GENTZEN-TYPE SYSTEMS FOR C, K AND SEVERAL EXTENSIONS OF C AND K; CONSTRUCTIVE COMPLETENESS PROOFS AND EFFECTIVE DECISION PROCEDURES FOR THESE SYSTEMS*

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§ 0. *Introduction*

In [1], page 42, M. Fitting gives an elegant Gentzen-type calculus for the system S4 of intensional logic. In this paper we will develop similar systems for C, K, CM, KM, C4, K4, CM4, KM4 (= S4), KD, KX, KB, KMB and KM4B (= S5). For each of these systems we can describe an effective procedure for searching a derivation of a given formula, which in a finite number of steps either will give a derivation of the formula or a countermodel for the formula. This yields us practical decision procedure and completeness for these systems.

As will become clear from the completeness proofs in section 2, for the semantics of these intensional logics we can restrict ourselves to frames $\langle I, R \rangle$ (or, in case of C-logics, normal frames $\langle I, R, N \rangle$), where I is (a subset of) the set of all finite sequences of natural numbers and R is defined as follows:

for C, K: iRj iff j is an immediate successor of i (i.e. if $i = \langle n_1, \ldots, n_k \rangle$, then $j = \langle n_1, \ldots, n_k, m \rangle$ for some natural number m),

for CM, KM: iRj iff i = j or j is an immediate successor of i,

for C4, K4: iRj iff j is a successor of i (i.e. if $i = \langle n_1, \ldots, n_k \rangle$, then $j = \langle n_1, \ldots, n_k, n_{k+1}, \ldots, n_{k+m} \rangle$ for some $m \geqslant 1$ and natural numbers $n_{k+1}, \ldots, n_{k+m}$),

for CM4, KM4: iRj iff i = j or j is a successor of i,

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for KB: iRj iff j is an immediate successor of i or i is an immediate successor of j,
for KMB: iRj iff i = j or j is an immediate successor of i or i is an immediate successor of j,
for KM4B (= S5): iRj for all i, j $\in$ I.
As shown in [7], the ideas used in this paper also can be used for a constructive treatment of the Counterfactual Logics VC and VCS of D. Lewis and R. Stalnaker respectively.

The procedures for searching derivations in the different systems and the completeness proofs in this paper are adaptations of the author's procedure and completeness proof for intuitionistic logic in [5].

As was pointed out to the author by S. Kripke, the results of this paper essentially are already in [6]. What is new, is the presentation, which, in the author's opinion, is less cumbersome and more perspicuous than the one in [6]; also more systems than in [6] are treated here, with great uniformity, and more Gentzen-type systems are explicitly presented.

In [4], page 75-82, K. Schütte gives constructive completeness proofs for the systems KM, KM4 (= S4), KMB and KM4B (= S5); however, he does not present Gentzen-type systems for these logics.

In [2], page 331-334, Hughes and Cresswell give systems of natural deduction for KM (= T), S4 and S5.

In [3], page 74-80, Prawitz gives systems of natural deduction for S4 and S5. However, neither Hughes and Cresswell, nor Prawitz give constructive completeness proofs with respect to these systems. (Hughes and Cresswell give constructive completeness proofs with respect to axiomatic systems, however only for T (= KM), S4 and S5).

In order to make this paper self contained, I will give all the necessary definitions below.

Definition 0.1: A *logic* is by definition a set L of formulas, which is closed under TFL (truthfunctional logic), INT (interchange of B and C, whenever B $\leftrightarrow$ C is in L) and SUB (uniform substitution of formulas for primitive formulas). A logic is *regular* if it contains $\Box$ (P & Q) $\leftrightarrow \Box$ P & $\Box$ Q and $\Diamond$P $\leftrightarrow \neg\Box\neg$P. A logic is *normal* if it is regular and it contains $\Box$ T.

Definition 0.2: C is the least regular logic.
K is the least normal logic.

$M := \Box P \to P$, $D := \Diamond T$, $B := P \to \Box\Diamond P$,
$4 := \Box P \to \Box\Box P$, $X := P \to \Box P$, $E := \Diamond P \to \Box\Diamond P$.

C – is defined as the least regular logic which contains – and K – is defined as the least normal logic which contains –.

Definition 0.3: A (Lemmon-Scott) *normal frame* is a triple <I, R, N>, where I is a non empty set, R is a binary relation on I and N is a subset of I. The elements of N are called normal worlds. A (Lemmon-Scott) *frame* is a normal frame <I, R, N> with N = I. Instead of <I, R, I> we will simply write <I, R>.

[ ] is an *interpretation* based on a normal frame <I, R, N> iff [ ] associates to each formula A a subset [A] of I, such that [ ] is truthfunctionally standard, i.e.

$[T] = I$, $[B \,\&\, C] = [B] \cap [C]$, $[B \vee C] =$
$[B] \cup [C]$, $[\neg B] = I - [B]$, $[B \to C] = [\neg B \vee C]$, and
$[B \leftrightarrow C] = [(B \to C) \,\&\, (C \to B)]$, and such that
$[\Box B] = \{i \in N \mid \forall j \in I\,(iRj \to j \in [B])\}$, and
$[\Diamond B] = \{i \in N \mid \exists j \in I\,(iRj \,\&\, j \in [B])\} \cup \{i \in I \mid i \notin N\}$.

Definition 0.4: Let [ ] be an interpretation based on a normal frame <I, R, N>·<I, R, N, [ ]> $\models_i$ A iff $i \in [A]$.
If not confusing, we write «$i \models A$» instead of «<I, R, N, [ ]> $\models_i$ A».<I, R, N, [ ]> $\models$ A iff [A] = I.

<I, R, N> $\models$ A (A *is true in* <I, R, N> iff for all interpretations [ ] based on <I, R, N>, [A] = I. For F a class of normal frames, $\models_F$ A iff for all <I, R, N> in F, <I, R, N> $\models$ A.

Definition 0.5: A normal frame <I, R, N> is a frame
for CM iff R is reflexive, i.e. $\forall i \in I[iRi]$,
for C4 iff R is transitive, i.e. $\forall i,j,k \in I[iRj \,\&\, jRk \to iRk]$,
for CM4 iff R is reflexive and transitive.

A frame <I, R> is a frame

for KM iff R is reflexive,
for K4 iff R is transitive,
for KM4 (= S4) iff R is reflexive and transitive,
for KD iff R satisfies seriality, i.e. $\forall i \in I\ \exists j \in I[iRj]$,
for KX iff R is trivial, i.e. $\forall i,j \in I[iRj \rightarrow i=j]$,
for KB iff R is symmetric, i.e. $\forall i,j \in I[iRj \rightarrow jRi]$,
for KMB iff R is reflexive and symmetric.
for KM4B (= S5 = KME) iff R is reflexive, transitive and symmetric.

Definition 0.6: $\models_C$ A iff A is true in all normal frames. $\models_K$ A iff A is true in all frames. $\models_{C-}$ A iff A is true in all normal frames for C −. $\models_{K-}$ A iff A is true in all frames for K −.

The following results are known (see e.g. [2] and [4]):
$A \in C$ iff $\models_C A$, $A \in C-$ iff $\models_{C-} A$,
$A \in K$ iff $\models_K A$, $A \in K-$ iff $\models_{K-} A$.

§*1. Gentzen-type systems for* C, K *and several extensions of* C *and* K

By a *signed formula* we mean an expression of the form TA or FA where A is a formula. A *sequent* S is a finite set of signed formulas.
S, $TB_1, \ldots, TB_n, FC_1, \ldots, FC_m$ will stand for
$S \cup \{TB_1, \ldots, TB_n, FC_1, \ldots, FC_m\}$.
Interpretation of $\{TB_1, \ldots, TB_n, FC_1, \ldots, FC_m\}$: if
$B_1 \& \ldots \& B_n$, then $C_1 \vee \ldots \vee C_m$. The rules below are to be read: if the situation(s) below is (are) the case, then the situation above is also the case.
Rules for &, ∨, ¬ *and* →:

T &:	S, TB & C S, TB, TC	F &:	S, FB & C S, FB ‖ S, FC

$$
\begin{array}{llll}
\mathrm{T}\vee: & \begin{array}{l}\mathrm{S},\ \mathrm{T}B\vee C\\ \mathrm{S},\ \mathrm{T}B\,\|\,\mathrm{S},\ \mathrm{T}C\end{array} & \mathrm{F}\vee: & \begin{array}{l}\mathrm{S},\ \mathrm{F}B\vee C\\ \mathrm{S},\ \mathrm{F}B,\ \mathrm{F}C\end{array}\\[2ex]
\mathrm{T}\neg: & \begin{array}{l}\mathrm{S},\ \mathrm{T}\neg B\\ \mathrm{S},\ \mathrm{F}B\end{array} & \mathrm{F}\neg: & \begin{array}{l}\mathrm{S},\ \mathrm{F}\neg B\\ \mathrm{S},\ \mathrm{T}B\end{array}\\[2ex]
\mathrm{T}\rightarrow: & \begin{array}{l}\mathrm{S},\ \mathrm{T}B\rightarrow C\\ \mathrm{S},\ \mathrm{F}B\,\|\,\mathrm{S},\ \mathrm{T}C\end{array} & \mathrm{F}\rightarrow: & \begin{array}{l}\mathrm{S},\ \mathrm{F}B\rightarrow C\\ \mathrm{S},\ \mathrm{T}B,\ \mathrm{F}C\end{array}
\end{array}
$$

Note that because $\{\mathrm{T}B \,\&\, C,\ \mathrm{T}B \,\&\, C\} = \{\mathrm{T}B \,\&\, C\}$, e.g.

$\begin{array}{l}\mathrm{T}B \,\&\, C\\ \mathrm{T}B \,\&\, C,\ \mathrm{T}B,\ \mathrm{T}C\end{array}$ is a derived rule.

Let A be a formula. A derivation of A in classical propositional calculus is by definition a finite set of sequents such that 1. the upper sequent is {FA} 2. the lowest sequents each contain TB and FB for some formula B and 3. each sequent, except the upper sequent, is the result of applying one of our rules to a sequent immediately above it.

Example: the following is a derivation of $\neg(P \,\&\, Q) \rightarrow \neg P \vee \neg Q$ in classical propositional calculus.

$$
\begin{array}{c}
\mathrm{F}\neg(P\,\&\,Q)\rightarrow\neg P\vee\neg Q\\
\mathrm{T}\neg(P\,\&\,Q),\ \mathrm{F}\neg P\vee\neg Q\\
\mathrm{T}\neg(P\,\&\,Q),\ \mathrm{F}\neg P,\ \mathrm{F}\neg Q\\
\mathrm{F}P\,\&\,Q,\ \mathrm{F}\neg P,\ \mathrm{F}\neg Q\\
\mathrm{F}P,\ \mathrm{F}\neg P,\ \mathrm{F}\neg Q\,\|\,\mathrm{F}Q,\ \mathrm{F}\neg P,\ \mathrm{F}\neg Q\\
\mathrm{F}P,\ \mathrm{T}P,\ \mathrm{F}\neg Q\quad\|\,\mathrm{F}Q,\ \mathrm{F}\neg P,\ \mathrm{T}Q
\end{array}
$$

If the rules $\mathrm{F}\neg$ and $\mathrm{F}\rightarrow$ are replaced by

$$\mathrm{F}\neg: \frac{\mathrm{S},\ \mathrm{F}\neg B}{\mathrm{S_T},\ \mathrm{T}B} \quad \text{and } \mathrm{F}\rightarrow: \frac{\mathrm{S},\ \mathrm{F}B\rightarrow C}{\mathrm{S_T},\ \mathrm{T}B,\ \mathrm{F}C} \text{ respectively,}$$

where $\mathrm{S_T} = \{\mathrm{T}A \mid \mathrm{T}A \in \mathrm{S}\}$, then one obtains a Gentzen-type system for intuitionistic propositional calculus (see [1], page 28-30). I have put a horizontal line in the intuitionistic $\mathrm{F}\neg$ and $\mathrm{F}\rightarrow$ in order to stress the transition from S to $\mathrm{S_T}$.

The systems C, K and others are obtained by adding to the rules, mentioned above for classical propositional calculus, the following rules T□ and F□. «◇A» is defined as «¬□¬A». In stating the rules, the following abbreviations are used:

$S_T = \{Tb \mid Tb \in S\}$, $S_{T\Box} = \{T\Box B \mid T\Box B \in S\}$,
$S_{T\neg\Box} = \{T\neg\Box B \mid T\neg\Box B \in S\}$, $S_{\Box} = \{TB \mid T\Box B \in S\}$

	T□	F□
C	$\dfrac{S, F\Box C_1, \ldots, F\Box C_n, T\Box A\ (n \geq 1)}{S_{\Box}, FC_i, TA\ (1 \leq i \leq n)}$	$\dfrac{S, T\Box B, F\Box A}{S_{\Box}, TB, FA}$
K	as for C	$\dfrac{S, F\Box A}{S_{\Box}, FA}$
CM	S, T□ A S, TA	as for C
KM	as for CM	as for K
C4	$\dfrac{S, F\Box C_1, \ldots, F\Box C_n, T\Box A\ (n \geq 1)}{S_{T\Box} \cup S_{\Box}, FC_i, T\Box A, TA\ (1 \leq i \leq n)}$	$\dfrac{S, T\Box B, F\Box A}{S_{T\Box} \cup S_{\Box}, T\Box B, TB, FA}$
K4	as for C4	$\dfrac{S, F\Box A}{S_{T\Box} \cup S_{\Box}, FA}$
CM4	as for CM	as for C4
KM4 = S4	as for KM	as for K4
KD	$\dfrac{S, T\Box A}{S_{\Box}, TA}$	as for K
KX	S, F□ C, T□ A S, FC, TA	S, F□ A S, FA
KB	$\dfrac{S, F\Box C, T\Box A}{S_{\Box} \cup S_1 \cup S_2, FC, TA}$	$\dfrac{S, F\Box A}{S_{\Box} \cup S_1 \cup S_2, FA}$

	where $S_1 = \{T \neg\Box \neg B \mid TB \in S\}$ and $S_2 = \{T \neg\Box C \mid FC \in S\}$	
KMB	as for KM	as for KB
KM4B = S5	as for KM	$\dfrac{S, F\Box A}{S_{T\Box} \cup S_{T\neg\Box} \cup S_1 \cup S_2, FA}$

Because $S, F\Box C, T\Box A = S \cup \{F\Box C, T\Box A\}, F\Box C, T\Box A$ and $S, F\Box A = S \cup \{F\Box A\}, F\Box A$, the following are derived rules:

KB

$$\frac{S, F\Box C, T\Box A}{S_{\Box} \cup S_1 \cup S_2, T\neg\Box\Box C,\ FC, T\neg\Box\neg\Box A, TA}$$

$$\frac{S, F\Box A}{S_{\Box} \cup S_1 \cup S_2, T\neg\Box\Box A, FA}$$

S5

$$\frac{S, F\Box A}{S_{T\Box} \cup S_{T\neg\Box} \cup S_1 \cup S_2, T\neg\Box\Box A, FA}$$

For similar reasons the following are derived rules:

CM	$\dfrac{S, T\Box A}{S, T\Box A, TA}$	
KX	$\dfrac{S, F\Box C, T\Box A}{S, F\Box C, FC, T\Box A, TA}$	$\dfrac{S, F\Box A}{S, F\Box A, FA}$

Intuitive Motivation. The rules given above can be read in two ways:

1. read upwards, as rules in the sense of Gentzen, interpreting the sequents rather than the signed formulas, the interpretation of $\{TB_1, \ldots, TB_n,\ FC_1, \ldots, FC_m\}$ being: if $B_1 \& \ldots \& B_n$, then $C_1 \vee \ldots \vee C_m$. For instance, rule KF $\Box$ with $S = \{T\Box B, TC, FE\}$ then says: if $B \supset A$, then $\Box B \& C \rightarrow \Box A \vee E$. In this reading a formula A is provable if $\{FA\}$ (to be read as $\rightarrow A$ or A) can be obtained by applying the rules to sequents of the form

{..., TB, FB, ...} (to be read as: if ... and B, then B or ...), which can be conceived as axioms.

2. read downwards, as semantic tableaux rules in the sense of Beth, interpreting the signed formulas rather than the sequents, the interpretation of T A being: A holds at world w, and of F A: A does not hold at world w. Rule K F□, for instance, now says: if □ A does not hold at world w, then there is a world w′, accessible from w, at which A does not hold; only if □ B holds at w, B will hold at w′; so in general we are not allowed to copy S below which is stressed by drawing a horizontal line between the upper and the lower sequent. In this reading A is provable if the supposition F A (A does not hold at w) turns out to be impossible.

Remarks. Note that rule C F□ is a consequence of rule CT □ and that rule C4 F□ is a consequence of rule C4 T□. Rule C T□, rule C4 T□ and rule K X T□ can only be applied if the T□ formula to which the rule is applied, is accompanied by some F□ formula. Rule C F□ and rule C4 F□ can only be applied if the F□ formula to which the rule is applied, is accompanied by some T□ formula. In rules C M T□, K X T□ and K X F□ there is no horizontal line between the upper and lower sequent, indicating that in applying these rules no formulas get lost.

Definition 1.1: Let A be a formula and let L be any of the systems described above (L = C, K, CM, ...). A *derivation of* A *in* L is by definition a finite tree of sequents such that 1. the upper sequent is {FA}, 2. each of the lowest sequents contains T B and F B for some formula B or, in case L is a K-system, contains F T, 3. each sequent, except the upper sequent, is the result of applying one of the rules for L to a sequent above it.
$\vdash_L$ A iff there is a derivation of A in L.

Theorem 1.2.: $\vdash_K \Box$ T, but not $\vdash_C \Box$ T.

1. For L = C or K, $\vdash_L \Box (P \;\&\; Q) \to \Box P \;\&\; \Box Q$ and $\vdash_L \Box P \;\&\; \Box Q \to \Box (P \& Q)$
2. For L = C or K, if n ≥ 1 and $\vdash_L B_1 \;\&\; \ldots \;\&\; B_n \to C$, then

$\vdash_L \Box B_1 \;\&\; \ldots \;\&\; \Box B_n \rightarrow \Box C$. Hence C and K are closed under INT.

3. If $\vdash_K A$, then $\vdash_K \Box A$.

4. For L = C or K, $\vdash_{LM} \Box P \rightarrow P$.

5. For L = C or K, $\vdash_{L4} \Box P \rightarrow \Box\Box P$.

6. For L = C or K, $\vdash_{LM4} \Box P \rightarrow P$ and $\vdash_{LM4} \Box P \rightarrow \Box\Box P$.

7. $\vdash_{KD} \Diamond T$.

8. $\vdash_{KX} P \rightarrow \Box P$.

9. $\vdash_{KB} P \rightarrow \Box\Diamond P$.

10. $\vdash_{KMB} \Box P \rightarrow P$ and $\vdash_{KMB} P \rightarrow \Box\Diamond P$.

11. $\vdash_{S5} \Box P \rightarrow P$, $\vdash_{S5} \Box P \rightarrow \Box\Box P$, $\vdash_{S5} P \rightarrow \Box\Diamond P$ and $\vdash_{S5} \Diamond P \rightarrow \Box\Diamond P$.

Proof: $\dfrac{F\Box T}{FT}$ is a derivation of $\Box T$ in K, but not in C.

1.

$F\Box(P \& Q) \rightarrow \Box P \& \Box Q$
$T\Box(P \& Q),\ F\Box P \& \Box Q$
$T\Box(P \& Q),\ F\Box P \;\|\; T\Box(P \& Q),\ F\Box Q$

$\dfrac{T\Box(P \& Q),\ F\Box P}{TP \& Q,\ FP}$	$\dfrac{T\Box(P \& Q),\ F\Box Q}{TP \& Q,\ FQ}$
$TP,\ TQ,\ FP$	$TP,\ TQ,\ FQ$

$F\Box P \& \Box Q \rightarrow \Box(P \& Q)$
$T\Box P \& \Box Q,\ F\Box(P \& Q)$
$\dfrac{T\Box P,\ T\Box Q,\ F\Box(P \& Q)}{TP,\ TQ,\ FP \& Q}$
$TP,\ TQ,\ FP \;\|\; TP,\ TQ,\ FQ$

2. Suppose L = C or K and suppose $\vdash_L B_1 \& \ldots \& B_n \rightarrow C$. Then there is a derivation of $B_1 \& \ldots \& B_n \rightarrow C$ in L. Such a derivation has the form

$FB_1 \& \ldots \& B_n \rightarrow C$
$TB_1 \& \ldots \& B_n,\ FC$
$TB_1, \ldots, TB_n,\ FC$
$\vdots$

Now the following is a derivation of $\Box B_1 \& \ldots \& \Box B_n \rightarrow \Box C$ in L:

$F\Box B_1 \& \ldots \& \Box B_n \rightarrow \Box C$
$T\Box B_1 \& \ldots \& \Box B_n,\ F\Box C$
$\dfrac{T\Box B_1, \ldots, T\Box B_n,\ F\Box C}{TB_n, \ldots, TB_n,\ FC}$
$\vdots$

The proof of 3 is analogous to the proof of 2. The proofs of 4, 5, 6, 7 and 8 are straightforward.

9.

$$\cfrac{\begin{array}{c} F\,P \to \Box\neg\Box\neg P \\ TP,\ F\Box\neg\Box\neg P \end{array}}{\cfrac{\begin{array}{l} T\neg\Box\neg P,\ F\neg\Box\neg P \\ F\Box\neg P,\ F\neg\Box\neg P \\ F\Box\neg P,\ T\Box\neg P \end{array}}{\begin{array}{c} F\neg P,\ T\neg P \\ TP,\ T\neg P \\ TP,\ FP \end{array}}}$$

The proofs of 10 and 11 are analogous to proofs already given

Theorem 1.3 (Soundness):

If $\vdash_{\overline{C}}$ A, then $\models_{\overline{C}}$ A. If $\vdash_{\overline{C-}}$ A, then $\models_{\overline{C-}}$ A.

If $\vdash_{\overline{K}}$ A, then $\models_{\overline{K}}$ A. If $\vdash_{\overline{K-}}$ A, then $\models_{\overline{K-}}$ A.

Proof: 1. Suppose $\vdash_{\overline{C}}$ A, i.e. there is a derivation of A in C. Because $\{FA\}$ is the upper sequent in the given derivation of A in C, it suffices to show by induction on the length of the given derivation that for each sequent $\{TB_1, \ldots, TB_n, FC_1, \ldots, FC_m\}$ in the given derivation of A in C it holds that $B_1 \& \ldots \& B_n \to C_1 \vee \ldots \vee C_m$ is true in all normal frames.
Basic step: The lowest sequent(s) in the given derivation contains (contain) TB and FB for some B and hence has (have) the form $\{TB, TB_1, \ldots, TB_n, FB, FC_1, \ldots, FC_m\}$.
And $B \& B_1 \& \ldots \& B_n \to B \vee C_1 \vee \ldots \vee C_m$ is true in all normal frames.
I discuss here only the induction steps for T$\Box$ and F$\Box$.
The other induction steps are straightforward.

Induction step for T$\Box$: $\dfrac{S,\ F\Box C_1, \ldots, F\Box C_n,\ T\Box A\ (n \geq 1)}{S_\Box,\ FC_k,\ TA\ (1 \leq k \leq n)}$

Let $S = \{TB_1, T\Box B_2, FC\}$. Then $S_\Box = \{TB_2\}$ and $S_\Box, FC_k, TA = \{TB_2, FC_k, TA\}$. The induction hypothesis says that $\models_{\overline{C}} B_2 \& A \to C_k$, i.e. $B_2 \& A \to C_k$ is true in all normal frames. We want to show that $\models_{\overline{C}} B_1 \& \Box B_2 \& \Box A \to C \vee \Box C_1 \vee \ldots \vee \Box C_n$. So

let [ ] be an interpretation based on a normal frame <I, R, N> and suppose $i \models B_1 \ \& \ \Box B_2 \ \& \ \Box A$. Then $i \in N$ and $\forall j \in I[iRj \rightarrow j \models B_2 \& A]$.
Hence, by the induction hypothesis $i \in N$ and $\forall j \in I[iRj \rightarrow j \models C_k]$, i.e. $i \models \Box C_k$.

Induction step for $F\Box : \dfrac{S,\ T\Box B,\ F\Box A}{S_\Box,\ TB,\ FA}$

Let $S = \{TB_1, FC_1, F\Box C_2\}$. Then $S_\Box = \varphi$ and $S_\Box$, TB, FA = {TB, FA}. The induction hypothesis says that $\models_C B \rightarrow A$. We want to show that $\models_C B_1 \& \Box B \rightarrow C_1 \vee \Box C_2 \vee \Box A$. So suppose $i \models B_1 \& \Box B$. Then $i \in N$ and $\forall j \in I[iRj \rightarrow j \models B]$. Hence, by the induction hypothesis, $i \in N$ and $\forall j \in I[iRj \rightarrow j \models A]$, i.e. $i \models \Box A$.

2. The other proofs are similar to the proof in 1.
Let us check for example the induction steps for $T\Box$ and $F\Box$ in the case of S5 (= KM4B = KME).

$T\Box : \dfrac{S,\ T\Box A}{S,\ TA}$ Let $S = \{TB_1, TB_2, FC_1, FC_2\}$

The induction hypothesis says that $\models_{S5} B_1 \ \& \ B_2 \ \& \ A \rightarrow C_1 \vee C_2$, i.e. $B_1 \& B_2 A \rightarrow C_1 \vee C_2$ is true in all frames with a reflexive, transitive and symmetric relation R. we want to show that $\models_{S5} B_1 \ \& \ B_2 \ \& \ \Box A \rightarrow C_1 \vee C_2$. So suppose $i \models B_1 \ \& \ B_2 \ \& \ \Box A$. Then, because R is reflexive, $i \models B_1 \ \& \ B_2 \ \& \ A$. Hence, by the induction hypothesis, $i \models C_1 \vee C_2$.

$F\Box : \dfrac{S,\ F\Box A}{S_{T\Box} \cup S_{T\neg\Box} \cup S_1 \cup S_2,\ FA}$

Let $S = \{TB_1, T\Box B_2, T\neg\Box B_3, FC\}$. Then $S_{T\Box} \cup S_{T\neg\Box} \cup S_1 \cup S_2 =$

$\{T\Box B_2,\ T\neg\Box B_3,\ T\neg\Box\neg B_1,\ T\neg\Box\neg\Box B_2,\ T\neg\Box\neg\neg\Box B_3,\ T\neg\Box C\}$.

The induction hypothesis says that

$\models_{S5} \Box B_2 \ \& \ \neg\Box B_3 \ \& \ \neg\Box\neg B_1 \& \neg\Box\neg\Box B_2 \ \& \ \neg\Box\Box B_3 \ \& \ \neg\Box C \rightarrow A$.

We want to show that $\models_{S5} B_1 \ \& \ \Box B_2 \ \& \ \neg\Box B_3 \rightarrow C \vee \Box A$.

So suppose that $i \models B_1 \,\&\, \Box B_2 \,\&\, \neg\Box B_3$ and $i \models \neg C$.

Because $i \models \Box B_2$ and R is transitive, $\forall j \in I[iRj \rightarrow j \models \Box B_2]$ (a).

Because $i \models \neg\Box B_3$, R is symmetric and transitive,

$\forall j \in I[iRj \rightarrow j \models \neg\Box B_3]$ (b). Because $i \models B_1$ and R is symmetric,

$\forall j \in I[iRj \rightarrow j \models \neg\Box \neg B_1]$ (c). Similarly,

$\forall j \in I[iRj \rightarrow j \models \neg\Box \neg\Box B_2]$ (d)

and $\forall j \in I[iRj \rightarrow j \models \neg\Box\Box B_3]$ (e). Because $i \models \neg C$ and R is symmetric,

$\forall j \in I[iRj \rightarrow j \models \neg\Box C]$ (f). By the induction hypothesis,

(a), ..., (f), $\forall j \in I[iRj \rightarrow j \models A]$, i.e. $i \models \Box A$.

Let L be any of the logics described above (C, K, CM etc.). If $\vdash_L A$, then by theorem 1.3 A is true in all frames <I, R, N> with R and N appropriately chosen. And it is well known (see e.g. [2] and [4]) that this implies that $A \in L$.

In theorem 1.2 we have shown that the axioms of L are derivable in the appropriate Gentzen-type system, mentioned above. Since it is not at all trivial to prove that Modus Ponens is a derived rule in our Gentzen-type systems (i.e. if $\vdash A$ and $\vdash A \rightarrow B$, then $\vdash B$), we cannot conclude from this that if $A \in L$, then $\vdash_L A$.

However, the completeness results of this paper (corollary 2.14) and the completeness results of for instance [2] and [4], together, yield that our Gentzen-type system for L is equivalent with the axiomatic characterization of L ($\vdash_L A$ iff $A \in L$). As a result, Modus Ponens is a derived rule in our Gentzen-type systems.

Another, more direct, way to prove the equivalence of the three notions $\vdash_L A$, $A \in L$, $\models_L A$, L being any of the logics described above, is to show the following

1. If $\vdash_L A$, then $A \in L$. The proof actually is similar to the proof of Theorem 1.3 (if $\vdash_L$, A, then $\models_L A$), replacing $\models_L A$ everywhere by $A \in L$.
2. If $A \in L$, then $\models_L A$ (soundness). The proof of this is straightforward.

3. If $\models_{\overline{L}}$ A, then $\vdash_{\overline{L}}$ $\tilde{A}$. This is our completeness result (see corollary 2.14).

Theorem 1.5: Let A, B and $\exists x[\Box A(x)]$ be sentences and let E be a formula of classical propositional calculus.

1. For L = K, KM, K4 and KM4 (= S4):
   a) Disjunction Property: if $\vdash_{\overline{L}} \Box A \vee \Box B$, then $\vdash_{\overline{L}} \Box A$ or $\vdash_{\overline{L}} \Box B$.
   b) Explicit Definability: if $\vdash_{\overline{L}} \exists x[\Box A(x)]$, then $\vdash_{\overline{L}} \Box A(v)$ and hence $\vdash_{\overline{L}} \forall x[\Box A(x)]$.
   c) If not $\vdash \neg E$ classically and $\vdash_{\overline{L}} E \to \Box A \vee \Box B$, then $\vdash_{\overline{L}} \Box A$ or $\vdash_{\overline{L}} \Box B$.
2. a), b) and c) do not hold for L = KB, KMB and KM4B (= S5).
3. For L = C, CM, C4 and CM4, not $\vdash_{\overline{L}} \Box A$, not $\vdash_{\overline{L}} \Box A \vee \Box B$, not $\vdash_{\overline{L}} \exists x[\Box A(x)]$ and not $\vdash_{\overline{L}} E \to \Box A \vee \Box B$.

Proof: 1. Suppose L = K, KM, K4 or KM4 and $\vdash_{\overline{L}} \Box A \vee \Box B$, i.e. there is a derivation of $\Box A \vee \Box B$ in L. Such a derivation starts with

$$F\Box A \vee \Box B$$
$$F\Box A, F\Box B$$

and then proceeds either with FA or with FB. Hence, either A or B is derivable in L. The proofs of b) and c) are similar.

2. For L = KB, KMB and KM4B, $\vdash_{\overline{L}} \Box P \vee \Box \neg \Box \Box P$, but not $\vdash_{\overline{L}} \Box P$ and not $\vdash_{\overline{L}} \Box \neg \Box \Box P$.
3. Follows immediately from the structure of the Gentzen-type systems for C, CM, C4 and CM4.

P.S. The rules for the quantifiers are as follows:

T $\forall$:	$\dfrac{S, T\,\forall x\,A(x)}{S, T A(a)}$		F $\forall$:	$\dfrac{S, F\,\forall x\,A(x)}{S, F A(a)}$ «a» new
T $\exists$:	$\dfrac{S, T\,\exists x\,A(x)}{S, T A(a)}$	«a» new	F $\exists$:	$\dfrac{S, F\,\exists x\,A(x)}{S, F A(a)}$

If we replace rule F $\forall$ by the following one

F $\forall$: $\dfrac{S, F\,\forall x\,A(x)}{S_T, F A(a)}$ («a» new), then one obtains a system for intuitionistic predicate calculus.

§ 2. *Effective Procedure for searching a derivation of a given formula, which in a finite number of steps gives either a derivation of the formula or a countermodel for the formula.*

The properties of the procedure mentioned above yield us the completeness of the Gentzen-type systems considered with respect to the corresponding semantics and in addition yield us a practical decision procedure for the systems considered.

The procedures for searching derivations in the different systems and the completeness proofs of this section are adaptations of the author's procedure and methods for intuitionistic logic in [5].

Definition 2.1: Let Γ be a set of signed formulas.

Γ is a Hintikka element iff

1. if $TB \& C \in \Gamma$, then $TB \in \Gamma$ and $TC \in \Gamma$,
2. if $TB \vee C \in \Gamma$, then $TB \in \Gamma$ or $TC \in \Gamma$,
3. if $TB \rightarrow C \in \Gamma$, then $FB \in \Gamma$ or $TC \in \Gamma$,
4. if $T \neg B \in \Gamma$, then $FB \in \Gamma$,
5. if $FB \& C \in \Gamma$, then $FB \in \Gamma$ or $FC \in \Gamma$,
6. if $FB \vee C \in \Gamma$, then $FB \in \Gamma$ and $FC \in \Gamma$,
7. if $FB \rightarrow C \in \Gamma$, then $TB \in \Gamma$ and $FC \in \Gamma$, and
8. if $F \neg B \in \Gamma$, then $TB \in \Gamma$.

Let L be any of the logics, considered in section 1 (C, K, C M, etc.)

Definition 2.2: We define a systematic procedure for searching a derivation of a formula E in L as follows:

Step 0: Consider $\{FE\}$. Apply all rules of L, which have no horizontal line, as many times as possible, without loosing any formulas in applying a rule, if not necessary.

Having finished step 0 of our procedure we have one (if $T\vee$, $F\&$ and $T\rightarrow$ have nowhere been applied) or more Hintikka elements Γ_0. Each Γ_0 is finite.

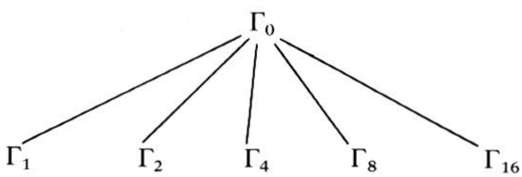
Motivation for step 1: Each Γ_0 may contain several $T\Box$ or $F\Box$ expressions. In each application of a $T\Box$ or $F\Box$ rule some formulas may get lost (that is why we have put a horizontal line in those rules), hence for each Γ_0 there may be several ways to go on. Each way may eventually give a sequent which contains TB and FB for some formula B or FT in the case of K-logics, so for each Γ_0 we have to consider all possibilities to go on.

Step 1: For L = C.

(a) For any F$\Box$ C, T$\Box$ A combination in Γ_0, say the m^{th}, we form a successor $\Gamma_{2^{m-1}}$ of Γ_0 as follows.

Consider $(\Gamma_0)_\Box$, F C, T A. Apply all rules of L (= C) which have no horizontal line, as many times as possible to $(\Gamma_0)_\Box$, F C, T A.

Having finished step 1 for L = C we have for each F$\Box$ C, T$\Box$ A combination in Γ_0 one (if T $\vee$, F & and T$\rightarrow$ have nowhere been applied) or more Hintikka elements $\Gamma_{2^{m-1}}$. Each $\Gamma_{2^{m-1}}$ is finite. If Γ_0 contains 1 + 1 F$\Box$ C, T$\Box$ A combinations, step 1 yields successors Γ_1, Γ_2, Γ_4, ..., Γ_{2^l} of Γ_0. So, having finished step 1 of our procedure for L = C, we have one (if T $\vee$, F & and T$\rightarrow$ have nowhere been applied) or more partial trees of the form



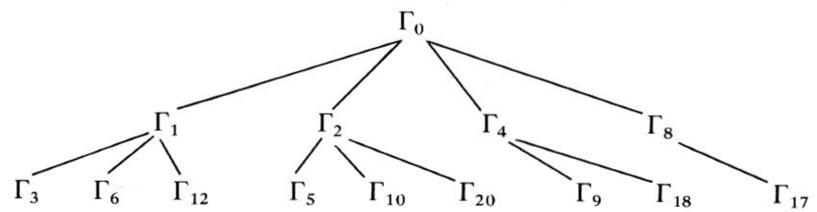
where each Γ_i is a finite Hintikka element.

Step 1 for the other logics L ($\neq$ C) is similar.
For L = K *replace* (a) *by* (b): For any F$\Box$ C, T$\Box$ A combination and for any F$\Box$ A expression in Γ_0, say the m^{th}, we form a successor $\Gamma_{2^{m-1}}$ of Γ_0 as follows. If a combination F$\Box$ C, T$\Box$ A occurs in Γ_0, consider $(\Gamma_0)_\Box$, F C, T A and proceed as in (a). If F$\Box$ A occurs in Γ_0, consider $(\Gamma_0)_\Box$, F A and apply all rules of L (= K) which have no horizontal line, as many times as possible to $(\Gamma_0)_\Box$, F A.
For L = CM *replace* (a) *by* (c): For any T$\Box$ B, F$\Box$ A combination in Γ_0, say the m^{th}, we form a successor $\Gamma_{2^{m-1}}$ of Γ_0 as follows.
Consider $(\Gamma_0)_\Box$, T B, F A. Apply all rules of L (= CM) which have no horizontal line (including rule T$\Box$), as many times as possible, to $(\Gamma_0)_\Box$, T B, F A.
For L = KM *replace* (a) *by* (d): For any F$\Box$ A expression in Γ_0, say the m^{th}, we form a successor $\Gamma_{2^{m-1}}$ of Γ_0 as follows. Consider $(\Gamma_0)_\Box$,

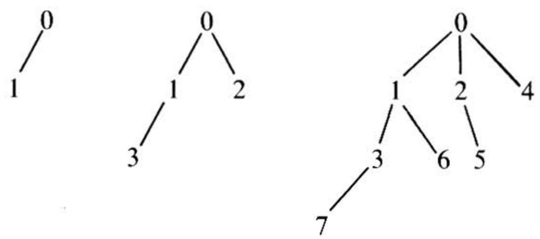
FA. Apply all rules of L (= KM), which have no horizontal line (including rule T□), as many times as possible, to $(\Gamma_0)_\square$, FA.
For L = C4, K4, and so on, the replacement of (a) in step 1 will be in accordance with the rules T□ and F□ of the particular system considered, similar to the replacement of (a) for K, CM and KM, considered above.

Step 2: Analogous to step 1, applied now to each Γ, which we have got from step 1.

Having finished step 2 of our procedure, we have one (if T ∨, F & and T → have nowhere been applied) or more partial trees of the form



The other steps of our procedure are similar.
The indexing of the Γ_i has been taken as follows:



and so on.

Remark: The systematic procedure for searching a derivation of a formula E in L, in definition 2.2, investigates all sequences of sequents which might be derivations of E. After finitely many steps it will become clear whether there is a derivation of E in L or not. If so, our procedure has constructed a derivation of E in L (theorem 2.7); if not, our procedure has actually constructed a countermodel for E, as we will show further on (theorem 2.13).

Definition 2.3: The trees we get by the procedure, described in definition 2.2, for searching a derivation of E in L, are called *search trees in* L *for* E.

Note there may be many search trees in L for E, because of the rules T ∨, F & and T →.

Example 2.4: $\vdash_{KD}$ □ (P & Q) → □ P & □ Q ?

Step 0 of the procedure of definition 2.2 yields two Hintikka elements Γ_0 and Γ_0':

F□ (P & Q) → □ P & □ Q	F□ (P & Q) → □ P & □ Q
T□ (P & Q), F□ P & □ Q	T□ (P & Q), F□ P & □ Q
T□ (P & Q), F□ P	T□ (P & Q), F□ Q

Step 1 yields two (partial) search trees in KD for
□ (P & Q) → □ P & □ Q:

F□ (P & Q) → □ P & □ Q		F□ (P & Q) → □ P & □ Q	
T□ (P & Q), F□ P & □ Q		T□ (P & Q), F□ P & □ Q	
T□ (P & Q), F□ P		T□ (P & Q), F□ Q	
T P & Q	T P & Q, F P	T P & Q	T P & Q, F Q
T P, T Q	T P, T Q, F P	T P, T Q	T P, T Q, F Q

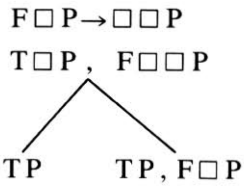
Note that we have found now a derivation of □ (P & Q) → □ P & □ Q in KD by taking the branch which contains T P and F P and the branch with contains T Q and F Q, together.

Example 2.5: $\vdash_{KD}$ □ P → □ □ P ?

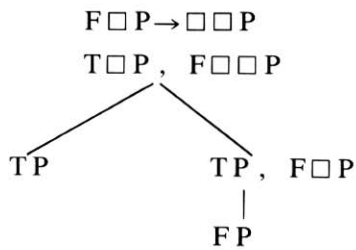
Step 0 of the procedure of definition 2.2 yields one Hintikka element Γ_0:

F□ P → □ □ P
T□ P, F□ □ P

Step 1 yields one partial search tree in KD for □ P → □ □ P:



And step 2' yields again one partial search tree in K D for $\Box P \rightarrow \Box\Box P$:



No further steps are possible and we have found that $\Box P \rightarrow \Box\Box P$ is not derivable in K D. We will see further on that we have actually constructed a frame for K D in which $\Box P \rightarrow \Box\Box P$ is not true.

Definition 2.6: A search tree in L for E is *closed* iff some node in the tree contains T B and F B for some formula B or, in case L is a K-logic, contains F T. A search tree in L for E is *open* iff it is not closed.

Theorem 2.7: If all search trees in L for E are closed, then $\vdash_L$ E.

Proof: Suppose all search trees in L for E are closed. Then there is a natural number N such that all search trees in L for E are closed before level (height) N. (If we would extend our treatment to the predicate calculus, we would need König's Lemma for this.) There are only finitely many different partial search trees in L for E of height N. Take from each partial search tree in L for E a finite branch which causes closure. These branches together yield a derivation of E in L.

Next we want to show that an open search tree in L for E actually is (or in case L = K B, K M B or K M 4 B, yields) a countermodel for E, i.e. a frame for L, in which E is not true.

A node j in a search tree is called an *immediate successor* (a *successor*) of a node i, if j results from i by one (one or more) step(s) in the systematic procedure of definition 2.2.

Lemma 2.8: Each (open) search tree τ in L for E has the following property: to each node (finite sequence of natural numbers) i in τ is associated a set of signed formulas which we will also denote with i, such that for each node i:

1. i is a finite Hintikka element
2. a) For L = C, C M, C4 and C M4: if T□ B, F□ A occurs in i, then there is an immediate successor j of i such that T B, F A occurs in j.
   b) For L = K, K M, K4, K M4, K D, K B, K M B and K M 4 B: if F□ A $\in$ i, then there is an immediate successor j of i such that F A $\in$ j.
   c) For L = K X: if F□ A $\in$ i, then F A $\in$ i.
3. a) For L = C, K, K D and K B: if T□ A $\in$ i, then for all immediate successors j of i, T A $\in$ j.
   b) For L = C M, K M and K M B: if T□ A $\in$ i, then T A $\in$ i and for all immediate successors j of i, T A $\in$ j.
   c) For L = C4, K4: if T□ A $\in$ i, then for all successors j of i, T A $\in$ j.
   d) For L = C M 4, K M 4 and K M 4 B: if T□ A $\in$ i, then T A $\in$ i and for all successors j of i, T A $\in$ j.
   e) For L = K X: if T□ A $\in$ i, then T A $\in$ i, provided for some C, F□ C $\in$ i.
4. For L = K D: for each i, if i contains a T□ or F□ expression, then there is an immediate successor j of i.

Proof: Immediate from the description of the systematic procedure for the different logics (definition 2.2) and the rules for the different logics.

Concerning 4, note that for example for L = K it may happen that T□ A $\in$ i and that there is no immediate successor of i because there is no expression of the form F□ C in i.

In case L is a B-logic (i.e. L = K B, K M B or K M 4 B), we can and have to modify an open search tree in L for E in order to be able to conceive of it as a frame for L, in which E is not true.

Definition 2.9: Let L be K B, K M B or K M 4 B and let τ be an (open) search tree in L for some formula E.

(a) For L = K B or K M B: if T$\Box$ A occurs in a node i of τ, add T A to the immediate predecessor of i and develop the new tree according to the prescriptions of the systematic procedure in definition 2.2.
(b) For L = K M 4 B (= S5 = K M E): if T$\Box$ A occurs in a node i of τ, add T A to each node in τ and develop the new tree according to the prescriptions of the systematic procedure in definition 2.2.

Let τ^* be the result of applying (a), respectively (b) to τ as many times as necessary.

Lemma 2.10: Let L be K B, K M B or K M 4 B and let τ^* result from τ as described in definition 2.9. Then τ^* has the following properties:

1. (a) For L = K B: if T$\Box$ A $\in$i, then T A $\in$j for all immediate successors and predecessors j of i.
   (b) For L = K M B: if T$\Box$ A $\in$i, then T A $\in$i and T A $\in$j for all immediate successors and predecessors j of i.
   (c) For L = K M 4 B: if T$\Box$ A $\in$i, then T A $\in$j for all nodes j in τ^*.
2. If τ is an open search tree in L for some formula E, then τ^* is also open (i.e. for no formula B, τ^* has a node which contains both T B and F B).

Proof: The proof of 1. is immediate from the definition of τ^*, and lemma 2.8.3. For 2, suppose L = K B, T$\Box$ A $\in$i, i a node in τ^* and T A in the immediate predecessor k of i would give a closure with a formula T B (F C) in k. Then T $\neg\Box\neg$B (T $\neg\Box$ C) and hence F$\Box\neg$B (F$\Box$ C) would occur in the node i in the original search tree τ and hence one of the successors of i in the original search tree τ would contain T A and F $\neg$B (F C) and hence T A and T B (T A and F C). Hence the original search tree τ would be closed.

The proofs for L = K M B and K M 4 B are similar.

Definition 2.11: For a search tree τ in L for E, let I_τ be the set of all nodes in τ, if L is not a B-logic and I_τ is the set of all nodes in τ^*, if L is a B-logic; let N_τ be the set of all nodes in τ, respectively τ^*, which contain T$\Box$ B for some formula B and

1. for L = C, K: iR_τj iff j is an immediate successor of i,
2. for L = C M, K M: iR_τ j iff i = j or j is an immediate successor of i,
3. for L = C4, K4: iR_τj iff j is a successor of i,
4. for L = C M4, K M4: iR_τj iff i = j or j is a successor of i,

5. for L = KD: $iR_\tau j$ iff either i does not contain any T□ or F□ expression or j is an immediate successor of i,
6. for L = KX: $iR_\tau j$ iff i contains F□ C for some formula C and j = i,
7. for L = KB: $iR_\tau j$ iff j is an immediate successor of i or i is an immediate successor of j,
8. for L = KMB: $iR_\tau j$ iff i = j or j is an immediate successor of i or i is an immediate successor of j,
9. for L = KM4B: $iR_\tau j$ for all i, $j \in I_\tau$.

Theorem 2.12: 1. Let τ be a search tree in C −. Then $<I_\tau, R_\tau, N_\tau>$ is a normal frame for C −.
2. Let τ be a search tree in K −. Then $<I_\tau, R_\tau>$ is a frame for K −.

Proof: Immediate from definition 2.11 and definition 0.5.
For KD we also need lemma 2.8.4.

Theorem 2.13: Let τ be an open search tree in L for E. Let $<I_\tau, R_\tau>$ ($<I_\tau, R_\tau, N_\tau>$) be the corresponding (normal) frame and let $[\ \ ]_\tau$ be the interpretation based on $<I_\tau, R_\tau>$ ($<I_\tau, R_\tau, N_\tau>$) defined by $[P]_\tau = \{i \in I_\tau \mid TP \in i\}$. Then for all $i \in I_\tau$ and for all formulas A,
1. if $TA \in i$, then $i \models A$
2. if $FA \in i$, then not $i \models A$.

Proof: By induction on A, using lemma 2.8 and lemma 2.10.
1. A = P: if $TP \in i$, then $i \models P$ by definition; if $FP \in i$, then because τ is open, $TP \notin i$ (lemma 2.10.2) and hence not $i \models P$.
2. The induction steps for ∨, &, → and ¬ are straightforward.
3. $A = \Box A'$.

(a) Suppose L = C and $T\Box A' \in i$. We have to show that $i \models \Box A'$, i.e. $i \in N_\tau$ and $\forall j \in I_\tau [iR_\tau j \rightarrow j \models A']$. Because $T\Box A' \in i$, it follows trivially that $i \in N_\tau$ and so, by the induction hypothesis, it suffices to show that $\forall j \in I_\tau [iR_\tau j \rightarrow TA' \in j]$.
And this follows from definition 2.11.1 and lemma 2.8.3a.

Suppose L = C and $F\Box A' \in i$. We have to show that $i \notin N_\tau$ or $\exists j \in I_\tau [iR_\tau j \ \&\ \text{not}\ j \models A']$. By the induction hypothesis it suffices to show that $i \notin N_\tau$ or $\exists j \in I_\tau [iR_\tau j \ \&\ FA' \in j]$. So suppose $i \in N_\tau$ i.e. i contains T □ B for some B. Then if follows from lemma 2.8.2a and definition 2.11.1 that $\exists j \in I_\tau [iR_\tau j \ \&\ FA' \in j]$.

(6) The proofs for the other logics are similar.

Theorem 2.7 and theorem 2.13 together yield the following *Corollary 2.14 (Completeness):* Let L be any of the logics mentioned above. If $\models_{L}$ E, then $\vdash_{L}$ E.

Proof: Suppose $\models_{L}$ E. Then for all search trees τ in L for E, $<I_\tau, R_\tau, N_\tau, [\ \ ]_\tau> \models_{io}$ E and F E ∈io, where i_o is the first node of I_τ. So by theorem 2.13, all search trees τ in L for E must be closed. And hence by theorem 2.7, $\vdash_{L}$ E.

Corollary 2.15 (Decidability): Let L be any of the logics mentioned above. Then L is decidable.

Proof: The decision procedure is afforded by attempting to construct a derivation of E in L according to the systematic procedure of definition 2.2. After finitely many steps this procedure either yields a derivation of E in L or a countermodel for E in L, i.e. a (normal) frame for L, in which E is not true.

Nijmegen, Princeton, June 13, 1977 H.C.M. de Swart

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