ELIMINARILITY, NON-CREARIVITY AND EXPLICIT DEFINABILITY

R. KLEINKNECHT

Shoenfield has shown in [1], chap. 4.6, by means of a syntactical proof, that every explicitly definable (predicate or function) constant is both eliminable and occurs in a formula which satisfies the condition of non-creativity. In the following it will be shown syntactically, that every eliminable constant is also explicitly definable.

Since, according to Beth's definability theorem, explicit definability is equivalent to implicit definability, if follows that the condition of eliminability is equivalent both to explicit and implicit definability.

§ 1. Syntactical Preliminaries

All the following considerations refer to first-order logic with identity. The *primitive signs* used will be object variables (ov's), object constants (oc's), n-ary predicate constants (pc's), n-ary function constants (fc's), the logical signs «$\neg$», «$\rightarrow$», «$\wedge$», the identity symbol «$\equiv$» and the parantheses «(»and«)». A *language* is any (possibly empty) class of constants. *Terms* and *formulas* are defined in the usual manner. If A is any formula, if $x_1, \ldots, x_n$ are distinct ov's and $s_1, \ldots s_n$ any terms, then $[A, x_1/s_1, \ldots, x_n/s_n]$ is that formula which results from A by simultaneous substitution of $s_1, \ldots, s_n$ for $x_1, \ldots, x_n$. If Σ is a language and K is a constant, then we will denote the language $\Sigma \cup \{K\}$ simply by $\Sigma + K$. Let Σ-*terms* (Σ-*formulas*) be terms (formulas) which only contain constants from Σ. We will denote the class of free ov's of a formula A by fr (A) and the class of all Σ-formulas by spr_Σ.

For further considerations we will need the concept of substitution of a constant in a formula by a formula. Here we can limit ourselves to the treatment of pc's and fc's, since the treatment of fc's can easily be transferred to oc's. The treatment of substitution is needed for lemmas 1 and 2 but not for theorems 1 and 2 (which are the heart of the paper).

§ *1.1. Substitution of Formulas for Predicate Constants*

Let p be an n-ary pc and E a formula which contains exactly n free ov's. Inductively, we now assign to every formula A a formula [A, p/E], which results from A by substitution of E for p.

(1a) If A ia an atomic formula of the form $ps_1 \ldots s_n$ and $fr(E) = \{x_1, \ldots, x_n\}$ (whereby $x_1, \ldots, x_n$ are distinct ov's and x_i $(1 \leq i \leq n)$ is the i-th free ov of E), then let $[A, p/E] = [E, x_1/s_1, \ldots, x_n/s_n]$.

(1b) If A is an atomic formula in which p does not occur, then let $[A, p/E] = A$.

(2) If A and B are any formulas and if x is an ov, then let $[\neg A, p/E] = \neg[A, p/E]$, $[A \rightarrow B, p/E] = [A, p/E] \rightarrow [B, p/E]$ and $[\wedge xA, p/E] = \wedge x[A, p/E]$.

§ *1.2. Substitution of Formulas for Function Constants*

Let f be an n-ary fc and E a formula which contains exactly n + 1 free ov's. We shall assign to every formula A a formula [A, f/E], which results from A by substitution of E for f. But since complex terms, i.e., terms which contain several fc's, can appear in A, [A, f/E] can not be formed as easily as can [A, p/E]. To construct the formula [A, f/E], first we form a *term normal form* $\overline{A}$ which is equivalent to A, in which the same constants and ov's occur as in A, and which has the property that each one of its atomic subformulas contains at most one constant. The second step consists of constructing a formula $[[\overline{A}, f/E]]$, which results from $\overline{A}$ by substitution of E for f. If A is an arbitrary formula, then [A, f/E] can be identified with the formula $[[\overline{A}, f/E]]$. In order to form $\overline{A}$, we first inductively assign to every term s and to every ov x an *identity formula* ifo(s, x):

(1) If s is an ov or an oc, then let $ifo(s, x) = s \equiv x$.

(2) If s has the form $gs_1 \ldots s_m$, then let $ifo(s, x) =$

$\vee u_1 \ldots \vee u_m (\bigwedge_{i=1}^{m} ifo(s_i, u_i) \wedge gu_1 \ldots u_m \equiv x)$, whereby $u_1, \ldots, u_m$ are the first m ov's which do not occur in $s_1, \ldots, s_m$.

To every formula A there will now be assigned by induction a term normal form $\overline{A}$:

(1a) If A is an atomic formula of the form $ps_1 \ldots s_m$, then let $\overline{A} =$

$\vee u_1 \ldots \vee u_m (\bigwedge_{i=1}^{m} ifo(s_i, u_i) \wedge pu_1 \ldots u_m)$, whereby $u_1, \ldots, u_m$ are the first m ov's which do not occur in $s_1, \ldots, s_m$.

(1b) If A is an atomic formula of the form $s_1 \equiv s_2$, then let $\overline{A} =$ $\vee u_1 \vee u_2 (ifo(s_1, u_1) \wedge ifo(s_2, u_2) \wedge u_1 \equiv u_2)$, whereby u_1, u_2 are the first two ov's, which do not occur in s_1, s_2.

(2) If A and B are any formulas and x is an ov, then let $\overline{\neg A} = \neg\overline{A}$, $\overline{A \to B} = \overline{A} \to \overline{B}$ and $\overline{\wedge x A} = \wedge x \overline{A}$.

By induction we now assign a formula [[A, f/E]] to every *term normal form* A.

(1a) If A is an atomic term normal form of the form $fy_1 \ldots y_n \equiv y_{n+1}$ and if fr (E) = $\{x_1, \ldots, x_{n+1}\}$ (whereby $x_1, \ldots, x_{n+1}$ are distinct ov's and x_i $(1 \leq i \leq n+1)$ is the i-th free ov of E), then let [[A, f/E]] = $[E, x_1/y_1, \ldots, x_{n+1}/y_{n+1}]$.

(1b) If A is an atomic term normal form, in which f does not occur, then let [[A, f/E]] = A.

(2) If A and B are any term normal forms and if x is an ov, then let $[[\neg A, f/E]] = \neg[[A, f/E]]$, $[[A \to B, f/E]] = [[A, f/E]] \to [[B, f/E]]$ and $[[\wedge x A, f/E]] = \wedge x [[A, f/E]]$.

If A is an *arbitrary* formula, then let $[A, f/E] = [[\overline{A}, f/E]]$.

§ 2. *Lemmas*

For the following considerations, we want to assume that a complete and correct deductive system is given. We write $T \vdash_{\Sigma} A$, if $T \cup \{A\} \subseteq spr_{\Sigma}$ and a deduction of A from T exists in which all terms are Σ-formulas. Instead of $\emptyset \vdash_{\Sigma} A$ we will simply write $\vdash_{\Sigma} A$. We will denote by $cn_{\Sigma}(T)$ the consequence class $\{A \mid T \vdash_{\Sigma} A \text{ and } fr(A) = \emptyset\}$ of T. We will furthermore say that T is an *Σ-theory* ($Th_{\Sigma}(T)$ for short), if $T = cn_{\Sigma}(T)$. If A is any formula and x is any ov, then let $\vee !xA$ stand for the unity formula $\vee x \wedge y ([A, x/y] \leftrightarrow x \equiv y)$, whereby y is the first ov which does not occur in $\wedge xA$. Now let Σ be any language, p an n-ary pc with $p \notin \Sigma$ and f an n-ary fc with $f \notin \Sigma$. In addition let $E \in spr_{\Sigma}$.

Lemma 1: Let $A \in spr_{\Sigma+p}$ and $fr(E) = \{x_1, \ldots, x_n\}$ (whereby $x_1, \ldots, x_n$ are distinct ov's). Then $\wedge x_1 \ldots \wedge x_n (px_1 \ldots x_n \leftrightarrow E) \vdash_{\Sigma+p} [A, p/E] \leftrightarrow A$.

Lemma 2: Let $A \in spr_{\Sigma+f}$ and $fr(E) = \{x_1, \ldots, x_{n+1}\}$ (whereby $x_1, \ldots, x_{n+1}$ are distinct ov's). Then $\wedge x_1 \ldots \wedge x_{n+1} (fx_1 \ldots x_n \equiv x_{n+1} \leftrightarrow E) \vdash_{\Sigma+f} [A, f/E] \leftrightarrow A$.

The proofs of both these lemmas can be easily obtained by means of induction of the degree of A.

The following lemma can be proved by elementary logical transformations.

Lemma 3: Let $fr(E) = \{x_1, \ldots, x_n+_1\}$ (whereby $x_1, \ldots, x_n+_1$ are distinct ov's). Then $\wedge x_1 \ldots \wedge x_{n+1} (fx_1 \ldots x_n \equiv x_{n+1} \leftrightarrow E) \vdash_{\Sigma+f} \wedge x_1 \ldots \wedge x_n \vee ! x_{n+1} E$.

§3. Eliminability, Non-Creativity and Explicit Definability

The concept of eliminability of a constant in a theory can be determined as follows:

K is *eliminable in* T *relative to* Σ (in brief: $Elim_{\Sigma}(K, T)$ iff

(1) K is a constant not contained in Σ;

(2) $Th_{\Sigma+K}(T)$;

(3) for every A in $spr_{\Sigma+K}$ there is a B in spr_{Σ} with $T \vdash_{\Sigma+K} B \leftrightarrow A$.

We will introduce the concept of non-creativity in two steps.

Relative to K, D *satisfies the requirement of non-creativity in* T *on the basis of* S, Σ (in brief: Ncr_{Σ} (D, K, T, S)) iff

(1) $Th_\Sigma(S)$;

(2) K is a constant not contained in Σ;

(3) $D \in spr_{\Sigma+K}$;

(4) $T = cn_{\Sigma+K}(S \cup \{D\})$;

(5) for every A in spr_Σ it holds that: if $T \vdash_{\Sigma+K} A$, then $S \vdash_\Sigma A$.

K *can be non-creativly introduced into* T *on the basis of* Σ (in brief: $Ncr_\Sigma(K, T)$) iff there exists a D and an S with $Ncr_\Sigma(D, K, T, S)$. It can easily be shown that $Ncr_\Sigma(K, T)$ iff there exists a D with $Ncr_\Sigma(D, K, T, T \cap spr_\Sigma)$.

We will define the concept of explicit definability first for predicate constants and then for function constants.

D is an *explicit definition of the predicate constant* p *in* T *relative to* Σ (in brief: $ExplDefPr_\Sigma(D, p, T)$) iff

(1) p is a pc not contained in Σ;

(2) D has the form $\wedge x_1 \dots \wedge x_n (px_1 \dots x_n \leftrightarrow E)$, whereby E is a Σ-formula with $fr(E) = \{x_1, \dots, x_n\}$ and $x_1, \dots, x_n$ are distinct ov's;

(3) $Th_{\Sigma+p}(T)$;

(4) $D \in T$.

p is an *explicitly definable predicate constant in* T *relative to* Σ (in brief: $ExplDfr\ Pr_\Sigma(p, T)$) iff there exists a D with $ExplDefPr_\Sigma(D, p, T)$.

Instead of the concept $ExplDefPr_\Sigma(D, p, T)$ many authors use the more general concept $ExplDefPr^*_\Sigma(D, p, T)$ which differs from the former only in that $fr(E) \subseteq \{x_1, \dots, x_n\}$ is required insted of $fr(E) = \{x_1, \dots, x_n\}$. This procedure is however somewhat artificial and is not conform with the usual manner of defining scientific concepts. Normally it is required that the same free ov's occur in the definiendum as in the definiens. Also technically there is no advantage to using the weaker requirement $fr(E) \subseteq \{x_1, \dots, x_n\}$. To show this we define:

$\text{ExplDfrPr}^*_\Sigma(p, T)$ iff there is a D with $\text{ExplDefPr}^*_\Sigma(D, p, T)$. Then it can easily be proved that $\text{ExplDfrPr}_\Sigma(p, T)$ iff $\text{ExplDfrPr}^*_\Sigma(p, T)$ (see proof of Theorem 1). Thus we see that the condition $\text{fr}(E) = \{x_1, \ldots, x_n\}$ is not too strong. The same holds true for the following definitions which refer to fc's.

D is an *explicit definition of the function constant* f *in* T *relative to* Σ (in brief: $\text{ExplDefFct}_\Sigma(D, f, T)$) iff

(1) f is a fc not contained in Σ;

(2) D has the form $\wedge x_1 \ldots \wedge x_{n+1}(fx_1 \ldots x_n \equiv x_{n+1} \leftrightarrow E)$, whereby E is a Σ-formula with $\text{fr}(E) = \{x_1, \ldots, x_{n+1}\}$ and $x_1, \ldots, x_{n+1}$ are distinct ov's;

(3) $\text{Th}_{\Sigma+f}(T)$;

(4) $D \in T$.

f is an *explicitly definable function constant in* T *relative to* Σ in brief: $\text{ExplDfrFct}_\Sigma(f, T)$) iff there exists a D with $\text{ExplDefFct}_\Sigma(D, f, T)$.

We shall now show that a constant in a given theory can be explicitly defined iff it satisfies the condition of eliminability in that theory. The explicit definability of a constant is thus equivalent to the eliminability of that constant.

Theorem 1: Let p be any pc. Then $\text{ExplDfrPr}_\Sigma(p, T)$ iff $\text{Elim}_\Sigma(p, T)$.

Proof: By Lemma 1 $\text{Expl DfrPr}_\Sigma(p, T)$ is a *sufficient* condition for $\text{Elim}_\Sigma(p, T)$. Conversely, let $\text{Elim}_\Sigma(p, T)$. Also, let n be the degree of p and let $x_1, \ldots, x_n$ be distinct ov's. Then $px_1 \ldots x_n \in \text{spr}_{\Sigma+p}$ and there exists, by the eliminability condition, a B from spr_Σ such that $T \vdash_{\Sigma+p} px_1 \ldots x_n \leftrightarrow B$.

Case 1: $\text{fr}(B) = \emptyset$. Let $D = \wedge x_1 \ldots \wedge x_n (px_1 \ldots x_n \leftrightarrow B \wedge \bigwedge_{i=1}^{n} x_i \equiv x_i)$. Then $T \vdash_{\Sigma+p} D$.

Case 2: $\text{fr}(B) \neq \emptyset$.

Case 2.1: $\text{fr}(B) \subseteq \{x_1, \ldots, x_n\}$. Let

$$B' = \begin{cases} B, \text{ if } \{x_1, \ldots, x_n\} \setminus \text{fr}(B) = \emptyset; \\ B \wedge \bigwedge_{i=1}^{n} y_i \equiv y_i, \text{ if } \{x_1, \ldots, x_n\} \setminus \text{fr}(B) = \{y_1, \ldots, y_m\}. \end{cases}$$

Then $\vdash_\Sigma$ $B \leftrightarrow B'$ and $\text{fr}(B') = \{x_1, \ldots, x_n\}$. Now let $D = \wedge x_1 \ldots \wedge x_n (px_1 \ldots x_n \leftrightarrow B')$. Then $T \vdash_{\Sigma+p} D$.

Case 2.2: $\text{fr}(B) \nsubseteq \{x_1, \ldots, x_n\}$. Let $\{y_1, \ldots, y_m\} = \text{fr}(B) \setminus \{x_1, \ldots, x_n\}$

and $D = \wedge x_1 \ldots \wedge x_n (px_1 \ldots x_n \leftrightarrow \wedge y_m B \wedge \bigwedge_{i=1}^{n} x_i \equiv x_i)$. Then $T \vdash_{\Sigma+p} D$.

In all cases it results that there exists a D with $\text{ExplDefPr}_\Sigma(D, p, T)$.

Theorem 2: Let f be an arbitrary fc. Then $\text{ExplDfrFct}_\Sigma(f, T)$ iff $\text{Elim}_\Sigma(f, T)$.

Proof: Similar to Theorem 1 (by Lemma 2).

What has been proved for fc's holds also for oc's. The theorems and proofs for the oc's will however not be formulated here. Since oc's can be treated as O-ary fc's all results which hold for fc's can be transferred to oc's.

We will now show that a constant which can be non-creatively introduced into a theory also satisfies the condition of eliminability in this theory.

Theorem 3: If $\text{Elim}_\Sigma(K, T)$, then $\text{Ncr}_\Sigma(K, T)$.

Proof: According to Theorem 1 and Theorem 2, it follows from $\text{Elim}_\Sigma(K, T)$ that there exists an explicit definition D of K in T relative to Σ. Let $T^* = T \cap \text{spr}_\Sigma$. Then $\text{Th}_\Sigma(T^*)$ and, by Lemma 1 and Lemma 2, $T = \text{cn}_{\Sigma+K}(T^* \cup \{D\})$. If D has the form $\wedge x_1 \ldots \wedge x_{n+1} (fx_1 \ldots x_n \equiv x_{n+1} \leftrightarrow E)$, then, since $D \in T$, by Lemma 3 $T^* \vdash_\Sigma \wedge x_1 \ldots \wedge x_n \vee ! x_{n+1} E$. As Shoenfield ([1], pp. 58-60) has shown, T is then a conservative extension of T*, i.e., $\text{Ncr}_\Sigma(D, K, T, T^*)$ holds. Consequently $\text{Ncr}_\Sigma(K, T)$.

REFERENCE

[1] SHOENFIELD, J.R., *Mathematical Logic* (Addison-Wesley, Reading, Mass., 1967).