Q, ENTAILMENT, AND THE PARRY PROPERTY

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In this paper I want to discuss how two different sets of logical and philosophical intuitions, issuing in two distinct non-standard logical systems, can be seen to converge, and perhaps to justify the fundamental soundness of each set. I also present the semantics, a suggested axiomatic base, and a general motivating rationale for a system that will combine the fruit of each of these intuitions, and to point to some of the implications this system will have for other areas of logic.

In 1933 W.T. Parry produced the system of analytic implication (PAI). (1) This system was evidently intended by him to give expression to a, or the, Kantian notion of analyticity, in whose terms a predicate in an analytic sentence will have been already contained in its subject. On the ground that entailment and valid argumentation ought to be a matter of conditionals (in the latter case with conjunctions of premises as antecedent, and conclusions as consequents) being analytically true in this sense, Parry developed his system with $\rightarrow$ taken as a primitive connective for 'entails' or 'analytically implies'. Analytic implication builds upon the propositional calculus. It may be presented as any consistent and complete axiomatic base for PC, together with the following axioms and rules: (2)

All instances of the following schemata are axioms:

A1 $(A \cdot B) \rightarrow (B \cdot A)$

A2 $A \rightarrow (A \cdot A)$

A3 $A \rightarrow \sim\sim A$

A4 $\sim\sim A \rightarrow A$

A5 $[A \cdot (B v C)] \rightarrow [(A \cdot B) v (A \cdot C)]$

A6 $[A v (B . \sim B)] \rightarrow A$

(1) W.T. PARRY, «Ein Axiomsystem für eine neue Art vom Implikation (Analytische Implikation)», *Ergebnisse eines Mathematischen Kolloquiums* Vol. 4, pp. 5-6; and presented again, in English, in Dunn [see next footnote] and in A.R. ANDERSON and N.D. BELNAP, *Entailment* (Princeton, 1975), pp. 429-434.

(2) Provided by J. Michael DUNN, «A modification of Parry's analytic implication», *Notre Dame Journal of Formal Logic,* Vol. 13, pp. 195-205.

A7	$[(A \rightarrow B).(B \rightarrow C)] \rightarrow (A \rightarrow C)$
A8	$[A \rightarrow (B.C)] \rightarrow (A \rightarrow C)$
A9	$[(A \rightarrow B).(C \rightarrow D)] \rightarrow [(A.C) \rightarrow (B.D)]$
A10	$[(A \rightarrow B).(C \rightarrow D)] \rightarrow [(A \vee C) \rightarrow (B \vee D)]$
A11	$(A \rightarrow B) \rightarrow (A \supset B)$
A12	$[(A \leftrightarrow B).f(A)] \rightarrow f(B)$
A13	$f(A) \rightarrow (A \rightarrow A)$

(In A12 and A13, f(A) is any sentence with A occurring as a subsentence in it, and f(B), in A12, is the result of replacing one or more occurrences of A in f(A) with B. $\ulcorner A \leftrightarrow B \urcorner$ is of course defined as $\ulcorner(A \rightarrow B).(B \rightarrow A)\urcorner$.) There are two rules of inference:

(1) Detachment for analytic implication, $A, A \rightarrow B \vdash B$.
(2) Adjunction, $A, B \vdash \ulcorner A.B \urcorner$.

(An interpretation for the → (different from those of Parry and Dunn) will be offered below.)

The striking thing about analytic implication is that all theorems with → as main connective turn out to exhibit what has been called the Parry property: none of them contains any sentential variable in the consequent that doesn't also occur in the antecedent. This was precisely what Parry intended and had in mind, as giving expression to the Kantian doctrine of analyticity.

The idea was that A → B only if A entails B, or the inference from A to B is valid; and perhaps that if the inference from A to B is *formally* valid, then A → B.

M. Dunn showed (3) that Parry's original system could and ought to be strengthened. That is, plausible wffs with →, that possess the Parry property, were shown not to be provable in PAI. Moreover, if they were added as axioms, an interesting result, a version of the Deduction Theorem, would then be provable for the resulting strengthened system. To Parry's axioms, Dunn adds:

A14	$(A.\sim B) \rightarrow \sim(A \rightarrow B)$
A15	$A \rightarrow (\sim A \rightarrow A)$

(3) M. Dunn, *ibid.*

The original motivation for PAI is not challenged or explored by Dunn, although he does provide alternative semantic for the →, together with proofs of completeness and decidability. Dunn's strengthened version of PAI may be called AI.

Anderson and Belnap (4) take up the cudgels against the possibility of giving a coherent interpretation for Kant's dicta on analyticity.

They are especially concerned to do so because, in virtue of the Parry property, AI is a rival candidate for the analysis of entailment; a rival, that is, to Anderson and Belnap's own systems of entailment (E) and relevant implication (R). The original motivation to the latter was the conviction that contradictions do not entail all statements; and that tautologies are not entailed by all statements. But if entailment were the strict implication of the Lewis modal systems, or if validity meant the impossibility of true premises with false conclusions ('A entails B' ↔ 'A ∴ B is a valid argument'), then these results would occur. All of this is of course well-known. If the 'paradoxes' of strict implication are not to be provable then at least one of the rules used in deriving them must be rejected. Anderson and Belnap opt for disjunctive syllogism: its repudiation launches E, and R.

Parry never entered this debate. But his system, and Dunn's, clearly will have consequences for it. For if Parry's → were the right analysis of 'entails' then the 'paradoxes'would be unprovable, since $\ulcorner(p\,.\,\sim p)\rightarrow q\urcorner$ and $\ulcorner p\rightarrow(q\vee\sim q)\urcorner$ obviously will be unprovable in a system of analytic implication.

In AI in other words we find grounds for rejecting disjunctive addition, and can keep disjunctive syllogism as a valid rule of inference. A happy result for those many who find the case against disjunctive syllogism that Anderson and Belnap and their party mount unpersuasive, but who yet want an entailment connective free of the 'paradoxes'.

In a quite distinct corner of the logical landscape, A.N. Prior found himself in the early 1950's developing tense logic and thinking about proper names for contingently existing objects. He found intelligible and took seriously Russell's much earlier doctrine of logically proper names. Motivated by Russell (as Parry was by Kant), Prior developed the notion that propositions with logically proper names in them are

(4) A.R. Anderson and N.D. Belnap, *Entailment*, pp. 155, 429-434.

importantly unlike all other propositions: they are such that, if there hadn't been referents for these names, these propositions wouldn't even have existed (there wouldn't have been any such propositions as these ones), much less been true or false. With such reflections, Prior developed a system of modal logic, which he called Q, which would give expression to the idea of contingently existing (as opposed to contingently true or false) propositions. Prior originally set out the bare bones of the idea in *Time and Modality*. (5) Q was later axiomatized and given a semantics and consistency and completeness proofs. (6)

In Q the modal operators '$\Box$' and '$\Diamond$' are not interdefinable (as they are in standard Lewis-style modal logic). (i) and (ii).

(i) $\sim\Diamond\sim A \supset \Box A$
(ii) $\sim\Box\sim A \supset \Diamond A$

are both rejected (though their converses are theorem schemata for Q): if A were 'Aristotle is self-identical' (7) then we can maintain that it is not true that it is possible that (i.e., that there is a possible world where) it is false that Aristotle is self-identical. But it doesn't follow from that and isn't true that it is a necessary truth (true in all possible worlds) that Aristotle is self-identical (since this proposition doesn't exist in all worlds). And if we let A be «Aristotle is diverse from himself» then we can show (ii) false.

Q can be set out as follows: (8)

(5) Oxford, 1957, Ch. V.

(6) Details in R.A. BULL, «An axiomatization of Prior's modal calculus Q», *Notre Dame Journal of Formal Logic*, Vol. 5, 1964, pp. 211-214, and G.E. HUGHES and M.J. CRESWELL, *An Introduction to modal Logic* (London, 1968), pp. 303-305.

(7) I use 'Aristotle' as an example of a logically proper name, something which fairly clearly would not have been accepted by Russell, the originator of the concept of a logically proper name. I do happen to think that 'Aristotle' really *is* a logically proper name in English, although obviously establishing this would take a great deal of argument. But what is wanted in this context is an example only. So 'Aristotle' may be imagined here as being used idiosyncratically, in some context presupposing a private baptism by each reader of this paper, for some object he is in a position to apply a logically proper name to.

(8) The account of Q provided comes largely from HUGHES and CRESWELL, *loc. cit.*

Vocabulary and Formation rules are as usual for an axiomatized system of propositional logic (taking, say, '.' and '∼' as primitive) adding '◇' and '□' as primitive singulary connectives. Truth conditions for propositions are defined by 3 valued matrices for the truth-functional connectives, 1 representing truth, 2 unstatability or non-existence, 3 falsehood. Everything works as usual, except wherever a compound has a 2 for a propositional variable, for the whole wff the result is 2.

For □ and ◇:
we associate with every propositional variable an infinite sequence of numbers (1, 2, 3) which may be taken to represent the alternative situation with respect to that proposition in each possible world.

Prior originally conceived these sequences of numbers as representing the state of things for a proposition at successive moments of *time*, and Hughes and Creswell follow him in this. Replacing moments of time with possible worlds seems an obviously appropriate adjustment in the semantics for a system of modal logic. (9) (We must, in doing so, make all propositions in Q be propositions expressed by 'eternal sentences' – i.e., we must treat all propositions as being eternally true or eternally false or eternally non-existent in the worlds they are being assessed in. Alternatively we can construe 1, 2, and 3 as values of functions whose arguments are not simply propositions and (possible) worlds, but propositions, worlds, and times in those worlds. Every proposition then will have as many numbers in its 'sequence' for a given world as there are times in that world – presumably therefore for «most» – all ? – worlds, an infinite sequence. On either of these redeployments of Prior's semantics – adaptations of his semantics from a tense logic to a modal logic – the sequences associated with propositions will have superdenumerably many numbers in them, as there are superdenumerably many possible worlds. With *times* one seems to have more options: depending on what it is desired a time be, it seems possible, at least for worlds like what we take the actual one to be, to have only as many as aleph null times.)

(9) Prior himself characterized Q in this way in «Notes on a group of new modal systems», *Logique et Analyse*, New Series, Vol. 1-3, 1958-60, p. 122.

Then:

Rules for $\Box$:

(a) If the sequence for α consists entirely of 1's, then so does that for $\Box\alpha$.

(b) If the sequence for α contains any 2's, then $\Box\alpha$ has 2 wherever α has 2, but 3 in every other place.

(c) If the sequence for α contains no 2's but does contain some 3's, then the sequence for $\Box\alpha$ consists entirely of 3's.

Rules for $\Diamond$:

(a) If the sequence for α contains no 1's at all, then the sequence for $\Diamond\alpha$ is the same as that for α.

(b) If the sequence for α contains some 1's and some 2's (whether or not it contains 3's, as well), then the sequence for $\Diamond\alpha$ has 2 wherever α has 2, but 1 everywhere else.

(c) If the sequence for α contains some 1's but no 2's, then the sequence for $\Diamond\alpha$ consists entirely of 1's.

A wff α is said to be valid in Q iff no matter what sequences are associated with the variables in α, the sequence for α itself contains no 3's. Axioms [due to Bull (10)]:

Some complete basis for PC +

Q1 $\sim\Diamond\sim p \supset p$

Q2 $\Box p \supset p$

Q3 $(\Box p . \Box q) \supset \Box(p . q)$

RQ$\Box$a. $\vdash(\beta \supset \gamma) \rightarrow \vdash(\beta \supset \sim\Diamond\sim\gamma)$, provided β is fully modalized and each variable in β occurs in γ.
[a wff is fully modalized iff every variable in it occurs within the scope of '$\Box$' or '$\Diamond$']

RQ$\Box$b. $\vdash(\Box\alpha \supset (\beta \supset \gamma)) \rightarrow \vdash(\Box\alpha \supset (\beta \supset \sim\Diamond\sim\gamma))$, provided β is fully modalized and each variable in β occurs in either α or γ.

RQ$\Box$c. $\vdash(\Box\alpha \supset(\beta \supset \gamma)) \rightarrow \vdash(\Box\alpha \supset(\beta \vee \Box\gamma))$, provided β is fully modalized and the variables of β and γ each occur in α (11).

(10) See Bull, *loc. cit.*, and HUGHES and CRESWELL, *ibid.*

(11) It should be noted that HUGHES and CRESWELL, *loc. cit.*, give Bull's RQLc rule incorrectly. They replace Bull's second restriction, indicated here, with the restriction

Q is contained in S5, but not in any of S1 - S4.

We note also that the rule of necessitation, $\vdash \alpha, \vdash \Box\, \alpha$ will fail for Q, (as, it may be seen, it ought to: e.g., 'Aristotle is wise or it is false that Aristotle is wise' is a tautology, a substitution instance of a theorem of Q, but does not express a necessary truth, since the proposition doesn't exist in all worlds) and also that the S5 axiom schema $\ulcorner \Diamond A \vee \Box \Diamond A \urcorner$, will fail too. But all instances of the S4 axiom schema, $\ulcorner \Box A \vee \Box\Box A \urcorner$ are theorems of Q.

Q may be interpreted as a three valued logic. This wasn't Prior's intention. The idea was not that propositions (some of them at least) are true, false, or have some third truth value. It was that they are true, false, or it might have been that they not exist at all. Nonetheless, Q can be interpreted as three-valued. ([12])

Q is a modal logic. AI is not. That is, it does not contain modal operators among its primitive or defined vocabulary. Moreover, AI is a logic of entailment or logical implication (whether adequate or otherwise). Q is not a logic of entailment. It lacks an $\rightarrow$ connective and introduces no principle for obtaining $\rightarrow$ or $\dashv$ (the obvious formulation of the latter being barred – or requiring executive decision – by the non-equivalence of '$\Box$' with '$\sim\Diamond\sim$' in Q).

Yet Q and AI arrive, by different routes and for very different reasons for travelling those routes, at the same place.

Informally, for every motive Parry has for rejecting inferences to conclusions with variables not to be found among their premises there will be a comparable motive for Prior. Any such inference will be one where the variable found only in the conclusion can be instantiated with a contingently existing proposition, missing from some world where all the propositions instantiated for the premises are found.

In fact, I think Prior's motives perhaps provide the best ones for Parry's conclusions, and although not answering because not addressing the puzzlement Anderson and Belnap express about Kantian

that 'each variable in β occurs in either α or γ', which would allow the derivation of wffs which are invalid according to the semantics for '$\Box$' and '$\Diamond$' (e.g., '$\Box\, p \supset (\Diamond \supset \Box\, (p \vee q))$' and '$\Box\, p \supset (\Box\, p \supset \Box\, (p \vee q))$').

([12]) So interpreted, in the semantics intended here, its truth-functional propositional component will be equivalent to the system (called B3) first developed by G. BOCHVAR, in 1939. (For details see N. RESCHER, *Many-valued Logic* (New York, 1969), pp. 29-34.)

'identity thought through the subject', may help reinforce a conviction that this puzzlement is misplaced.

If I am understanding them correctly, Anderson and Belnap seem to make([16]) an odd complaint of AI, and on behalf of disjunctive addition. Slightly modifying their example, they say they think the inference from (say) 'Jones is a brother' to 'Jones is a brother or sister' is valid and cannot see how anyone could rationally suppose otherwise; hence (apparently) the case against disjunctive addition collapses.

If intended to justify the validity of disjunctive addition, this does not seem successful. It appears to be like trying to justify the validity of the inference pattern ⌜p.∵.q⌝ by pointing out that 'Jones is a bachelor' entails 'Jones is unmarried'. I.e., *every* inference pattern has valid substitution instances, and disjunctive addition is of course no exception. What requires justification is the claim that all substitution instances of disjunctive addition, including ones with wholly different subjects in the disjuncts in the conclusion, are valid.

The result of these reflections then will be a system, which we can call AI-Q, which unites AI with Q. It will consist of any complete base for PC, and '□', '◇', and '→' taken as primitive. We will utilize the semantics provided for Q, and conjoin with it a similar interpretation for the →.

That is, we use the same model, of infinite sequences of truth values (any of 1, 2, 3, for truth, unstatability (or non-existence), and falsehood, respectively) associated with a wff, to provide truth conditions for the →.

We determine values for a wff of the form $\alpha \rightarrow \beta$, in a world w as follows:

(1) For all worlds w, α has value 2 in w or β has value 2 in w iff $\alpha \rightarrow \beta$ has value 2 in w.

(2) a) If there is a world w such that α has value 1 in w, then if every world where α has 1 is a world where β has 1 then in all worlds where neither α nor β take 2, $\alpha \rightarrow \beta$ has value 1.

b) If there is a world where α has value 1 then, if there is a

([13]) ANDERSON and BELNAP, *op. cit.*, pp. 155, 430.

world where α takes 1 and β takes either 2 or 3 then in all worlds where neither α nor β take 2, $\alpha \rightarrow \beta$ has value 3.

(3) a) If there is no world where α takes value 1, but there is a world where α takes 3 then, if there is a world where α has 3 and β has 2 then in all worlds where neither α not β take 2, $\alpha \rightarrow \beta$ has value 3.

b) If there is no world where α takes value 1, but there is a world where α takes 3, then if there is no world where α has 3 and β has 2 then in all worlds where neither α nor β take 2, $\alpha \rightarrow \beta$ has value 1.

A wff with $\rightarrow$ as main connective will be valid in AI-Q iff it never takes value 3.

Some comments on these truth conditions:
First, they are true to the intuition that for any pair of propositions, α, β, either it is the case that α entails β or it is false that α entails β: Second, we assume that truth leads only to truth for a genuine entailment, i.e., if α entails β then, where α is true, β must be true also. Otherwise, all is, I think, intuitive and straightforward. For the cases of propositions incapable of being true (propositions taking 3 in every world they exist in, some of these propositions only existing in some worlds), the doctrine incorporated in the above principles is: propositions incapable of truth entail all (and only) propositions existing in the worlds *they* (the propositions unable to be true) exist in. This will clearly not endear these semantical principles to advocates of relevant implication, for it will imply that all necessary falsehoods entail all propositions that exist in all worlds. (Nor, in the same spirit, will another consequence of these principles: They will imply that any necessary truth is entailed by any proposition existing in the same world as that necessary truth.)

It is to be observed that these remarks apply to propositions, not to propositional variables. Thus, the wff $\ulcorner(p \,.\, \sim p) \rightarrow q\urcorner$ will not be valid in AI-Q (since it can take value 3), but it will have true substitution instances – sentences that result from replacing its propositional variables with sentences – in which the replacements for p and q are irrelevant to each other. So long as the propositions expressed by the sentences with which p and q are replaced exist in all and only the same worlds, the resulting proposition – the substitution instance of ⌜$(p . \sim p) \rightarrow q$⌝ with which we are concerned – will be true.

Will this result – that according to these semantics a contradictory proposition will imply any proposition existing in the same world (even if it won't be true that every contradictory propositional form will imply every other propositional form – i.e., wff of AI-Q) – will this result be false to the original philosophical or semantical motivation Parry had for AI? I don't think so. AI is a system of formal logic and as such is blind to the meanings of particular instances of its wffs. In so far as Parry might have wanted his system to capture meaning connections between subjects and predicates (or antecedents and consequents of conditionals) it could not have done so, precisely because it is a formal logic (without relevance conditions). That Jones is a bachelor entails that Jones is unmarried; but the best a system of propositional logic can do with this is represent it as an instance of ⌜$p \rightarrow q$⌝ which is of course an invalid wff in AI. Analogously, that ⌜Jones is a bachelor and not a bachelor entails that Jones raises sheep⌝ is true (or will be true according to the principles my semantics for AI-Q express), but can only be represented in AI-Q (as in AI), by ⌜$(p . \sim p) \rightarrow q$⌝, which is likewise invalid (in both AI and AI-Q). If Parry's Kantian intuitions incline him to suppose this proposition untrue, then something other than AI has to be designed to declare it so: and in fact nothing else (unless it took account of the meaning or content of particular propositions) could do this. So, the *formal* desiderata Parry has for designing AI are fully satisfied by AI-Q and its semantics.

Generalizing, we may say that AI-Q is a logic that makes minimal alterations to standard logic. Roughly, it makes adjustments in standard logic only as they are prompted by the idea of a contingently existing proposition. AI-Q will modify its AI component also. Instead of Parry's A12 we will have

> A12* $[(A \leftrightarrow B) . f(A)] \rightarrow f(B)$, provided that B in ⌜$f(B)$⌝ does not occur within the scope of a non-truth-functional or a non-modal sentence connective.

We modify Parry's (and Dunn's) axiom in this way because as it stands in their formulations of analytic implication, the axiom is untrue, as reflection on the well-known phenomena of substitutions into 'intentional' contexts and what they can lead to will testify. Of course it may be said that, in the language in which AI-Q is expressed, the only connectives available are truth-functional or modal ones, so the restriction we impose is already taken care of. But AI-Q is intended not just to be a self-contained system, consistent and complete with respect to its own notion of validity. It is also intended to be the 'true' logic of alethic modality and entailment. That is, we want all substitution instances of its theorems in a natural language to be true (and only such substitutions to be 'logically' true).

In constructing an axiomatic base for AI-Q it will not be sufficient simply to add Q to AI, for many wffs with both '$\rightarrow$' and '$\Box$' or '$\Diamond$' will be valid according to the conjoined semantics for the connectives but will not be provable from the conjunction of the axiomatic bases. A strengthening of the axioms of Q seems the appropriate direction to take. The result of replacing all occurrences of '$\supset$' in axioms of Q with '$\rightarrow$' produces wffs that are valid according to the combined semantics, and since A11 of AI – $(A \rightarrow B) \rightarrow (A \supset B)$ – ensures that all axioms of Q will be derivable from this strengthening, it will be appropriate to offer as the axiomatic base of AI-Q, the axioms and rules of AI (with A12* replacing A 12), plus:

A14 $\sim \Diamond \sim A \rightarrow A$
A15 $\Box A \rightarrow A$
A16 $(\Box A . \Box b) \rightarrow \Box (A . B)$

(i.e., conforming to AI, where we replace Q's axioms with axiom schemata).

The rules of Q we leave intact, except for RQ$\Box$ c, which we replace with

RQ$\Box$ c$'$ $\vdash (\Box \alpha \supset (\beta \supset \gamma)) \rightarrow \vdash (\Box \alpha \rightarrow (\beta \rightarrow \Box \gamma))$, provided
β is fully modalized and the variables of β and γ each occur in α.

So AI-Q will consist, in addition to the rules of AI, of RQ$\Box$ α, RQ$\Box$ b, and RQ$\Box$ c$'$. I have not yet proved that this axiomatization for AI-Q is complete, although I think that it is.

But AI-Q is readily shown consistent. That is, no substitution instance of any of A1 – 16 (replacing A12 with A12*) can ever take value 3 in any world, and every substitution instance of each member of this set takes 1 in at least one world.

A3 for example – $\ulcorner A \rightarrow \sim\sim A \urcorner$ – can be shown valid as follows. Whatever propositional variables occur in A, A will have a sequence of values associated with it. Wherever a 1 occurs, $\ulcorner \sim A \urcorner$ will have value 3 (in that world), hence $\ulcorner \sim\sim A \urcorner$ will have value 1. Wherever a 2 occurs, $\ulcorner \sim A \urcorner$ will have 2 also, so $\ulcorner \sim\sim A \urcorner$ will have 2. Wherever a 3 occurs, $\ulcorner \sim A \urcorner$ will have 2 also, hence $\ulcorner \sim\sim A \urcorner$ will have 3. (These results will follow from the truth conditions for '∼'.) So, A never takes value 1 while $\ulcorner \sim\sim A \urcorner$ has values 2 or 3, and A never takes value 3 while $\ulcorner \sim\sim A \urcorner$ has value 2. By the truth conditions for '→' therefore $\ulcorner A \rightarrow \sim\sim A \urcorner$ must always take 1 or 2. Hence, by definition, it is valid in AI-Q.

We can show A11 – $\ulcorner (A \rightarrow B) \rightarrow (A \supset B) \urcorner$ – valid as follows. Suppose A11 is not valid. Then there is a sequence of values associated with A and B which are such that there are cases of worlds in the sequence where $\ulcorner A \rightarrow B \urcorner$ has value 1 and $\ulcorner A \supset B \urcorner$ has value 2 or 3; or where $\ulcorner A \rightarrow B \urcorner$ has value 3 and $\ulcorner A \supset B \urcorner$ has value 2. (For these will be the only possible cases where A11 will be able to turn out 3.) In the former case: if $\ulcorner A \rightarrow B \urcorner$ has value 1 (in a world) then either A has 1 and B 1 also (in that world), or A has 3 and B has 1 or 3. (Since $\ulcorner A \rightarrow B \urcorner$ can't otherwise take value 1.) But $\ulcorner A \supset B \urcorner$ has 2 if and only if A or B has 2. So $\ulcorner A \supset B \urcorner$ cannot take 2. Hence $\ulcorner A \supset B \urcorner$ takes value 3. But if $\ulcorner A \supset B \urcorner$ takes 3 then if A has 1, B would have to have 3 (since it can't take 2). But if A takes 1 and B 3, then $\ulcorner A \rightarrow B \urcorner$ would have value 3 (by the semantics for '→') which contradicts the hypothesis that it takes 1. So it is impossible that $\ulcorner A \rightarrow B \urcorner$ takes value 1 and $\ulcorner A \supset B \urcorner$ takes values 2 or 3. And in the case where $\ulcorner A \rightarrow B \urcorner$ takes 3 and $\ulcorner A \supset B \urcorner$ takes 2: If $\ulcorner A \supset B \urcorner$ takes 2 then either A or B takes 2. But if either A or B takes 2, then $\ulcorner A \rightarrow B \urcorner$ will take 2 also, which contradicts our hypothesis that $\ulcorner A \rightarrow B \urcorner$ takes 3. So it is impossible for $\ulcorner (A \rightarrow B) \rightarrow (A \supset B) \urcorner$ to take any other value than 1 or 2, in any possible world. So all of its substitution instances are valid.

By similar arguments it can be shown that all of the other axioms of AI-Q are valid.

With those provisos, I close by commending to the reader the idea that AI-Q is the true system of modal logic, and the true system of entailment.[14]

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