PARADOXES IN ARABIC GEOMETRY AN ARCHEOLOGY OF SCIENTIFIC DISCOVERY

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Introduction:

We shall be concerned in this paper with giving a translation of a small passage from a manuscript attributed to Biruni, which contains a compendium of four geometrical, trigonometrical, and astronomical treatises. This manuscript is in the possession of the Oriental Public Library of Bankipore (Arabic MS 2468, 42, 36, 37 and 38). It was also published and edited by Dairatu'l-Ma'rif'il-'Osmaniah-Hyderabad 1948. The part that we shall be concerned with contains three geometric constructions on the foundations of geometry. For contextual reasons, the passage was suggested to be an insertion from another letter of Biruni. The first letter is attributed to Kindi; and the other two, to Biruni himself. A phrase of praise made by Biruni in the honor of Kindi, stylistically suggests the contemporality of both authors. This is chronologically forbidden, thus raising doubts as to the authenticity of the authorship of the manuscript. Although further research might throw new light on this subject, we shall consider the authorship provisionally authentic; which is, furthermore, confirmed by an analysis of the contents of the manuscript as we shall see later.

These three examples deal with Euclid's parallels postulate. Although no explicit declaration is made to this effect, the authors apparantly forsee difficulties arising from this postulate in the presence of infinity. ([1])

Thus it is our main objective to give a historical evaluation of this work. This turns to be quite a difficult job, since the very controversial subject of the infinite is taken up in such a brief space and in such a succinct presentation. Yet we are very careful in avoiding the misconceptions resulting from a logistical([2]) writing of history (of logic), as carefully criticised by Jacoby in «Die Anspruche der Logistik auf die Logik und ihre Geschichtschreibung» [19]; where the author presents many of the pitfalls in which eminent logisticians found themselves in tracing back the history of logistics to ancient sources. We repeat here the example of Lukasiewicz, since it is a good example of a fallacy that we shall be tempted into. Lukasiewicz considered Aristotle as a forrunner of symbolic logic because he used propositional variables. According to Jacoby, Lukasiewicz was looking at ancient thinking from a modern perspective, and putting thoughts into the mind of Aristotle that he never imagined. Aristotle's variables were a shorthand notation rather than a pure syntactical calculus of abstract operations. In view of these negative results, we shall suggest a methodology based upon phenomenological philosophy to analize historicity, which we apply to our particular examples.

If the reader would agree to the following criteria, he should accept the legitimacy of our conclusions.

1. Ancient authors must have followed a causal chain of reasoning in reaching a certain goal and starting with the state of the discipline at that period. This chain could be interrupted by psychological gaps. They are either constructive, due to sound intuition, or fallacious results of psychologism. ([3])

2. Non-rigorous constructions in ancient mathematics, could not be dismissed as such, by our modern standards. At worst, they should be given the benefit of heuristics.

Before starting with a translation of the text, we shall commence with a brief history of the geometrical works of the authors.

Al'Kindi (?-873), the first philosopher of the Arabs, was very found of geometrical reasoning. He must have been in contact with Greek geometry through Arabic and Syriac translations; since his knowledge of Greek must have been very poor [14]. He was especially interested in applying geometry to other disciplines; such as cosmology and theology, which he wrote down in quite original letters. His introduction of geometry in philosophical argumentation would manifest his interest in foundational problems. Kindi's and Biruni's works on the parallels postulate were already mentioned in [8].

Our second author, Biruni (973-1048), also had a wide range of interests. His known contributions to geometry are few, but highly original. He wrote more on trigonometry and astronomy. He was deeply acquainted with Greek geometry, as manifested in his published treatises [4]. No less remarkable was his acquaintance with Indian mathematics, which he made known through his celebrated book 'India'. His interest in Geometry went beyond technical mastership, to methodology and philosophy. As remarked by Arnaldez [3], Biruni tried to apply geometrical reasoning to scientific discourse in the manner of Cartesian deductions. We shall also encounter certain stipulations of Biruni disclosing his awareness of psychological and philosophical factors in geometric reasoning, and his position towards the atomistic theory of his days. These observations about Biruni's scholarship agree with the foundational nature of the treatise under consideration, and legitimatises our analysis of his works in such a perspective.

In the following translation we have tried to keep as close as possible to the original text at the expense of style and sometimes, intelligibility.

...Line hs (fig. 1) is drawn between both lines. It might be perpendicular to both of them, or it might not be so. Between points s,d a point p is drawn. Line atp is constructed.

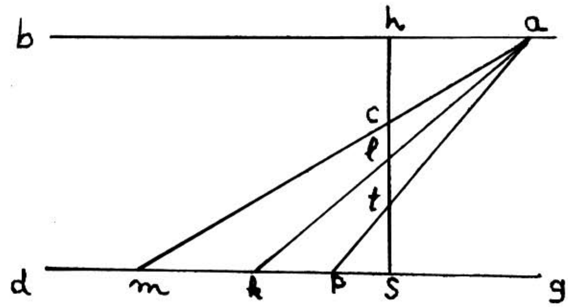

Fig. 1

It is known that if points like k, m are drawn on line gd beyond point p; and if a is joined to each of these points; then the joining lines intersect th between ab and the other line joining a to the point nearest to p; like line alk. This line intersects th at l between line atp and ahb.

Similarly acm intersects it (hs) at c, between lines alk and ahb. The point traced on line pd could be infinite. Similarly line th is not finished or exhausted by the lines joining a, to each of the trace points. For if it was exhausted, then the final (exhausting) line would coincide with line ab. It is therefore, parallel to gd. But it intersects with it (gd) at some point. We have obtained a line parallel to another and intersecting with it simultaneously in a certain direction. This is a contradiction. Therefore, th is divided into infinitely many quantities, decreasing in magnitude. This was put forth by al-Kindi.

As for his objection, may God keep his glory, that two parallel lines could approach each other and never meet is unpleasant to hear, unless they approach each other with both ends. Although I believe two straight nonparallel lines will meet on one of their sides, I say that I am indifferent to the condition of surprise or to its elevation.

If he could show for the parallel lines ab, gd (fig 2) that they could

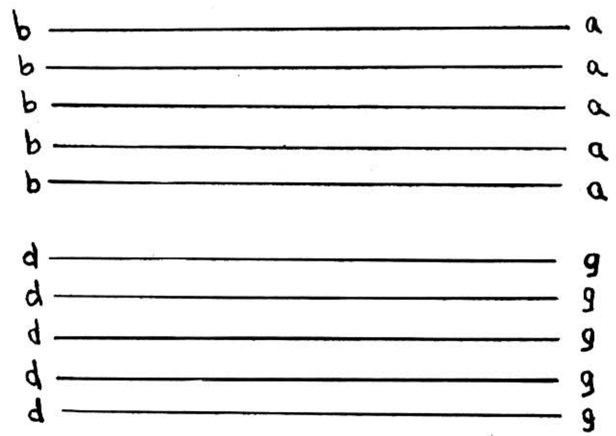

Fig. 2

approach each other, as before, without meeting, then lines ab (these are line ab in different positions due to motion), and lines gd (these are line gd in different positions due to motion). It is known that lines ab and lines gd increase infinitely, in number while a distance between them remains, fixed, which is not traversed by either of them. If the matter was so, that we could mark points h, s, p, t, and k on lines ab, which are joined by a straight line. We also mark points l, m, n, c, and g on lines gd in the same manner. Is it ever possible for these two joining straight lines to meet at all? Thus I have cleared up the puzzle, let him (refute this? words-missing).

If he was prevented from doing this (i.e. refuting) because of motion, I will deprive him of this (motion, i.e. remove it). I say suppose there existed infinitely many decreasing quantities. Let them be, for example, lines g, d, h, s, p, t, k, l, and m. If we erect the longest line, say g, on line ab at a (fig 3), and erect line d, the next in size, near

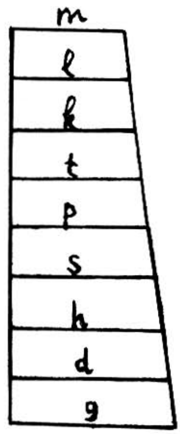

Fig. 3

it and parallel to it. Then we erect line h next and parallel to d. We pass at the other ends of these lines a single straight line touching all these infinite decreasing lines which were erected in the described positions. These lines, being infinite, never terminate. Does the straight line joining the ends ever meet line ab? This ought to be cleared out.

The existence of these decreasing quantities, and the demonstration of two parallel straight lines which approach each other but never meet, has another aspect. Let us construct the square abgd. We extend the two sides of adsg in the two directions of ag. On the extended line da, we mark point h. From this point we construct a line...

At a first glance, one is hardly inclined to assign to these arguments any historical or scientific value, because of the lack of mathematical rigor. This is, however, not the case. These arguments are good examples of mathematical intuition of the infinitely large and the infinitely small in geometry, and of philosophical issues on the foundations of geometry. This is not surprising regarding the high caliber of the involved scientific personalities. It should be kept in mind that there are common concepts in all three arguments upon which we shall dwell. These are analytic and infinitistic methods in geometry, and the historical concept of the Euclidean space. The analysis will proceed piecewise, in the light of the new methodologies that will be employed in this analysis.

We start with Kindi's argument. Since the beginning of the manuscript is missing, it might be difficult to arrive at the intention of the argument, especially because it reaches us second hand. We shall list and analize three different possible intentions.

a) Giving an analytic argument in geometry.
b) (Philosophical motivation) Relating the infinitely large to the infinitely small.
c) Presenting geometric paradoxes in connection with infinite divisibility and the parallels postulate.

We shall now look at each of these motivations separately:

a) This is indeed the contention of the closing clause. Kindi has arrived at dividing a finite (bounded from two sides) segment, into an infinity of monotonely decreasing segments. This is done in the following manner:

Kindi constructs an unbounded sequence of points on gd and joins them to point a intersecting with th. He then establishes an isomorphism between the points on gd and the traces of the transversal on th, by establishing a correspondence between the set of infinite points on gd and the points on th, he regards th as a set of point (which could be exhausted).

Kindi remarks that the resulting segments are decreasing in length He gives no proof for this statement. Since the divisions are infinite, he obviously perceives some sort of convergence. It would have been more in the spirit of Greek geometry if he had attained such a division using numerical rather than geometrical methods, such as using fractional sequences: $^1/_2$, $^1/_4$, $^1/_8$...

b) In this isomorphism he is making a correspondence between the infinitely large and the infinitely small. These two kinds of infinities were elaborated upon by Aristotle (e.g. in Physics).

Looking back at Kindi's argument, we see that the concept of geometric convergence, (of the infinitely decreasing magnitudes) which is not deductively analized by Kindi, could only result from an intuitive application of a pigeon-hole-principle. It is not immediately obvious from the construction that the magnitudes bisected on th are decreasing in magnitude, i.e. convergent in the modern language. This is so because, we can conceive of the case where the intervals on gd are monotonely increasing in magnitude; which leads to believe that the corresponding traces on th will have this property too. Thus, a pigeon-hole-principle is necessary for Kindi's deduction on convergence. Yet Kindi seems to be omitting such (evident) parts of the construction, which we should like to bring to the consciousness.

The metaphysical grounds of this principle are to be found in Kindi's book «on the first philosophy»; which was translated by Ivry into English, as 'al-Kindi's methaphysics'. More exactly, these grounds are traced back to the discussions on the relations between the 'one' and the 'many'.

Clearly no pigeon-hole-principle is necessary in the reverse process, where an infinite division is obtained by successive substractions of a certain proportion (of the whole) from the whole, which is the standard analytic intuition behind the mathematical and philosophical argumentations of Greek scholars. In order to make a correspondence, however, between the infinitely large and the infinitely small, such a principle is conjured up.

We look back on the motivations for this work. We find that Biruni gives a) as a motivation. This was also suggested by the authors of [8], who remarked that these three problems on the theory of parallels, could not have belonged to the treatise in which they appeared in [4]. They must belong to another treatise of Biruni's which was lost. This treatise bears the title 'Maqala fi anna lawasim l'maqadire la ila nihaya qariba min amril'khattain il'athain yaqruban wa la yal taqlyan fil istib'ad'. This is translated as'the article on the property of infinite divisibility of magnitudes, is near to the situation of two lines approaching each other but not meeting in any distance».

We shall investigate if this theory is acceptable, by looking at each of the three examples separately in the light of the subject suggested by the title. The subject partially agrees with Biruni's remark that Kindi established by this construction the infinite divisibility of th. We shall show that this could not have been the main motivation. If it was indeed, then Kindi would have started with the line th (or hs), that had to be infinitely divided. Then he would have constructed the parallels. This, however, does not carry sufficient grounds for the refutation of the thesis, in the light of which Kindi's construction would be summarised as follows:

An infinite sequence of points is drawn on gd. These points are joined successively to a. If the traces of this sequence on th were finite, then we arrive at a contradiction with the parallels postulate. Therefore the traces have to be infinite and the infinite divisibility is established. Kindi has thus arrived at refuting finite atomism in geometry, which we shall discuss in detail in a later section. We shall see there, that finite atomism has been refuted a long time before Kindi's in Greek philosophy, therefore, this could not have been Kindi's intention. This argument is again not totally conclusive for Kindi was found of introducing purely geometric argumentation in philosophy. He could have wanted to establish a logical relation between finite atomism and the parallels postulate. Yet in the construction he never referred to th as being finitely exhausted, but only exhausted.

However, we notice that Kindi is careful in introducing a 1-1 correspondence between the points on lines gd and th. To each new point on gd he carefully constructs a point on th located in an interval subtended between two lines meeting at a. This is out of place in the earlier finitistic argument, where this correspondence is not used (where, in fact, the whole construction on gd is not necessary). For if he used this correspondence he would have immediately obtained the infinite divisibility at th without recurrence to the parallels postulate, by simply realizing the limitlessness of gd. This is too obvious, to be missed by Kindi. He would be scratching his right ear with his left hand. Furthermore, there are other simple means for obtaining the same result. Another intuitive, application of the pigeon-hole-principle to the lines connecting the infinite set of points on gd to the finite set on th, would have resulted in a violation of Euclid's first postulate. This would have been too obvious, even to be mentioned in his proof as he did in the other application of the pigeon-hole-principle, as we have seen earlier.

The above given three reasons, in addition to the fact that the first thesis (i.e. the case of finite atomism) does not tie up with the nature of the two other examples given by Biruni, lead us to support the motivation established in c.). The first thesis does not possess the spirit of a paradox, which is declared by Biruni in the beginning of his introduction to his constructions.

We are therefore, led to the interpretation that will be given in c.), in which an infinitistic concept is concealed, in opposition to the first thesis of finite atomism. This is not new to Kindi. We have shown in [12,14], THAT Kindi has introduced a formal reasoning for operations on infinite magnitudes. This he has done in four letters.

We shall see in c-) that it is possible to arrive at a formulation of Kindi's example that fits under the new title of the letter, as in the case of Biruni's following examples, c) follows naturally from the above arguments.

c-) Kindi arrives at showing that such a correspondence does not exhaust line th as an infinite set of points. For if the oblique transversals exhausted line th, the last oblique line would be simultaneously parallel to gd and intersecting with it, because it would coincide with ab (4) (not realizing that a last line does not exist). We add the following argument, if the transversals do not exhaust th, there are points on th, other than h, which are not crossed by these oblique lines. A line joining point a to such a point would neither be parallel to gd nor intersecting with it. Thus we are in the same situation as in Biruni's examples, i.e. the existence of lines approaching each other but not meeting, as suggested by the new title.

We move on to Biruni's examples. In the first one he constructs a sequence of points on two sets of moving lines. These points when joined together, determine two lines. (He does not bother to check whether these lines are straight, although pure intuition denies this). He declares that these lines will never meet, since otherwise, this would deny the nonintersectability of the moving parallel lines. The intention of this example and the next one is to construct two lines which are neither parallel not intersecting; based upon the infinite divisibility property of straight lines. Although Biruni declares his counter-belief, namely that two non-parallel lines must meet in a point (in standard situations). He thus betrays a belief in the existence of lines of a third kind, rather than the non-validity of the parallel postulate. We shall study this in more detail and analize the fallacy behind this construction.

As for Biruni's last example, this is a construction motivated by a visual concept, and is supposed to serve the same purpose as the earlier example. Biruni constructs two lines which are non-parallel, since the distance between them is constantly decreasing. They are non-intersecting, for the distance never vanishes. (An equivalent formulation of the parallels postulate is implicitely conjured). This construction could have been motivated by the following constructions, due to Proclus.

Proclus shared Ptolmoy's doubt about the independence of the parallel postulate from the other postulates of Euclidean geometry. He declared that the «statement that since (the two lines) converge more and more as they are produced, they will sometime meet; is plausible but not necessary». He offered the example of a parabola (fig. 4).

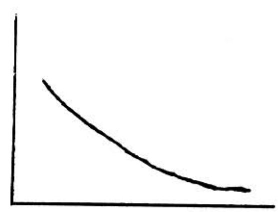

Fig. 4

He also used analytical tools and motion (kinematics) in proving the parallels postulate. See [15].

A line l is drawn between two parallel lines l and m intersecting at p. (fig. 5). Line q is perpendicular to m as line q moves to the right, point Y approaches line l.

It will eventually cross over l. Thus it must intersect line l.

Thus Biruni's geometrical techniques are not novel. Both of his

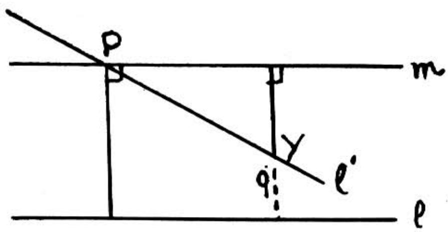

Fig. 5

examples are false and lack mathematical rigor. In that respect they should have no historical value. However if we analize the situation more carefully, we might arrive at another conclusion.

There are several reasons to believe that Biruni is aware of a geometrical paradox, which he pretends to clarify. His controversy and challenges of Kinidi's views, giving at least three variations of the same construction, betrays Biruni's doubts about the efficacy of his construction.

A formal analysis of the paradoxes.

We call the preceding paradoxes, paradoxes of the infinite; because they arise from the adjunction of the notions of the infinitely large and the infinitely small in geometry. We shall give a formalisation of these paradoxes:

1. Kindi's paradox was already formulated in c), in the infinitistic situation.
2. Biruni's first paradox: The two lines joining points on the infinite positions of two parallel lines, will never meet, as the lines come nearer to each other.

But they should obviously meet according to their construction in a euclidean space. We have called this a paradox since it arises under the same psychological convictions as in the Eleatic paradoxes, as we shall see later. (Proclus also gave a proof of the parallels postulate, resulting in Achilles-type paradox). However, this should be called a fallacy, since its validity is not restorable as in the case of the other paradoxes, except on the basis of relativistic considerations as we shall see in the last section.

Biruni's second paradox: Biruni constructs two lines which are non-parallel, so they must intersect (in the finite). Nevertheless, they fail to intersect since they are separated by a nonvanishing distance.

Resolution of the paradoxes:

The first paradox (Kindi's) could be resolved by taking up «infinity» as a point on the Euclidean line, obtaining an extended euclidean space. We shall come back to this later.

The second paradox is fallacious. It could be resolved in a Poincaré-type model of hyperbolic geometry.

The third paradox could be resolved using non standard analysis, by constructing a model of the Euclidean space in which two lines migth have an infinitesimal slope relative to each other. They will never meet in the finite. We hope to demonstrate this in a forthcoming article on the foundations of geometries.

We have learnt from the history of modern mathematical logic that the role of paradoxes in the developement of logical concepts could only be understated. This is confirmed by the great number of publications on logical paradoxes in the turn of this century. We have also seen how the resolution of Burali-Fortis Paradox led Cantor to formulate his transfinite arithmetic; and the resolution of paradoxes of set theory led to a refinement of set and type theories. Are we, therefore, vindicated in claiming that Kindi and Biruni anticipated noneuclidean geometry, simply because the paradoxes which came out implicitely, from their work led to a refined concept of the euclidean space in which the usual euclidean concepts prove to be discontenting?

On the one hand, at extended euclidean space concept is not non euclidean, since distances and angles remain unchanged. However, the introduction of lines of the third kind, as suggested by Biruni, clearly violates the fifth postulate in the euclidean (finite) plane, since more than one parallel could be constructed. Nevertheless, Biruni did not enunciate these conclusions; nor did he make any comments on validity, independence, or consistency of axiom systems. His work is non-axiomatic. Yet we are tempted to declare him a for-runner of non-euclidean geometry, because he was not convinced of the conclusiveness of the euclidean concept of space. For this he gave at least three examples. The example of Proclus deals rather with dependence of axiom system, and his example of the converging parabolic line does not justify his claim. Biruni's examples are non-constructible in the standard euclidean methodology.

However, non euclidean geometry in the modern sense of mathematical logic is an axiomatic system; i.e. a purely deductive system. The mathematical model plays a secondary role, other than demonstrating the consistency of the system and guarding the intuitions in formal deductions. In that sense none of these works is non-euclidean. Thus neither Biruni nor Kindi could have realized the significance of his works relative to Euclid's.

We realize that we are unable to settle these conflicting views without introducing a proper methodology. This will be done with the help of modern philosophical and psychological theories.

It is clear from the history of science in its present status that it falls short of a most demanding responsability, namely, that of producing proper tools for the evaluation of scientific discoveries. Historians differ in opinion in the attribution of credit to various historical personalities. This perplexity is not confined to particular scientists, but applies to whole cultures as well. We shall give an example from one of the personalities appearing in this paper, namely, al'Kindi's. Kindi was regarded by some historians as one of the greatest twelve men of science in the history of mankind. To others he was a mere intelligent commentator on Aristotle. This goes, as well, for the evaluation of specific works of Kindi. Compare [13,23]. On the other hand, great errors could be committed by prominent historians, and even specialists in their field, in making such an appraisal. We have seen many examples in [19] from the history of logistics. It is time, we feal, that general criteria be established that would enable a scientific and homogeneous evaluation of these phenomena. This we call an archeology of scientific discovery.

It is a puzzling problem in the history of science, that many ideas remained inert and inproductive in the history. This could be characterised by introvertness, although these ideas contained the neuclii of great discoveries; while other formulations proved to be active and productive (extrovert) and gave birth to important results. Although I am pessimistic that a particular methodology could be reached without the investment of serious efforts; it is worthwhile to delve into construction.

I will take the example of the liar paradox from Greek philosophy to illustrate my pessimism. It is obvious that such a general almost naive formulation of a linguistic phraseology, could not have been very relevant; and indeed it would never have been so, had it not been for mathematical logic and Gödel's construction of such a paradox to prove one of the most celebrated theorems of logic in our century. Could Gödel have discovered his theorem without making use of the liar paradox? Does it admit other formulations? Should we reasses its value because of Gödel's work? Indeed other substitutes were discovered only recently and required hard-core mathematics. Yet the relation between the old and new formulations might be purely coincidential.

Some ideas exist for centuries in incubation until they emerge suddenly into full life; but it is difficult to establish direct causality between these phenomena. What we are looking for, are the critical factors that induce the transition from the sterile to the productive state; from an isolation in itself of an idea to an openness onto new being. We are led by this language to existential philosophy, which we shall apply to the study of the above examples. We do not pretend to put down the foundations for a theory of knowledge in which these questions are naturally worked out, nor to develope a phenomenology of scientific discovery; but will limit our discussion to the bounds of this paper.

Husserlian Phenomenology

Phenomenology, as a new school of philosophy, was grounded by Husserl who was a mathematician. Husserl wanted to introduce mathematical rigor into philosophy and was guided by his mathematical intuition. It is particularly suitable to analize historical data as a scheme according to which essences reveal themselves in Phenomena, as done in phenomenology and later in existentialism.

For Husserl philosophy is a science which discovers the historical processes of the revelations of the universal consciousness which is inborn in humanity. When this statement is relativised to a particular science like geometry, it leads to a search process of the intuition as revealed in historicity. In his «Ideen» [17], Husserl declares the aim of phenomenology in reaching the intuition of the essences and the reflection on this intuition.

More recently, the Swiss psychologist, Jean Piaget, applied his methods of psychological and epistemological genetics which were inspired by Sartre's existential psycho-analysis; to the intuition, which we shall further investigate in connection with the intuition of the geometric space.

It is worth remarking that one of the founders of existentialism, René Descartes, also brought about a revolution in mathematics. It was the study of abstractions and essences that culminated in great developments in mathematics. This is the goal of phenomenology.

We hope that we have sufficiently motivated the philosophical methodology that will be employed in our analysis. Our aim was to reconstruct the historical moment in scientific discovery in view of the mental and psychological disposition of the discoverer.

Psychology of perception

Biruni realizes the importance of psychological processes in geometry. He says in [4], Ifrad l'magal p.3, that «the discussion of visual perception and geometry of vision - is philosophical, connected with psychological research and abstract imagination...». He may be taken to anticipate modern theory of perception.

Husserl founded a phenomenology of perceptive space which inspired a Gestalt theory, as developed by later psychologists, to deal with the perception of space and time. We shall come back to this later. In existential philosophy Sartre developed a theory of imagination in [26].

The importance of the psychology of perception in geometry is confirmed in modern times by the remarks of the great mathematician, Henri Poincaré [22] (and W. Kohler for that matter), that the perceptive space is different from the euclidean; thus negating the celebrated kantian thesis on the a priori nature of the euclidean space. In fact it was found out that perceptive geometry was hyperbolic [18]. These observations are confirmed by our examples; namely, that the intuition of the euclidean space finds its limit in the finite. It seizes to function in the domains of the infinitely large and the infinitely small (potential and actual infinities). This results into nonhomogenities of the euclidean space as we shall discuss later.

Genetical epistemology and the intuition:

We shall start with an analysis of Birunis examples in the frame of a genetical theory of intuition, due to J. Piaget [21]. The intuitions involved in Birunis examples revolve arround two types of intuitions, symbolizing and operational. The first arises from the «representation imagée», and the second from an interiorisation of the operational act which permits us to visualize in abstractum. Making use of his symbolizing intuition, Biruni would have concluded the intersection of his pairs of controversial parallels (which is clear from the figures). However, he is prevented from maintaining this conclusion by falling a victim of psychologism; thus permitting his operational intuition to take over. It was observed by Piaget that the introduction of the transfinite in mathematics was due to this intuition, which permits a progressive liberalisation of the perceptive model.

In the first example, the infinite sequence of (converging) points on the converging parallels, seize to meet. In this domain of the infinitesimal, the symbolizing intuition has already seized to function; for it is not possible to make a visual image of an infinitistic process. We have already remarked that the psychologies involved in this paradox resemble these of the eleatic paradoxes, and shall be analized in the context of existential psychoanalysis. We shall discover that Biruni was not completely liberated from the psychologism associated with the moving lines as he declares when he replaces the moving kinematics by a static model. A moving body is not the same as a body in various positions. The space of the trajectory of the points on the moving parallels is not the ordinary euclidean space. New spatio-temporal intuitions come into play.

The situation is similar in Biruni's second example (except for kinematics). Here, the infinitely small comes up with the infinitely large. The operational intuition takes the form of mathematical induction. In the domain of the infinitely large, the symbolizing intuition once more seizes to function. The slanting line, joining the heads of the perpendiculars, which was visualized to be straight in the finite, continues to appear so in the infinite due to the operational intuition. We shall look back on this in context of psychology of perception. Biruni did not need to introduce trigonometry, of which he was a master, into the science in order to discover his error, for he was working on the heuristic level. In view of a modern theory of infinitesimals, his intuition has not failed him.

Looking back upon Kindi's and Biruni's example from the view point of a genetical theory of intuition, we conclude that the geometrical intuitions are essentially euclidean. Yet we encounter an extension or a refinement of the euclidean intuition via Piaget's operational intuition which allows for infinitistic concepts in geometry. Such an extended intuition was also called a «transintuition», by M. Winter. Based upon formal reasoning (transintuitive definitions and axioms), a new equally acceptable concept of the euclidean space is reached. Indeed, neither Kindi nor Biruni went that far as to introduce the concept of infinity in geometry which was done for the first time by Keppler, as we shall see. They have developed, however, an intuitive awareness of this extended system with the aid of an operational intuition which appeared in the form of infinite sequences (of operations). It was due to the conflict between both intuitions, the symbolizing and the operational, that the authors disclosed the conflict in the form of camouflaged paradoxes.

In so far, infinitistic and non-euclidean methods could be regarded to be born in the same genetical moment, in seaking a generalisation of the euclidean intuition and a liberalisation from the myth of its universality. What is missing from the scene, is a syntactical or axiomatic formulation. I have already shown in [12] that Kindi introduced a formal system (axiom system) in which he showed that the notion of the infinite was contradictory with a set of acceptable facts or axioms. Is this the reason behind his failure to delve into the infinite in his example, which he could have done very neatly?

Existential Psychoanalysis:

It could not be denied that symbols in geometry play a fundamental role. The relations between the symbol and the symbolized is equally dominant in modern mathematical logic in the form of a semantical-syntactical duality. The deciphering of symbols and their relations to the symbolized, is the object of study in existential psychoanalysis. This is done by first attacking the pre-ontological state which is perceived in non reflected handling. This process is very valuable in analizing geometrical reasoning, where certain principles were used only implicitly; or subsconciously such as in Euclid's elements. Mathematicians became gradually aware of these pitfalls and developed sound deductive systems (compare Hilbert's axioms with Euclid's).

In an axiomatic system the symbol is linguistic rather than figurative. The developement of a proper geometrical language and symbolism is no doubt the key to rigorous geometrical reasoning. It is certain that the absence of an infinitistic mathematical language, prevented our authors from producing a convincing argumentation. They were, therefore, unable to put to paper all that they had in mind, so that we are obliged to recure to philosophy and psychoanalysis to reconstruct the mental and psychological processes behind the transcription. We have many examples from the history of mathematics where important ideas remained stagnant for centuries until a proper language was discovered that carried them to full growth. Paradoxes of logic are a good example of this phenomenon. In this paper we have given several examples of such paradoxes. A constructive study of these phenomena was made possible through phenomenology and existentialism.

Existential psychoanalysis was also applied to objects «psychoanalyse der Gegenstandlichen Qualitäten». We are not aware if this kind of psychoanalysis was applied to the history of scientific discoveries; but such a possibility is suggested in a declaration made by Sartre in «l'être et le néant» «... if a historical fact is taken as a factor of the psychic evolution and as a sympbol of that evolution .... existential psychoanalysis seaks to determine the original choices».

We shall give a sketch of an application of Sartre's existential psychoanalysis to the resolution of paradoxes of space and motion.

The same analysis applies to the understanding of Biruni's first example, the fallacy of the moving parallel lines. This is taken from the passage of «l'être et le néant» where Sartre discusses temporality and motion and tries to explain the Eleatic paradoxes, particularly the paradoxes of the arrow and that of Achilles. We have explained earlier in this paper that Biruni's fallacy results from motion in geometry. He replaced the concept of motion by that of change when he worked with his figure. This distinction between motion and change (or displacement) was elaborated upon in Sartre.

Biruni is actually tracing the trajectories of two points on two sets of moving lines, with the consequence that these two moving line trajectories never meet. Thus the fallacy is not in the curvedness of the alleged straight lines under construction; as is the case in his second paradox. Let us sum up Sartre's remarks. Sartre starts with a phenomenology of motion, and argues that 'motion is not derivable but given...motion determines space and is not determined by it... It is the mathematical tendency to treat a moving body as a being at rest, that would change the length of a line without drawing it out of its state of rest'. This fact accounts, in our case, for the non-intersection of the lines. Sartre arrives at the conclusion that Zeno's arguments arise from a naive concept of space, but he makes no attempt at clarifying this concept.

Thus in this deformed euclidean space, standard arguments fall short, and Biruni sees no contradiction in concluding the non-intersectability of the lines. Exactly how motion determines space is seen from Sartre's following remark, «The foundation of space is, therefore, the reciprocal exterioricity which comes to being through the for - itselfs, whose origin is the fact that being is what it is ... There is space in so far as the This is revealed as exterior to the Thises». Another fundamental entity in the case of the moving lines, is the trajectory of the points which describe a line. Sartre defines trajectory as «the nothing that measures and signifies exterioricity-to-itselfs; as the constitution of the exterioricity in the unity of a single being». The temporal identification of the moving body with itself across the constant positioning of its own exterioricity, causes the trajectory to reveal itself, that is to cause space to arise in the form of an evanescent becoming'.

Geometric atomism

Since the examples of Kindi and Biruni are connected with Greek geometric atomism, we shall give a brief sketch of its history. Atomism was introduced for the first time in Greek philosophy by Leukipp. It was formulated by his student Demokrit; it is said, in defiance to the Eleatic thesis of the impossibility of motion and emptiness. The counter part of this theory was introduced in geometry by the Pythagoreans, and persisted in the History of mathematics in the form of a dualism between the discrete and the continuous. Yet the Demokritian concept of a line as composed of an infinity of infinitely small atoms became untenable in the presence of a theory of incommensurables [6]. According to H.G. Zeuthen it was the merit of the Pythagoreans, trying to save the atomistic theory, to conceive of atoms that had infinitesimal dimension; But that the rejection of the theory infinitesimals in favour of the infinite divisibility in Greek geometry was due to the Eudoxian elements.

Clearly in an axiomatic foundation of an infinitesimal theory of magnitudes as formulated for the first time by A. Robinson [24], there is no fear of contradiction with infinite divisibility, nor, for that matter, with any other theory of mathematical analysis. This is exhibited by the introduction of a model, as we shall explain in a while.

The reduction of the atomistic theory to a theory of infinitesimals, would have quite understandably, brought about many controversies in the history of mathematics and was a source of paradoxes. However, it seems that historians of mathematics were quite incongruous about several issues in the history of Greek atomism. We fail to find a logical explanation to their hypothesis regarding the theory of infinitesimals, atomism, the Eleatic paradoxes, and the theory of incommensurables; which appear to be unjustified in the light of Robinson's work.

It is quite clear that a theory of finite atomism is immediately refutable by a theory of rationals; and that the Eleatic paradoxes fead upon the belief that if infinite atomism existed, then certain processes, such as motion, are prohibited; as they require the accomplishment of an infinite task. Nevertheless, there is no direct relation between the phenomenon of an infinite process and a theory of incommensurables, or of infinitesimals as declared earlier by historians.

Geometric atomism confronted Greek mathematicians with several dilemma's. How could the Archimedean (or the Eudoxian) axiom be applied to infinitesimals? How could a line of finite extent be composed of elements (points) of no dimension (non-existant)? Does not this problem bring back the old problem of the creation ex nihili?

We shall look at Aristotle's answer to some of these questions since he was well studied by Arabic scholars. Points according to Aristotle, are no more entities (constituents making up the continuous) but rather, idealised designations permitting the identification of intervals «physics» [2].

This explanation is quite plausible. A tract on indivisible lines, could not have been composed by Aristotle. According to [2], it should be attributed to Theophrastus or to some pupil of Aristotle. What we are sure of, is that Aristotle was aware of the second Eudoxian discovery and its relation to the infinite divisibility. This he had enunciated on several occasions in «Physics».

Atomism persisted in disguised form in the works of Kindi and Biruni. The relation of Biruni's first example to atomism is the same as that of the Eleatic paradoxes, which we already discussed in some detail. Biruni's awareness of the dormant philosophical problems regarding atomism and the foundations of geometry is quite justified in view of our earlier remarks about Biruni's statement in Ifrad l'magal.. His personal opinion on this subject is not very clear. According to Sezgin [28], he disagreed with the atomists as well as with their opponents, the geometricians. «Den Atomisten sind auch nicht wenige (zweifelhafte) Behauptungen eigen die den Geometern wohl bekannt sind, doch sind die Worte deren die den Atomisten Wiedersprechen, noch weniger annehmbar». It should be quite worthwile to work through Biruni's doubts and critique of the views of contemporary geometricians and atomists.

As for Kindi, we have seen that in the geometrical example he has given, the finite line th is composed of points (which although infinite, might be exhausted) in which case a contradiction with the parallels pos.arises. The atomism here is only implicit, th though finite in extent, is made up of indefinitely many points (atoms).

In the case of Kindi's paradox, it is not the implicit atomism, which we accused him of, that makes up the core of the paradox, but rather the isomorphism or the correspondence between two infinite processes. Thus this paradox does not anticipate a theory of infinitesimals but a theory of infinite elements in geometry. Such a theory was introduced by Johann Keppler in the sixteenth century; and then systematized by Gerard Désargues. An extended euclidean plane was obtained in which the new concepts of points at infinity, lines at infinity, and the plane at infinity were shown to be logically consistent [11]. The concept of isomorphism was discoved by Leibniz, and was called «similitude». It received its modern formulation in the nineteenth century [6].

Kindi's example stands against Aristotle's remark, that mathematicians do not actually need the concept of the infinite, unboundedness is sufficient; in «Physics» bk III-ch7. We have mentioned our second example on infinite magnitudes.

In conjuction with the problem of the infinitely large we run into the philosophical problems about the non-empirical character of geometry. A distance geometry, «Geometrie der Ferne» was proposed by physicists, which is non-euclidean in the distance; under certain situations. We refer the reader to [10], for a criticism of this geometry. This is particularly significant to Biruni's second example, where there is a definite schism between the «distant» and the «near».

It is not surprising to detect this atomism in Kindi. We find it in modern set theory and under the same circumstances as Kindi's, in relation with Cantor's isomorphism and set theories. Cantor's set theory ran into paradoxes because of negative definitions, such as the empty set and infinite ordinals. In geometry the concept of a line as a set of points of no dimensions, runs into the following foundational problems. We have already seen that Aristotle anticipated such problems. The line in the euclidean sense coincides with our modern set of real numbers, designated by R. Because of continuity and completeness properties of R, this line has a universal character. This situation was abolished by the introduction of a formal theory of infinitesimals, due to Abraham Robinson [24]. From his so called nonstandard model construction, the identification of points with elements of R turns to be only relative. Looking at an «enlargement» or «nonstandard» model, R* of R; we see that R has infinitely many holes (the infinitesimals) when viewed in R*. Thus the concept of a point in the euclidean sense is relative.

In the so called modern geometry, we meet atomism once more. A line is an abstractly defined set of points (atoms). The problem of paradoxes in mathematics remains unsettled so long as we have negative definitions; infinities, points etc. In that sense we are agreeing with the intuitionistic school of logic. These fears are confirmed by the history of logic. Modern logic tries to overcome psychologism for the benefit of logicism. It is questionable whether this is completely possible. In other words, we are confirming Poincaré's view on the role of psychology in geometry [22], a view which was opposed by Russel.

Mathematical heuristics:

In the second Criterium which we have given in the beginning of this paper, we agreed to the benefit of heuristics in evaluating mathematical works. The literature abounds on the subject of heuristics including systematic studies for the establishment of such heuristics. In the history of science, heuristics reflects itself in the traditional dichotomy between the deductive and the inductive thinking in the form of ars inviendi and ars disserendi. There is no reason to under-estimate the role of heuristics as a form of inductive or intuitive reasoning. In fact great discoveries in science could be attributed to such reasoning; because once such ideas are born it is almost routine work to check their validity a forterio. Once, a famous mathematician expressed himself to the effect that, it takes much luck to become a great mathematician; but only great mathematicians obtain such a luck.
There is, therefore, good reason to coin the notion of «historical heuristics» in the genetical development of mathematical thinking. In that sense non-rigorous proofs or constructions, such as the ones we just presented, could be regarded as the initial stages of a scientific discovery; even if no direct causality is detected between the historical event and the mature (formalized) theory. Heuristics are, therefore, a mode of being in mathematical reasoning. It is not excluded that guided heuristics could be utilised for the benefit of scientific discoveries; as done in brain-storming techniques.

We hope that we have reached the goals set forth in the beginning of the paper; and that the philosophical and psychological theories have greatly illuminated the ignorance encapsulating the scientific value of antique mathematical works. As a by-product or preciprocal fead-back, such an analysis of mathematical works should enrich philosophy itself. Great philosophers have always availed themselves of mathematical examples in formulating an epistemology. We only mention the examples of Aristotle, Kant, and Husserl.

Experimental and Gestalt psychology:

It is a remarkable fact indeed that Biruni's prophecy on the importance of psychology for visual perception has been confirmed experimentally in the very recent years. We have already mentioned that experimental and Gestalt psychologists have developed theoretical and experimental techniques to investigate the perception of space, time, and motion.

Luneberg was the first to put mathematical foundations for binocular vision in 1947 [19]. He showed that the perceptive space is Riemanian with constant negative curvature. His work attracted the attention of many scientists who extended his results in several directions. Experimentalists confirmed Luneburg's theory that in the case of most subjects the visual space was hyperbolic [18]. In the case of kinematics, the relativistic effect in the trajectory of moving bodies was confirmed in Caelli's article on the Lorentz transformations and the perception of motion. This is an unique result indeed, concerning Sartre's most intricate existential philosophy of being, which led him to the hypothesis that «the motion of objects determines space ... causing space to arise in the form of an evanescent becoming», as we have seen earlier. This is a case of relativistic phenomenon predicted by an existential philospher. A similar phenomenon is to be found in Heidegger «Sein und Zeit» [16]. Thus Sartre's remarks are confirmed by experimental psychologists and by Biruni's first fallacy as we pointed out. Perceptive relativistic effect prevents the infinite crossing of the parallels.

A similar situation is responsible for our observations on the deformations in the (mental) perception of the parallel lines in Biruni's second example in the distant space. In a very recent article by Indow in the Journal of Mathematical Psychology [18], the autor presents theoretical and experimental results on parallel alleys in the horizontal and horopter planes. He shows that parallel and equidistant lines are not the same in these planes. Theoretical work shows that in the horopter plane, convexity prevails in the distance. This is not yet confirmed experimentally. These deformations in addition to the fact that in hyperbolic geometry infinitely many parallels could be constructed parallel to a given line from a point outside that line, could account for Biruni's lines of the third kind and the fact that his curved line was thought to be straight. In this case, Biruni fell a victim of psychological fallacies, he anticipated, and of the atomistic theory that the challenged. It would be quite valuable to carry such experiments to full spectrum, hoping to explain imperfections and fallacies in antique geometry on psychological grounds. This requires a great number of historical data, as well as developed empirical theories; neither of which is available at present. Furthermore, many fine choices on which we have to decide, come into play, such as the type of planes, the nature of the so called euclidean mappings, the frame and the no frame situations etc. see Indow [18]. How are we to decide exactly which case of visual space was responsible for the psychological choices of our mathematicians?

We also suggest the establishment of experimental tricks to treat the phenomenology of the perception of the infinitesimal in geometry.

Special theory of relativity and the infinite

Let us compare, Achilles paradox with Biruni's first paradox. We may imagine Achilles and the tortorise to be following each other by jumping over the set of points from one parallel line to the next (as they converge towards a limit). As Achilles jumps over to the line on which the tortoise stands, the tortoise jumps over the next (and nearer) line. They will never meet since the number of parallels is infinite. We have already looked at Sartre's concept of motion as determining the space. Sartre's concept is non-euclidean and relativistic at once. We may say that in a Poincaré model of hyperbolic geometry the same situation reappears. The concept of distance is deformed and contracting; as the lines approach the boundaries. What was infinite in the euclidean space is now finite and bounded relative to the euclidean metric. Thus clearly, the Poincaré model demonstrates a relativistic phenomenon; exactly by mapping the unbounded (infinite) sequence onto an infinite sequence converging to a finite bound, with proper metric transformations. This goes in parallel with relativising unbounded velocities to that of the speed of light which is taken to be a universal constant in the special theory of relativity. This is done using Lorentz transformations.

Does Biruni's paradox add any thing to the Achilles paradox of shrinking space (if we may call it so)? The answer is yes; for in the Achilles case there is no direct bearing upon Euclid's axioms, more exactly the parallels postulate, as in the case of modern non-euclidean geometry. In the case of Biruni, this phenomenon of a shrinking space (which he did not utter as such, of course, but as an infinite divisibility) resulted in the constructibility of lines of the third kind; which defies the nature of the euclidean space and the euclidean postulate.

This observation would explain the fact that Biruni was against the atomistic view (5). The impossibility of traversing the infinite sequence is not due to atomicity any more, but rather to relativity. This would also explain the fact that he saw no contradiction with the parallels postulate in his example, for he was visualizing a new concept of space; a new contracting metric which prevents lines joining sequences of points on two set of parallels, from meeting.

Based upon this argument, we may safely conjecture that it was due to the direct ties between ancient and arabic works on the parallels postulate, and modern non-euclidean geometry, which led to formulations based upon this postulate. This is only «coincidential» and presents a good example of the role of historicity in the formation of concepts. There are many roads to non-euclidean geometry; via the introduction of infinities and infinitesimals, or relativistic considerations that we just described, or via axiomatics. There is nothing absolute about the nature of the parallels postulate. Indeed, it admits of several formulations which are certainly non-degenerate (i.e. logically non-equivalent). The role of the parallels postulate is only due to a particular axiomatisation. Other axiomatisations appeared making use of different notions and axioms such as Tarski's [29]. From the physical (versus logical) point of view, it is not the axiomatisation that matters but rather the mathematical model behind it (compare this with our earlier remarks on the same point).

We may, therefore, safely say that Biruni, by considering the relationship between infinite divisibility and the parallels postulate, put his finger on the most critical point in the history of geometry and applied mathematics in general. Biruni made no direct comments other than the ones we mentioned, and which demonstrate only indirect awareness of Biruni, of the universal implications of his constructions. The extent of his awareness could hopefully become clearer upon the discovery of the rest of the manuscript, and other works of Biruni's. Instead of an absolute evaluation of Biruni's work and originality we shall be content with a relative one. At this point we leave the problem open, for it will require a great deal of historical data in order to make such a comparison of developing awareness of concepts.

Detailed analysis of the role of infinitesimals in non-euclidean and relativistic concepts goes beyond the limits of this paper and is worthy an investigation of its own, which we hope to undertake.

In the mean-while, we do hope that we have not overstrained our imagination in interpreting Biruni's work, and fallen into the trap that we have warned ourselves against in the beginning of this paper. If the reader follows our reasoning carefully, and does not hasten to conclusions, we doubt that he will raise such accusations against our views. We are no less puzzled than he might find himself to be; that so much could come out of this work. We have just made it clear that it is not our intention to «clear the Puzzle» as Biruni thought he was doing to Kindi. We do not claim that Biruni's fallacy anticipate geometric relativity. We want, rather, to point to the mysterious relationship between psychology, intuition and reality.

In concluding, we hope that this paper has also shown the role of inter-disciplinary work in the future of scientific discovery. We have here an example of several non-related disciplines coming together to build an integral totality. The introduction of puristic and experimental philosophical, psychological and psycho-analytic theories, the physical theory of relativity and mathematical logic all stood together to explain an apparantly naive work in the history of geometry.

Moreover, we should like to stress the value of puristic and metaphysical reasoning in philosophy and psychology in predicting empirical results. There is a great tendency by modern philosophers and psychologists to foresake those routes ans dismiss such philosophers as pedantic and esoteric. Pure mathematics had to face similar accusations until, most unexpectedly, its methods proved to be a most valuable source of models for the physical world.

We are, therefore, entering a higher state in the inter-relations between he pure and the applied in philosophy and mathematics. A state in which our intuitions seem to fail us completely. But exactly in that state we hold witness to a new kingdom of scientific «prophecy».

NOTES

(1) By infinity we mean the potentially infinite in mathematics, dismissing the actually infinite, which was formalised for the first time by Cantor, and allegedly proved to be a contradictory notion by Kindi [12, 13]. We rule out the metaphysical controversies arising from these terms and ad here to a metaphorical usage in an informal language of early Arabic mathematics.

(2) Logistics refers to mathematical logic, this term was used until the beginning of our century.

(3) Psychologism is a tendency to make intervene psychological factors in deductive reasoning. According to Husserlian Phenomenology it is a new term for nominalism.

(4) In other words, due to the isomorphism, to point h on th, corresponds a point on gd, contradicting the parallels postulate.

(5) Aristotle spoke of Zeno's paradox in the De Generatione as the argument which convinced the atomist of a contradiction in infinite divisibility.

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