the words «intension» and «intensional», the treatment will be entirely extensional (only expressions are considered and what is denoted by them). If the terms «evening star» and «morning star» denote different objects of the intensional universe, then they are, *from a purely extensional standpoint*, not any longer replaceable without running the risk of changing the truth value of the sentences in which te replacement is made.

Any functional system with identity (pure or applied, first order or higher order) may be considered. Only its interpretation is different from the usual ones: it applies to intensional universes and the identity symbol will denote the intensional identity. Formally there are no differences. (2)

In the following pages we will speak indistinctly of one-place propositional functions and classes, two-place propositional functions and two-place relations, etc. and use sometimes the class notation instead of the functional one, e.g.:

$$F \subset G$$

instead of:

$$(x)\ (Fx \supset Gx).$$

2. Special subclasses of intensional universes.

Normally the individuals of a model of everyday life may be classified in persons, physical objects, etc. The corresponding intensional universes with their greater number of individuals allow special classifications, some of which are interesting enough to be mentioned.

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In order to give an example, let us take the class of the persons of a given intensional universe. It might be subdivided in classes like:

the proper persons (the persons according to their proper names: Aristotle, Plato, etc.),
the quantity persons (the persons according to some measured characteristics: the man who weighs 200 kg, etc.)

the quality persons (the persons according to some characteristic exterior property: the baldest king, the girl with the golden helmet, etc.),
the relation persons (the father of Mr. X, the youngest brother of Mr. Y. etc.),
the date persons (the persons according to some location in time or space: the man who was born January 1st, 1958 in Aubervilliers, etc.),
the activity persons (the man who conquered Gaul for Rome, etc.),
the passivity persons (the man who was killed in the Roman Senate, etc.),
etc.

Similarly, the class of the physical objects may be subdivided in classes like:

the proper objects (Titanic, the sword Hartung, etc.),
the quantity objects (the tower which is 80 m high, etc.),
the quality objects (the rose-colored palace, etc.),
the relation objects (the car owned by Mr. X, etc.);
the date objects (the ship launched January 1st, 1978, etc.),
the activity objects (the airplane that crossed the channel for the first time, etc.),
etc.

In order to give another type of example, the class of the natural numbers may be subdivided in this way:

the proper numbers (9, 15, etc.),
the relation numbers (the successor of 8, the product of 5 and 3, etc.),
the applied numbers (the number of planets, etc.),
etc.

It is easy to see that, extensionally speaking, the same object appears in different classes ($9 =_e$ the successor of $8 =_e$ the number of planets), but in an intensional context these objects may be intensionally different, because someone believes something about 9 but not about the successor of 8 or the number of planets; thus there are intensional universes in which the mentioned classes have no common elements.

In this way our classification may be exclusive (with respect to =) and possibly also exhaustive, e.g. if one adds a class which contains all the corresponding individuals (persons, physical objects, natural numbers, etc.) not contained in the preceding classes.

With respect to $=_e$, the same object may appear not only in different classes, but also twice or more times in the same class, as Voltaire and F.M. Arouet (Voltaire $=_e$ Arouet, and in certain intensional universes: Voltaire $\neq$ Arouet) in the class of the proper persons, or, e.g., the father of Mr. X and the youngest brother of Mr. Y ($=_e$ and $\neq$) in the class of the relation persons.

3. Kernel classes and broadening.

A starting point in the formation of classes over an intensional universe could be with very restrictive classes, which will be called «kernel classes». Let us suppose that «Ceasar» and «the man who conquered Gaul for Rome» have different intensions in a given context. Then accordingly, the man who conquered Gaul for Rome and not Caesar will belong to the kernel class of the conquerors; in a similar case, the man who was killed in the Roman Senate and neither Caesar nor the man who conquered Gaul for Rome will belong to the kernel class of those being killed in a senate.

Naturally one would like to be able to say that Caesar was a conqueror.This can be done without difficulty (if one disposes of «$=_e$») by defining the «broadening» of a class. Supposing that F *as any class, not necessarily a kernel class, we define «the broadening of* F" («b(F)») as «the class of the individuals extensionally identical with an element of F":

$$b(F) =_{df} \lambda y((Ex)y =_e x . Fx).$$

In an example with F as the kernel class of the conquerors, which contains only certain activity persons, b(F) would contain also certain proper persons, date persons, etc. If we want to express that Caesar

(c) was a conqueror, we will write:

$$b(F)c.$$

or:

$$c \in b(F)$$

By the definition (supposing that $=_e$ is an equivalence relation) we get immediately results like:

$$F \subset b(F)$$

$$b(F) = b(b(F))$$

etc.(3)

While it is easy to give a formal definition of «broadening», the same does not hold for «kernel class». Informally it is quite clear what is meant by the kernel class of the conquerors, but a formal definition (which is not intended here) supposes the analysis of the predicate «being a conqueror» and the choice of some basic kernel classes.

Two instructive definitions are that of «extensional subclass» («$\subset_e$») and «extensional identity between classes» («$=_e$» between class symbols, which should not be confused with the «$=_e$» of the preceeding pages; the latter one appears only between individual symbols):

$$F \subset_e G =_{df} b(F) \subset b(G)$$
$$F =_e G =_{df} b(F) = b(G)$$

Let us suppose, e.g., that for each activity person of the class *Ap* we have at least one extensionally identical proper person of the class *Prop*, then we get «$Ap \subset_e Prop$» without having «$Ap \subset Prop$». If the inverse relation holds too, we get «$Ap =_e Prop$» without having «$Ap = Prop$»; *Ap* and *Prop* have quite different elements, while the broadening of *Ap* coincides with that of *Prop*.

All the previous considerations apply also to the n-place propositional functions with $n > 1$, so that one has kernel relations, the broadening of relations, extensional subrelations, etc.

In the second definition on the right side, we used «$=$» between class symbols, where «$F = G$» is defined in the usual way by «$(x)(Fx \equiv Gx)$», *but all this over intensional universes.* For many systems we can consider a relation introduced in this way as the intensional identity of classes (and relations).

As in the case of the individuals, we see that we can have $=_e$ without having $=$, but if two classes (relations) are intensionally identical they are extensionally identical, too. All this, together with the fact that only the intensional identity allows replacement, may help us to solve some intensional problems concerning classes and relations (oblique use of class names and relation names). Thus somebody may believe something about the class F, without believing the same thing about the class G, with $F =_e G$. If $F \neq G$, there will be no replacement and the corresponding problem is eliminated.

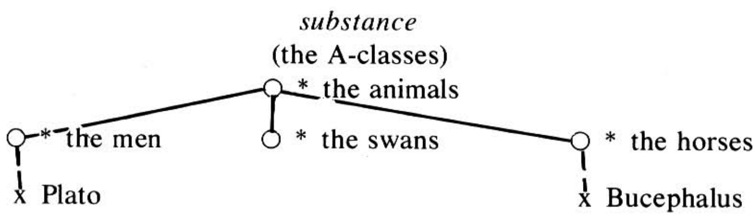
However, this technique might not work immediately in all cases of oblique use *of class names and relation names*. Let us suppose that we had «$(x)(Fx \equiv Gx)$» over an intensional universe and that, nevertheless, the expressions «F» and «G» did not have the same intension (the intensional universes do resolve the problem for all individual names but not automatically for all class names, etc.). In this case we could have recourse to one of the special systems mentioned in [4], in which the axioms of extensionality do not hold. In these systems «$F = G$» (for the intensional identity of the classes F and G) cannot be defined by «$(x)(Fx \equiv Gx)$»; F could be (intensionally) different from G, in spite of the fact that F and G have the same elements of the intensional universe.

4. Aristotle's categories.

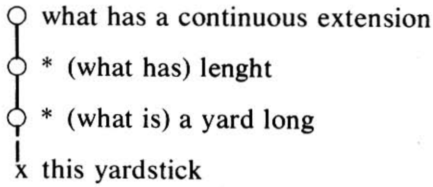
Let us suppose we divided the (intensional) universe of the general objects as we did with the physical objects, and that the persons and the (natural and real) numbers are included among the general objects. Now we may subdivide *the class of the general proper objects* in the (kernel) classes of the human beings, the birds, the physical objects, etc. and call each of these classes an «A-class». Similarly we subclassify *the general quantity objects* in those which weigh more than 100 kg, those which are higher than 1 m, etc. and call each of these (kernel) classes a «B-class». In the same way the C-classes (the white objects, the colourless objects, etc.) subclassify *the general* *quality objects*, the D-classes (the parents of somebody, the owners of something, etc.) subclassify *the general relation objects*, etc.

These subdivisions have been realized because they might suggest a connection with some philosophical categories. Surely divisions like Porphyry's tree do not need an intensional analysis. But that seems not te be the case with Aristotle's categories (substance, quantity, quality, relation, place, time, position, state, action and affection). There is a certain correspondence between the secundary substances of Aristote and the A-classes, the elements of the category quantity and the B-classes, the elements of the category quality and the C-classes, etc. Naturally the parallelism should not be enforced too much, but I think Aristotle did not only classify linguistic phenomena (even if the Greek grammar influenced him strongly in his classification); he might have classified the individuals of something like an intensional universe or, more precisely, the classes of these individuals. In classifying classes on might think that a universe constructed only on the base of $=_e$ could do the job, but only something like an intensional universe will make the classification really instructive. In order to show this, some examples for Aristotle's categories: substance, quantity, quality, relation and action will be indicated. As to the following drawings, in

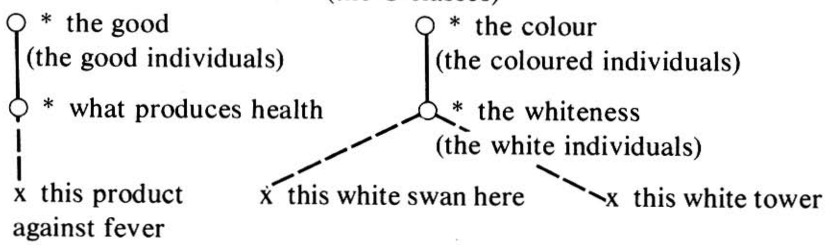
the circle below represents a subclass (species) of the cicle above (genus), the broken line represents the connection between an individual and a species, «*» indicates examples taken from Aristotle:



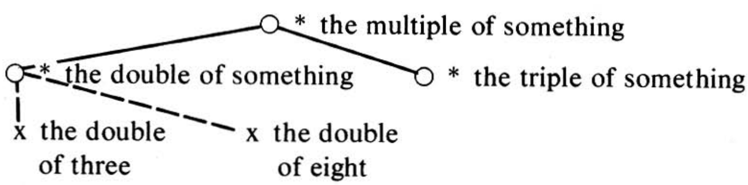
quantity
(the B-classes)



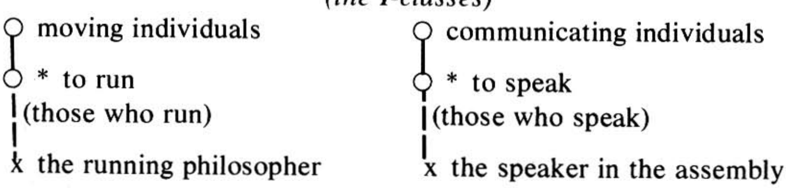
quality
(the C-classes)



relation
(the D-classes)



action
(the I-classes)



With all this we can get a two-order classification: in the higher order, classes are classified (explicitly); in the lower order, individuals could be classified (indirectly) into those belonging to an A-class, those belonging to a B-class, etc. (even more refined lower-order classifications are possible, e.g. according to the chosen «lowest species»). In both cases exclusivity can be obtained easily with intensional universes, while the usual universes do not give exclusivity at all in the lower order(4) and are not very helpfull if we want to establish a not entirely artificial exclusive classification of classes in the sense of Aristotle's categories(5).

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FOOTNOTES

(1) Nonintensional and intensional universes may have individuals which are not real objects, like complex numbers, Pegasus, etc. (see [3]).

(2) Naturally there are differences (concerning the well-formed expressions and the theorems) if a specific system of epistemic logic (see e.g. [6]) is introduced.

(3) If the operator b is added to the Boolean algebra of classes one gets a closure algebra (see e.g. [2]).

(4) It seems that Aristotle did not bother too much about exclusivity concerning individuals (see e.g. «Categories» [1] 11a, 37-39).

(5) As to the usual universes, only artificial restrictions could prevent, e.g., some A-classes from coinciding with some B-classes, etc.

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