FORMAL SEMANTICS FOR TEMPORAL LOGIC AND COUNTERFACTUALS

Igal KVART

I. *Introduction*

1. This work will offer a formal analysis of the concept of *the time a statement pertains to,* or *is about* [1], and extend it into a formal semantics for counterfactuals, based on a treatment of mine offered elsewhere [2]. We shall be concerned with this concept of the time a statement pertains to, to be also called its *reference time,* in the sense that, e.g., the sentence

(1) J. F. Kennedy was the president of the U. S. from January 20th, 1961 to November 17, 1963.

pertains to the time interval between January 20th, 1961 and November 17, 1963. Similarly, in the sentence

(2) Stalin used to drink every once in a while during his lifetime.

the temporal interval to which the sentence pertains is that which covers Stalin's lifetime. We shall attempt to provide a regimentation for such sentences, which will involve a temporal quantificational structure. The languages of the propositional calculus and the first-order calculus will be extended to include temporal parameters, which will enable the features of the times statements pertain to to be incorporated in such formal languages. Semantical systems for these two languages will then be offered. In other works of mine [2] I have proposed an analysis for counterfactual statements, which made an extensive use of the concept of the reference-time of statements.

(1) I am indebted to Prof. N. D. Belnap for various suggestions concerning this paper.

(2) See my forthcoming monograph «Towards a Theory of Counterfactuals» and my «Counterfactuals».

The outlines of this analysis will be briefly sketched, and a full-fledged formal semantics for counterfactuals, which will reflect this analysis, will then be presented on the foundation laid by the semantical systems for languages incorporating reference-times. It should be emphasized that the concept of reference-times does *not* call for a *tense*- analysis, but rather for a *temporal* analysis: of course, in context-dependent sentences the reference time will be a function of the context in a way which may be signaled by tense-indicators. But its analysis is in order for temporally context-independent sentences as well, and it is, in general, to be separated from tense considerations.

2. The theory of counterfactuals, whose formal semantics will be presented in Section II, has been presented in detail in the place mentioned in the second footnote. Here the crux of the analysis will be presented. But before—two paragraphs of notational conventions and the like:

We shall represent the time to which a statement A refers as t_A (and similarly t_B, t_C, etc.). We shall say that one temporal interval is *earlier* (or later) than another if every temporal point in it is earlier (or later) than every point in the other. We shall say that a statement A pertains to a time prior to that of a statement B in case t_A is earlier than t_B. One interval will be said to be *weakly earlier* (or weakly later) than another, if its starting point is earlier (or later) than the starting point of the other. If t′, t″ are temporal intervals, we shall designate by *'(t′, t″)'* the temporal interval whose starting point is the same as that of t′, and whose ending point is the same as that of t″. A temporal interval t will be said to be *between* t′ and t″ (or alternatively, within the (t′, t″) interval), when t′ is weakly earlier than t″, if t is weakly later than t′ but earlier than the ending point of t″.

Proper substituends for the letters 'A', 'B', 'C', etc. will be names of statements. A counterfactual statement of the form 'If A had been true, B would have been true' would be symbolized as: *A>B.* (In order for it to qualify as a counterfactual statement, its antecedent has to be false). Thus, '>' is taken as the counterfactual symbol. A would be called the *antecedent,* and B the *consequent* of the counterfactual statement

A>B. Logical symbols would signify themselves, and a juxtaposition of names of expressions would signify the concatenation of the expressions named by them.

In the account of counterfactuals mentioned above, attention is centered on those counterfactuals of the form A>B whose antecedent A (which is not law-contravening) is compatible (through natural laws) with the history of the world prior to the time to which it pertains, the latter time being weakly earlier than the time to which the consequent pertains. Counterfactuals, according to this view, make predictions in hypothetical cases: the antecedent constitutes the contrary-to-fact-assumption while the consequent describes the predicted event. The prediction, however, is not made on the basis of the contrary-to-fact-assumption alone, but rather, in addition, on the basis of the actual-course-of-events modified in a certain way in accordance with the contrary-to-fact-assumption. In the case of the kind of counterfactuals we are discussing here, the modified actual-course-of-events includes those events in the actual course whose occurrence is not risked by the transition to a course-of-events incorporating the contrary-to-fact-assumption. The description of the modified course-of-events includes, therefore, in the first place, the history of the world prior to the time of reference of the antecedent (which is the contrary-to-fact-assumption), i.e., all those true statements which pertain to times prior to that of the antecedent; it also includes those statements which pertain to times within the temporal interval (t_A, t_B), which describe actual events to whose occurrence the antecedent-event bears no negative causal relevance whatsoever. Those, therefore, will be either events to whose occurrence the antecedent-event is either causally irrelevant or purely positively causally relevant.

It is argued that, if that history of the world which pertains to times prior to t_A is to be denoted by 'W_{tA}' and the set of laws by 'L' then:

$$\text{(I)}\quad \{A\} \cup W_{t_A} \cup L \cup \left\{\begin{array}{l}\text{the set of true non-lawlike state-}\\ \text{ments C, such that the A-event}\\ \text{is either causally irrelevant or}\\ \text{purely positively causally rele-}\\ \text{vant to the C-event, and } t_C C(t_A,\\ t_B)\end{array}\right\} \rightarrow B$$

This analysis, therefore, reduces counterfactuals of this type to these two concepts: that of causal irrelevance and that of purely positive causal relevance. If we define semifactuals, i.e., counterfactuals with true consequents, in which the antecedent is causally irrelevant and purely positively causally relevant to the consequent as *irrel-semifactuals* and *pp*-semifactuals respectively, we can reformulate the last member of the union in the antecedent of (I) as:

$$\left\{\begin{array}{l}\text{the set of consequents of true irrel-semifactuals and pp-}\\ \text{semifactuals with antecedent A, such that } t_C C(t_A,\ t_B))\end{array}\right\}$$

and call it *CIP(A,* W_{t_B}*)* [3] (for set of consequents of *i*rrel-semifactuals and *p*p-semifactuals, etc.)

As an example for a counterfactual of this type consider Jones, a modest investor in the stockmarket, who deliberated long whether to sell his stocks for personal reasons, and eventually did. A few weeks later, prospects for a new source of energy became public and the stockmarket skyrocketed. The counterfactuals 'Had Jones not sold his stocks for another few weeks, he would have become rich' is thus true, and of the type mentioned. The antecedent-event was causally irrelevant to the uprise in the stockmarket. This counterfactual is amenable to the above analysis, as the uprise in the market would be described by statements which belong to the last member of the union in (I), to CIP(A, W_{t_B}).

In so far as the concept of causal relevance is concerned, a probabilistic account for when the event described by a state-

(3) For the determination of the arguments of CIP, consult analysis II and analysis III below and formula (10) in Section III.

ment A (=the *A-event*) is causally irrelevant to that described by a statement C is suggested as follows (A pertains to times weakly earlier than C. 'Event' here applies also to processes, states-of-affairs, etc.) [4]: The attempt is to provide a criterion which will be a function of W_{t_A} only, in so far as factual information is concerned. While the condition

$$P(C/A \,\&\, W) = P(C/{\sim}A \,\&\, W)$$

is necessary but not sufficient for causal irrelevance [5], due to the possibility of mutually neutralizing causal chains leading to C which are differently affected by the transition from $\sim$A to A, an adequate account can be provided in terms of events occurring within the (t_A, t_C) interval, which play a role in transferring to C the effects of previous causal chains. The A-event would be causally relevant to the C-event if and only if such actual events in the (t_A, t_C) interval, which transmit to C the effects of previous causal chains in either the A-course-of-events or the $\sim$A-course, would be affected by the transition from $\sim$A to A in so far as either their occurrence or their transmitted effect to C is concerned. Thus, the following condition for causal irrelevance is proposed:

The A-event is causally irrelevant to the C-event if and only if

(II) {W,A,C,L} is consistent and:
For every actual event e in (t_A, t_C), if
either (i) $P(C/e \,\&\, {\sim} A \,\&\, W) \neq (PC/{\sim}A \,\&\, W)$
or (ii) $P(C/e \,\&\, A \,\&\, W) \neq P(C/A \,\&\, W)$
then both
(iii) $P(e/A \,\&\, W) = P(e/{\sim}A \,\&\, W)$
and (iv) $P(C/e \,\&\, {\sim}A \,\&\, W) = P(C/e \,\&\, A \,\&\, W)$.

(i) and (ii) indicate that e makes a difference for the occurrence of C in the $\sim$A-course-of-events and the A-course-of-events

(4) See my forthcoming «Causal Relevance».

(5) The '&' sign in conditional probability is to be interpreted here as either the union of the sets involved or the conjunction.

respectively by affecting its probability to occur in them. (iii) guarantees that the transition does not affect e's probability to occur, and (iv) that it does not alter the effect which e transmits to C.

There remains the concept of purely positive causal relevance, to be analyzed too in probabilistic terms according to the following approximations ([5]). First, for the transition from the ~A-course-of-events to the A-course to be purely positively causally relevant to C, the probability of C must increase in it:

$$P(C/A \,\&\, W_{tA} > P(C/{\sim}A \,\&\, W_{tA})$$

This however, is not a sufficient condition, since it only guarantees that the overall effect of the transition is positive, not that there are no negative component effects. More should be required: that events which transmit positive causal effects to C in the ~A-course, continue to do so in the A-course and that their probability to occur does not decrease in the transition; that events which transmitted no causal effect in the ~ A-course at all should not shift to transmit negative causal effects in the A-course; and that for any of these two types of events, their combined effect on C (with A or ~A) should not decrease in the transition from ~A to A substantially. But events which enhance a negative effect, if any, on C in the ~A-course—their combined negative effect (with A or ~A) should not become substantially stronger in the transition to the A-course and, in case they remain amplifiers of negative effects on C in the A-course, their probabilities to occur should not increase in comparison with the ~A-course. This condition can thus be expressed probabilistically as follows:

The A-event is purely positively causally relevant to the C-event if and only if:

(III) A. {W,A,C,L} is consistent:

B. $P(C/A \,\&\, W) > P(C/{\sim}A \,\&\, W)$

([5]) See my forthcoming «Purely Positive Causal Relevance».

C Eevery actual event e in (t_A, t_C) maintains that:
  a. if (P(C/e & ~A & W) $\geqq$ P(C~A & W), then:
    1 P(C/e & A & W) $\geqq$ P(C/A & W)
    (with strict inequality in case of strict inequality in a)
    2 P(C/e & A & W) $\geqq$ P(C/e & ~A & W)
    3 In case of strict inequality in *a:*
    P(e/A & W) $\geqq$ P(e/~A & W)
  b. *1* If P(C/e & ~A & W) $\leqq$ P(C/~A & W), then
    (1*) P(C/e & ~A & W) $\leqq$ P(C/e & A & W)
    2 If P(C/e & A & W) $<$ P(C/A & W), then
    (2*) P(e/A & W) $\leqq$ P(e/~A & W).

In the formal semantics of Section III we shall attempt to reflect these truth-conditions for counterfactuals (of the type considered), which involve these conditions for causal irrelevance and purely positive causal relevance.

Section II. Canonical Representation

1 The notion of the reference-time of a statement will be dealt with primarily for syntactically simple sentences, that, is, sentences which do not contain subsentences (7), thought not exclusively for them. (We shall assume throughout that we work with a given language, say English.) The times to which statements pertain are temporal intervals (including temporal instants, which are degenerate temporal intervals). In sentences which are context-independent in every respect, the time to which they pertain is specified explicitly, as in the examples (1) and (2) above; it will be natural to regard the time of reference of a statement from the perspective of its temporal quantificational structure. Syntactically simple sentences would be associated with temporal quantifiers, and the time intervals to which they pertain would then be their domains of quanti-

(7) In terms of transformational grammar we could say that the tree-diagram of a syntactically simple sentence contains only one category-symbol 'S'.

fication. The temporal quantificational features of a sentence resemble in various ways those of ordinary quantification. Ordinary language sentences which involve quantificational structure are normally represented symbolically by formulas whose interpretation in turn requires a domain of quantification. Quite often the original ordinary-language sentence does not explicitly state that domain. But normally, if the sentence is unambiguous in its context, together they determine the domain, this being a normal phenomenon of context-dependence. Similarly, when the temporal quantificational structure of ordinary language is concerned, the quantificational domain may not be specified explicitly by the sentence itself, but is to be understood with the help of the contextual information.

Following a well-established usage, we shall take statements to be context-independent in every respect (or eternal sentences). This applies to the time to which they pertain as well: it should not be context-dependent. In discussing examples, however, for the sake of convenience, we shall occasionally allow context-dependence when there is no danger of confusion, even when strictly-speaking statements are called for. We shall also allow ourselves, where there seems to be no danger of confusion, to use 'sentence' in the above sense of 'statement.' But the analysis to be proposed in the sequel is to deal with statements in the above strict sense.

To facilitate regimentation involving reference-time, we shall define a set of sentences which can be called '*testors*' Those will be sentences which describe events or states-of-affairs which occur at a certain time-instant, and which do not logically imply any events at other temporal instants. Thus, the event described by the sentence 'John slept through the afternoon on 1/1/74' did not occur within a single time instant, while the event described by the sentence 'John slept at 3 p.m. 1/1/74' did. Similarly, while the sentence 'The second human landing on the moon occurred at t' [8] logically implies the occurrence of another event at another time—the first landing

[8] t is a temporal instant in which the second landing occurred.

on the moon, the sentence 'Human beings landed on the moon at t' does not.

Consider then, syntactically simple English sentences. They may include temporal locutions of various kinds: locutions specifying time-delimitations within which the event described by the sentence took place (e.g.: 'On 3/7/74', tenses of verbs) and locutions which characterize temporally the occurrence of the event described within such temporal delimitations (e.g.: 'frequently,' 'occasionally,' 'during...,' 'all through...'). In order to transform such sentences into temporal testors which describe events or states-of-affairs at a certain temporal instant without any logical temporal implication beyond it, we shall first require that all such locutions be deleted, and that the tenses of the verbs in the sentences will be changed to present tense (which we shall construe as tenseless). The result of such an operation may not be a well-formed sentence at all. In such a case, the original English sentence will not serve to form a temporal testor. In many cases, however, the result will be an English sentence. In others, an English sentence could be produced from it with minor grammatical adjustments, insignificant from a temporal point of view (and which may coincide with the result of putting some other English sentences through the above procedure). Such sentences would normally serve similar functions to what Rescher & Urquhart called 'chronologically pure' sentences or 'purely phenomenological' characterizations (9), and the reader is referred to their discussion there. The name we shall use will be '*temporally pure*'. A temporally pure sentence, produced from a sentence p along the lines described above, we shall call a *temporally pure projection of p*. Thus, it describes an event-type of which the event described by p is an event-token. Thus, a temporally pure projection of a statement p describes an event-type abstracted from the event-token described by p along the lines of complete temporal abstraction, and only along them.

The temporally pure sentences have two properties which are important to us here: first, they are undetermined with respect

(9) 'Temporal Logic', Springer-Verlag, N.Y. 1971, pp. 144-151.

to the line of occurrence of the events they describe. Secondly, conjoined with an instant occurrence time, they lack strictly logical implications concerning events in other times.

Example 1:

Consider the following syntactically simple sentence:
(3) J. F. Kennedy was occasionally sick during his adult life.

The locution 'during his adult life' belongs to the first type of locutions mentioned above, which temporally delimits the occurrence time of the event(s) described. The locution 'occasionally' belongs to the second type there, characterizing the temporal position of the event(s) described within the temporal limits set by the first locution. Omitting those two locutions and changing the tense of the sentence, we obtain:

(3') J. F. Kennedy is sick

which is a temporally undetermined sentence.

Up to now we have considered as temporally undetermined only syntactically simple sentences. We could now like to extend this type to a category of sentences which are not syntactically simple. Those will be sentences of the form 'John believes that...,' 'Tom hopes that...' etc., that is, sentences which express propositional attitudes. We shall not put any condition on the embedded sentences, but the embedding ones must be themselves temporally undetermined, according to the characterization given above. The embedding sentences must be syntactically simple. Thus, in a sentence like 'John believes that p', no constraints are put on p, while the embedding sentence 'John believes' fulfills the requirements of temporal undetermining and syntactical simplicity.

Sentences of this type describe events or states-of-affairs which pertain to the mental. Their time of occurrence is independent of the embedded sentence, and is determined uniquely by the embedding one. The content of one's propositional attitude may pertain to any time; but it puts no purely logical constraints upon the time of occurrence of an event describing an individual having this propositional attitude. This latter time is determined by the embedding sentence alone. Thus, sentences of this kind, whose embedding sentences qualify as temporally undetermined, we shall count as temporally undetermined as well. These, however, will not include propositional verbs known as 'factives', such as 'know', 'realize', etc. which, in addition to describing a certain mental state, involve an implication concerning the truth of its content.

2 We now move to a more detailed analysis of the time of reference of statements. We shall limit our discussion to syntactically simple statements which are context-independent in every respect (unless we specify otherwise) ([10]).

In Section I above we have seen a few examples of the time intervals to which statements pertain. We can now notice, that a syntactically simple sentence typically involves a temporal quantifier associated with that time-interval. Consider, for instance,

(4) It rained all day on 1/20/74.

The time interval to which this statement pertains is the duration of the day indicated. However, one could assert, with respect to this day, that it rained all through it, as sentence (4) does, or only during some time in it. The distinction clearly indicates a structure analogous to that of quantification over a domain of individuals, where here temporal points ([11]) play this role instead. We can therefore interpret the above sentence as involving universal quantification. Similarly, a sentence like

(5) Exchanges of fire occurred in the Mideast intermittently during 1969

([10]) We shall not be concerned with tense considerations at all; once the time of reference is given, tense indicators are reducible to it and the time of utterance. We are not concerned here with the pragmatic analysis of statements and, dealing with context-independent sentences, we can afford to abstract from the time of utterance.

([11]) An equally attractive interpretation is to choose not temporal instants but rather temporal intervals.

pertains, again, to the time-interval indicated—the year 1969. It should, moreover, be interpreted as involving an existential temporal quantifier, since the event-type described by it, the exchanges of fire, did not take place at every moment of this interval—only at some of them. The temporal quantification thus covers the time-interval to which the statement pertains.

That the distinction of universal and existential quantifiers does not catch the variety of quantifying expressions in natural languages is notorious (thus, expressions like 'most', 'a few', 'many' etc.). The same applies to temporal quantification, and thus, since this phenomenon is not unique to it, we need not dwell on this issue in the context of temporal analysis.

3 Consider the time of occurrences of the event (or state-of-affairs) described by a statement p. The information that p may provide concerning this time of occurrence may vary in its degree of specificity. The statement may specify a time-interval during every instant of which the event described is alleged to have taken place; or else, it may specify a time interval *within* which the event described is said to have occurred. This time-interval T, specified by the statement p, may thus *coincide* with the time of occurrence of the event described, according to the statement under consideration, or it may *contain* it. In the first case, the event is said to have taken place at *every* instant of T; in the second—only at *some* instants of T. In this first case, when p specifies by T the time of occurrence of the event described by it or a part of it, the import of p is that *p* is realized at every t in T ([12]). In the other case, when what is specified is only an interval T *within* which the event described by p is said to have occurred, the import of p is that *p* is realized at *some* t in T.

This clearly indicates a logical structure of a temporal quantificational character, which involves a temporal quantifier—'every' or 'some', which covers the temporal interval T. Let us, then, represent p in the first case as *((t)p,T)*, that is, as an ordered pair whose last element specifies the temporal inter-

([12]) t is to be substitutible for temporal instants here; T for temporal intervals.

val T, and whose first element is the concatenation of a universal quantifier symbol and the temporally pure sentence *p*. In the second case, we shall symbolize p as *((Et)p,T)* analogously. We shall take T as *the time of reference* of the statement p. In the first case, the event described by p is said to have occurred at every instant of T; in the second case—at some instant of T. T then functions here as the domain of temporal quantification, for a universal temporal quantifier in the first case, and an existential one in the second. The *time to which a statement pertains, then,* is its *domain of temporal quantification* ([13]).

Thus, we have arrived at a regimentation of a syntactically simple context-independent sentence. Such a sentence is represented by an ordered pair, the first element of which is a concatenation of a quantifier-symbol and a sentence which is a temporally pure projection of the original sentence, and the second element of which is a phrase describing the time of reference of the original sentence, which is its domain of temporal quantification. Thus, in general, we may represent such a sentence p by:

$$(6) \qquad \left(\begin{Bmatrix} (t) \\ (Et) \end{Bmatrix} \begin{matrix} \text{a temporally pure} \\ \text{projection of p,} \end{matrix} \quad \begin{matrix} \text{a description of the} \\ \text{reference-time of p} \end{matrix} \right)$$

We shall refer to this regimentation as the *canonical representation of p.*

A translation into English of a canonical representation ((t)*p*, T) (where '*p*' stands for a temporally pure sentence and T is a temporal domain) will thus be: throughout the period of T, *p* (with tense-adjustments in *p* according to the relation between

([13]) When the determination of the temporal interval to which a statement pertains is context-dependent, we are faced with the familiar pragmatic problem of how to functionally determine the contextual information. It requires the elaboration of pragmatic bridge-rules which connect the temporal interval to the overt features (linguistic or non-linguistic) of the context, whose nature is still obscure. This, however, is no unique problem to temporal features, but common to all context-dependent characteristics (thus compare with the analogous case of the context-dependence of the ordinary domains of quantification).

T and the utterance time). Similarly, ((Et)*p*,T) would be translated as: sometimes within the period of T, *p* (likewise, with tense adjustments).

4 Example 3:

In Section I, we have considered the example
(1) J. F. Kennedy was the president of the U. S. from January 20, 1961 to November 17, 1963.

We have indicated there that the time of reference of this sentence was the time interval described by it. Eliminating this temporal characterization, and adjusting the tense yields a sentence which is a temporally undetermined projection of (1):

(*1*) J. F. K. is the president of the U. S.

which describes an event-type of which (1) is an event-token. The event described by (1)—J. F. K.'s being a president—is said to have taken place throughout the interval described, which is (1/20/61, 11/17/63); thus the quantification in question is universal, the time of reference being the quantificational domain. Thus, we can present the canonical representation of (1) as:

(8) ((t)(*7*), (1/20/61, 11/17/63)).

Example 4:

We have considered above the sentence

(5) Exchanges of fires occurred in the Mideast intermittently during 1969.

and observed that the reference-time of (5) was the year 1969. Abstracting from the temporal characterization of (5) (and adjusting the tense), we obtain a temporally undetermined projection of it:

(5) Exchanges of fire occur in the Mideast.

which describes an event-type of which the events described in (5) are event-tokens. Clearly, the event-type of (5) is not said in (5) to have taken place continuously through the time of reference of (5), but rather at some times within it; thus, the temporal quantificational character in (5) is existential, the time of reference, as usual, being the quantificational domain. Hence, we can present the canonical representation of (5) as:

(9) ((Et)(5), the year 1969)

5 A few comments are in order. First, the temporal quantificational structure of statements thus yields two components normally associated with quantificational structures: a quantifier and a domain of quantification. However, in the regimentation proposed above (which shortly will be further developed in the formulation of a formal language) there are two distinctive features, uncommon in standard symbolizations of quantificational structures: First, the domain of quantification is represented by a component in the regimentation (and later—by a variable in the object language), rather than being kept behind the scenes in the semantics. Secondly, the quantifier symbols, though clearly in accordance with the import of standard quantifiers, do not behave syntactically as quantifiers: they do not bind any variables, since we do not have bindable temporal variables. In these two respects, the temporal-quantification structure differs from the standard regimentation of individual-quantificational structures.

Secondly, we have previously extended the concept of a temporally undetermined projection of a sentence to context-independent sentences describing propositional attitudes, whose embedding sentence is syntactically simple. Similarly, the concept of canonical representation described above applies to such sentences as well, mutatis mutandis, that is, when the operations are performed on the embedding sentence only, leaving the embedded sentence intact. However, this extension will be of no particular importance to the purposes of constructing a first-order temporal formal system and a formal semantics for the counterfactual construction, as will be clear later.

Thirdly, we have restricted the application of the canonical representation to sentences which are context-independent in every respect. This requirement as well can be relaxed. Thus, for instance, if a sentence is context-independent in every respect except for the specification of individuals referred to in it, its canonical representation is well-defined according to the above guidelines, provided its temporally undetermined projection, which will be a component in its canonical representation, is; and the temporally undetermined projection of a syntactically simple sentence, context-independent in every respect except for the one mentioned above, produced along the very same lines, would also be context-dependent in this particular respect. A case in point is that of complex English sentences, whose structure involves individual-quantifiers (to be distinguished from temporal-quantifiers) whose scopes cover more than one syntactically simple sub-sentence. If each sub-sentence is syntactically simple and context-independent in every respect except for reference to individuals, they can thus be provided with canonical representation (14).

Finally, in so far as the times of reference of the antecedents and consequents of counterfactuals, which are themselves complex sentences, are concerned, it seems that they should be taken as the 'stretched' unions of the times of their syntactically simple components—that is, as the intervals whose lowest (highest) point is the lowest (highest) point of any of the intervals corresponding to the component sentences. This remark, however, goes beyond the formal analysis of counterfactuals whose components are syntactically simple, the formal presentation of which is the chief goal of this work, and thus we shall not dwell on this point in greater detail here.

(14) If, moreover, the propositional connections between them are representable in a *standard* propositional language, then the whole sentence can be symbolized in, say, an extended first-order language (such as we shall soon offer), provided that the temporally undetermined projections of its subsentences can be symbolized in it (since then, as we shall soon see, these subsentences can be so symbolized as well).

Section III: Formal Semantics

1 In this section we shall develop formal semantics for Temporal Logic which will incorporate the concept of reference-time, and extend it to a semantics for counterfactuals. The formal temporal languages that we shall develop will be based on the standard propositional and first-order languages so as to enable the formalization of statements into them in a way that will reflect their temporal features along the lines of the temporal canonical representation developed in the previous section. The semantical systems for these languages will be provided. The analysis for counterfactuals that we have surveyed in Section I made use of the concept of reference-time, an analysis for which was provided in the last section. The formal system for counterfactuals to be developed below will be built on the basis of the formal systems for Temporal Logic. We shall couch the counterfactual connective in the temporal first-order language which we shall develop. Its semantics will be based on the semantics for that temporal first-order language and will reflect the analysis of counterfactuals presented in Section I.

We shall present first a propositional language which incorporates the above temporal analysis of statements, and provide a semantics for it. This language will consist of sentences whose structure reflects the temporal canonical representation mentioned above. We shall then proceed to represent its first order analogue (language and semantics) and then extend it by adding a counterfactual connective, and provide a semantics for it. We shall thus discuss three formal systems. The three languages, respectively, will be *PTL*—propositional temporal language; *QTL*—quantificational temporal language, and *CQTL*—counterfactually-enriched QTL.

The temporal characteristics of the first two languages will be introduced at the level of atomic formulas. Since the atomic formulas of a first order language represent syntactically simple sentences of English, it will be sufficient, for the purpose of presenting these temporal features at the level of a propositional language, to let the propositional variables correspond to syntactically simple English sentences as well. Thus, the structure of atomic sentences of the propositional language will reflect the temporal canonical representation discussed above. (Since we have shown the temporal canonical representation not only of syntactically simple sentences but also of sentences which describe propositional attitudes whose embedding sentences are syntactically simple, the atomic formulas of the propositional language can be viewed as formal representations of both types of sentences. This extension of representation, though, will of course not be reflected in the first-order formal language). From the viewpoint of providing a formal analysis of temporal structures, the propositional language will encompass only the representation of these two categories of English sentences, together with the modes of sentence-composition representable in a standard language of this kind.

2 The temporal canonical representation discussed above had the form $\left(\left\{\begin{matrix}(t)\\(Et)\end{matrix}\right\} p, T\right)$, where p was a temporally pure sentence, and T was a temporal domain. Accordingly, we shall symbolize temporally pure sentences by two types of variables, which will be called *propositional component variables* and *temporal component variables* respectively.

The Language PTL.

I. *Primitive Symbols.*

1. Indenumerable number of propositional component variables over which the syntactical variables p^1, p^2,...vary.
2. Indenumerable number of temporal component variables, over which the syntactical variables t_1, t_2, ... vary.

3. Two places predicates: $\subseteq$, $<$, $\leqq$, $\overline{<}$
4. A two-place functional symbol: U.
5. The following symbols: (,), , , ~, &, (t).

Def.: $t_1 \mathrm{U}\, t_2 \mathrm{U} \ldots t_n$ $(n \geqq 1)$ is a *temporal term*.

We shall use the syntactical variables t^1, t^2, ... to vary over temporal terms.

II. Formation Rules.

$((t)p^1, t^1)$, $((t)\sim p^1, t^1)$, $(\sim(t)p^1, t^1)$, $(\sim(t)\sim p^1, t^1)$, are wffs of PTL, called *atomic propositional wffs.*

2. $t^1 \subseteq t^2$, $t^1 < t^2$, $t^1 \leqq t^2$, and $t^1 \overline{<} t^2$ are wffs of PTL, called *atomic temporal wffs.*
3. If α, β are wffs of PTL, so are $\sim\alpha$, $\alpha \,\&\, \beta$. (If α, β, are propositional wffs (temporal wffs), so are $\sim\alpha$, $\alpha \,\&\, \beta$).

III. *Semantics for PTL.*

A PTL-model is an ordered pair $<Q, V>$ such that: Q is a function from propositional component variables into **PR** (R — the set of real numbers; **PR** — the power set of R), and V is a function which maps all the temporal component variables into **PR** and all the wffs of PTL into $\{0,1\}$, and fulfill the following conditions:

1. $V(t^1 \cup \ldots \cup t^n) = V(t^1) \cup \ldots V(t^n)$.
2. $V((t)p^1, t^1) = 1$ iff $V(t^1) \subseteq Q(p^1)$
   $V(t) \sim p^1, t^1) = 1$ iff $V(t^1) \cap Q(p^1) = \emptyset$.
3. If $(\sim\alpha, t^1)$ is a wff, then: $V(\sim\alpha, t^1) = 1$ iff $V(\alpha, t) = 0$.
4. $V(t^1 \subseteq t^2) = 1$ iff $V(t^1) \subseteq V(t^2)$.
5. $V(t^1 < t^2) = 1$ iff for every τ^1 and τ^2, if $\tau^1 \in V(t^1)$ and $\tau^2 \in V(t^2)$, then $\tau^1 < \tau^2$.
6. $V(t^1 \leqq t^2) = 1$ iff there is τ^1 in $V(t^1)$ such that for every τ^2 in $V(t^2)$: $\tau^1 < \tau^2$.
7. $V(t^1 < t^2) = 1$ iff there is τ^2 in $V(t^2)$ such that for every τ^1 in $V(t^1)$: $\tau^1 < \tau^2$.
8. If α is a wff of PTL, then
   $$V(\sim\alpha) = 1 \quad \text{iff} \quad V(\alpha) = 0.$$
9. If α, β are wffs of PTL, then:

$$V(\alpha \,\&\, \beta) = 1 \quad \text{iff} \quad V(\alpha) = V(\beta) = 1.$$

Def.: $(Et) =_{df} \sim(t)\sim$

Thus, it follows that:

$$V((Et)p^1, t^1) = 1 \quad \text{iff} \quad V(t^1) \cap Q(p^1) \neq \emptyset.$$

The function of Q for a propositional component variable p^1 is analogous to that of assigning to a temporally undetermined sentence *p* the set of time instants in which event-tokens of the type described by *p* occur: The valuation function assigns 0 or 1 to atomic propositional wffs in part in accordance with whether the time assigned by Q to the propositional component variable intersects or includes the time assigned by V to the temporal term in it; this is analogous, in the case of sentences in canonical representation, to basing the decision as to truth-value of the sentence involved on whether the occurrence-times of event-tokens of the type described by *p* include or intersect with the time to which the sentence pertains (subject to the temporal quantifiers and negation symbols involved.) The temporal relations 'earlier' and 'weakly earlier', defined in 2, Section I above, are the analogues of the above relation-symbols '$<$' and '$\leqq$' respectively, under the above interpretation.

3. The second language which we shall discuss, QTL, differs from PTL in one respect: while PTL symbolized temporally indetermined projections propositionally as propositional component variables, QTL will symbolize them in a first-order way. Thus, the temporally undetermined sentences in the temporal canonical representations will be symbolized along the lines of standard quantification theory. The scope of symbolic representation of QTL is thus in a certain respect more limited than that of PTL: sentences whose temporally undetermined projections in their canonical representations do not have a structure symbolizable by standard quantificational theory cannot be symbolized in QTL, while they might be symbolizable in PTL (example: sentences describing propositional attitudes whose embedding sentence is syntactically simple). The deficiency in symbolizational force is of course compensated by the added

richness of the logical structure symbolized. This relation between QTL and PTL is thus quite analogous to that between a standard first-order language and a standard propositional language, and thus reflected in the relation between the propositional component variables of PTL and the first-order wffs which replace them in QTL.

The Language QTL

I. *Primitive Symbols.*

1. Unlimited number of individual variables. The syntactical variables x, y, z, ... vary over them.
2. Predicate variables of various degrees (unlimited number). Syntactical variables P_1, Q_1, R_1,... (for the first degree), P_n, Q_n, R_n,... (for the nth degree, $n \geqq 2$) vary over them.
3. Unlimited number of individual constants: their syntactical variables are e_1, e_2,...
4. Unlimited number of temporal component variables with the syntactical variables t_1, t_2, ...
5. 2-place predicate symbols: $=$, $\subseteq$, $<$, $\leqq$, $\overline{<}$.
6. A 2-place function-symbol: U.
7. The following symbols:), (, , , $\sim$, &, (t).

Def.: As in PTL: $t_1 \cup t_2 \cup \ldots \cup t_n$ ($n \geqq 1$) is a *temporal term.* The syntactical variables t^1, t^2, ... vary through them.
Def.: x, y, z, ..., e_1, e_2, ... are *individual terms.*
We shall have the syntactical variables a_1, a_2, ... b_1, b_2, ... vary through them.

Formation Rules:

I. *Preliminary Definitions*:

A. 1. $\pm a=b$ and $\pm P_n a_1,\ldots,a_n$ are *atomic quasi-formulas* ('... $\pm\alpha$ ...' is tantamount to 'both ...α... and ...$\sim$α...').
2. If α is an atomic quasi-formula, so are $\pm(x)\alpha$.

3. If α is an atomic quasi-formula, then $(\pm(t)\alpha, t^1)$ are *temporal atomic quasi-formulas.*
4. If α is a temporal atomic quasi-formula, so are $\pm\alpha$.
5. If α, β are temporal atomic quasi-formulas, then $\pm\alpha$ and α & β are *temporal quasi-formulas.*
6. If α is a temporal quasi-formula, so are $\pm(x)\alpha$.

B. 1. $t^1 \subseteq t^2$, $t^1 < t^2$, $t^1 \leq t^2$ and $t^1 \lessdot t^2$ are *temporal wffs.*
2. If α, β are temporal wffs, then so are $\sim\alpha$, α & β.

II. 1. If α is a temporal quasi-formula in which no individual variable is free, or if α is a temporal wff, then α is a *wff of QTL.*
2. If α, β are wffs of QTL, so is α & β.

Semantics for QTL.

A *QTL-model* is an ordered pair $\langle D, V \rangle$, where D is a set and V is a function which maps all the individual terms into D; all the temporal component variables into **PR**; all the predicate variables into $\mathbf{P}(D^n \times \mathbf{PR})$ (so that for every predicate variable P_n: $V(P_n) \subseteq D^n \times \mathbf{PR}$); and all temporal quasi-formulas and temporal wffs of QTL into $\{0,1\}$ and fulfill the following conditions:

1. $V(t^1 \cup \ldots \cup t^n) = V(t^1) \cup \ldots \cup V(t^n)$.
2. $V((t)a_1 = a_2, t^1) = 1$ iff $V(a_1) = V(a_2)$.
3. $V((t)P_n a_1, \ldots a_n, t^1) = 1$ iff there is $\tau \in \mathbf{PR}$ such that $V(t^1) \subseteq \tau$ and $\langle V(a_1), \ldots, V(a_n), \tau \rangle \in V(P_n)$.
4. If $\alpha = ((t)(x)\beta, t^1)$ is a temporal atomic quasi-formula, then $V(\alpha) = 1$ iff for every value-assignment function V', differing from V at most at x: $V'((t)\beta, t') = 1$.
5. If $((t)\alpha, t^1)$ is a temporal atomic quasi-formula, then: $V((t)\sim\alpha, t^1) = 1$ iff for every value-assignement function V', which differs from V at most at t^1, and such that $V'(t^1)$ is included in $V(t^1)$: $V'((t)\alpha, t^1) = 0$; and $V(\sim(t)\alpha, t^1) = 1$ iff $V((t)\alpha, t) = 0$.
6. If $(x)\alpha$ is a temporal quasi-formula, then $V((x)\alpha) = 1$ iff for

every value-assignment function V', differing from V at most at x, V'(α) = 1.

7.-12. are the same as conditions 4.-9. in the semantics for PTL (after changing 'PTL' into 'QTL' in rules 8., 9. there).

(The function that Q fulfilled in PTL is here fulfilled by V when applied to predicate symbols).

Def.: A set of wffs of QTL is *QTL-consistent* if and only if there is a QTL-model <D,V> such that for every wff α in the set: V(α) = 1.

4. We now move to display a formal language for counterfactuals—CQTL—and its semantics. The language will be that of QTL, enriched by a counterfactual connective '>'. In constructing the semantics, we shall follow the main features of the analysis of counterfactuals which have been displayed in Section I, and attempt to express them in our formal machinery.

The analysis provided there had quite emphatic temporal features. Thus, the classification of the types of counterfactuals to be analyzed relied on the concept of 'the history of the world prior to a certain time', and the truth-conditions themselves involved this notion and a derivative of it 'the history of the world between t and t'')· The temporal limits of such histories, however, were determined by the times of reference of the antecedent and consequent of the counterfactual under examination. We have in Section II provided an account of the time of reference for statements, concentrating on two types (syntactically simple statements and propositional-attitudes statements with a syntactically simple embedding sentence). This account was mirrored formally by the two languages which we have developed hithertho—PTL and QTL, which symbolized the temporal structure of statements as elaborated by the previous account, the first for a propositional language, and the second for a quantificational one.

Since the quantificational language QTL is the richer one in logical structure, we shall introduce the counterfactual connective into it. Thus, the components of a counterfactual formula would be formulas in QTL. The resulting language will thus be a counterfactually-enriched temporal first-order language. Moreover, since the truth-conditions for counterfactuals involve the notion of logical implication, we shall use in the formal account the best approximation available in the language we are using: that of QTL-consistency. The wish to have a stronger account of consistency leads to seeking the richer underlying formal language. However, to the extent that a first-order language reflects certain, however important, structural aspects of sentences of natural language and ignores others, the notion of consistency definable for such a language will do only limited justice if projected back to sentences of a natural language which were symbolized into the first-order language. This aspect introduces an element of approximation into our formal account; the notion of logical consistency (and that of logical implication) is that capturable by the base language, which is a first-order one.

The account of counterfactuals which we shall attempt to capture in our formal semantics is that presented in formula (I) in Section I above, for counterfactuals whose antecedents are compatible with the prior history of the world (through the laws), which yield that

A counterfactual A>B (of the type discussed) is true if and only if

(10) $W_t \cup L \cup \{A\} \cup \{C | C \in W_{tA,\,tB} \,\&\, R_{c.ir.}(C, A, W_{tA}) \vee R_{p.p.c.r.}(C, A, W_{tA}\} \rightarrow B.$

where $R_{c.ir.}(C, A, W_{tA})$ is the relation of causal irrelevance of A to C on the basis of W_{tA}, and $R_{p.p.c.r.}(C, A, W_{tA})$ is the relation of purely positive causal relevance of A to C on the basis of W_{tA}. We have in Section I called the last member of the union of the antecedent in (10) CIP(A, W_{tB}). We shall use this name for the corresponding set of formulas in the formal semantics as well. The counterfactual connective '>' which appears in the formal language CQTL would symbolize counterfactuals of the type analyzed only.

Since we deal here with formal languages which are temporally adequate in that the reference-time of simple formulas are displayed in them, a way of expressing W_t—the history of the world prior to t— is readily available. Clearly, a simple sentence would not belong to W_t unless its reference-time is strictly prior to t. As to complex sentences, the most natural move is that such a sentence would not belong to W_t unless the reference time of every subsentence of it is strictly prior to t. In a formal language, such as QTL, the reference times of all syntactically simple sentences, whether components of other sentences or not, are displayed by the temporal terms. Thus, this condition is readily statable in terms of the temporal terms which appear in the candidate for membership in W_t. Thus, for that purposes we shall define a temporal function of a sentence which will yield the 'stretched' interval of the temporal intervals which constitute the reference times of its simple constituents (that is, the smallest interval which includes them all). Clearly, in addition, in order to qualify for W_t, a sentence will have to be non-lawlike and true ([15]). This line of thinking will be naturally extended for the definition of $W_{t,t'}$ — the world-history between t and t'.

Since in that analysis for counterfactuals we have presupposed, rather than analyzed, the notion of lawlikeness, its formal correlate will reflect this situation. We shall take in the semantics as the set of laws simply a set L of true wffs of QTL. The approximation achieved thereby can be corrected once an account of lawlikeness is available, by further conditions to be superimposed on the set L.

As to the last component in (10) above, we shall use the analyses above in Section I, since '$R_{c.ir.}$ (S, A, W)' and '$R_{p.p.c.r.}$ (C,A,W)' in (10) are to be replaced by conditions for the causal

([15]) In a forthcoming paper «World Histories» I attempted to provide an analysis for such world histories which are sets of sentences of a natural language through a different strategy, so as to avoid the cumbersome task of extending the definition of reference time to all the sentences of a natural language, however complex—a task which is so easy in adequate formal languages such as PTL and QTL. Clearly, this account could be used as well in the present formal system (using the notion of QTL-consistency to yield equivalent results). These two strategies, however, are to be expected to yield equivalent results; but a further discussion of this claim will not be provided within the limits of this work.

irrelevance and purely positive causal relevance of A to C on the basis of W respectively. Those analyses were given there in qualitative probabilistic concepts, which involved only the relations of 'greater than' and 'equal to' between two conditional probabilities, and no probabilistic metric. No attempt was made to provide an analysis of the concept of probability itself. Accordingly, in the formal semantics we shall introduce two probability relations—of equi-probability and of 'greater than' between probabilities. Since conditional probability has two arguments, the relations of 'equal to' and 'greater than' between conditional probabilities are 4-place relations. Thus, if $P_=$ is to stand for the concept of equi-probability, then '$P(A/\alpha) = P(C/\gamma)$', where A and C are statements, α and γ are sets of statements (or conjunctions thereof), would be expressed as $P_=(A, \alpha, C, \gamma)$; and similarly for the 'greater than' relation $P_<$.

The probabilistic conditions of the analyses of causal irrelevance and purely positive causal relevance will thus be expressed in terms of these two 4-place probabilistic relations.

5 *The Language CQTL.*

We shall define CQTL as an extension of QTC.

I. *Primitive Symbols*:

Those of QTC, plus the symbol '$>$'.

II. *Formation Rules*:

1. If α is a wff of QTL, it is a wff of CQTL.
2. If α, β are wffs of QTL, then $\alpha > \beta$ is a wff of CQTL.

Def.: If α is a wff of QTL, then $T(\alpha)$ is the term $t^1 \cup t^2 \cup \ldots \cup t^n$, where $t_1, \ldots, t^n$ are all the second (temporal) components in the temporal atomic quasi-formulas in α in their order of appearance in α.

III. Semantics for QTL.

A *CQTL-model* is an ordered quintuple $<D, V, L, P_{=}, P_{<}>$, which fulfills the following conditions:

1. D is a set. 2. V is value assignment function which fulfills, with respect to the symbols and wffs of QTL in CQTL, the conditions for V in a QTL-model. ($V(\alpha>\beta)$ will be defined later). 3. L is a set of CQTL-wffs which are also QTL-wffs, such that if $\alpha \in L$, $V(\alpha) = 1$. 4. $P_{=}(\alpha, A, \beta, B)$ and $P_{<}(\alpha, A, \beta, B)$ are 4-place relations, where α and β are wffs of QTL, and A, B are sets of wffs of QTL, such that:

For every wffs α, β and sets of wffs A, B of QTL:

1. Exactly one of the following hold: $P_{<}(\alpha, A, \beta, B)$, $P_{<}(\beta, \beta, \alpha, A)$, $P_{=}(\alpha, A, \beta, B)$.

Before we move to $V(\alpha>\beta)$, we shall define W_{t}, , $W_{t1, t2}$ and CIP.

Def. 1:

$$W_{t1} = \{\alpha | \alpha \text{ is a wff of QTL}; V(\alpha) = 1; \alpha \notin L; V(T(\alpha)<t^1) = 1\}$$

Def. 2:

If $V(t^1 \leqq t^2) = 1$, then:

$$W_{t1, t2} = \{\alpha | \alpha \text{ is a wff of QTL}; V(\alpha) = 1; \alpha \notin L; V(T(\alpha)\overline{<}t^2 = 1, V(t^1 \leqq T(\alpha)) = 1\}$$

Def. 3:

$$P_{\leqq}(\alpha, A, \beta, B) = P_{=}(\alpha, A, \beta, B) \text{ or } P_{<}(\alpha, A, \beta, B).$$

Def. 4:

If α, β are wffs of TQL, and $V(T(\alpha) \leqq T(\beta)) = 1$, and $W_{T(\alpha)}$, L, α, are QTL-consistent, then:

$CIP(\alpha, T(\beta)) = \{\gamma | \gamma \text{e} W_{T(\alpha), t(\beta)}; \{W_{T(\alpha)}, L, \alpha, \gamma\}$ is QTL-consistent: either of the following conditions I or II obtain:

I. For every wff ε of QTL, if $\varepsilon \in W_{T(\alpha), T(\gamma)}$ and if either not $P_{=}(\gamma, \{\varepsilon, \sim\alpha, W_{T(\alpha)}\}, \gamma, \{a, W_{T(\alpha)}\})$, then:

$$P_{=}(\varepsilon, \{\alpha, W_{T(\alpha)}\}, \varepsilon, \{\sim\alpha, W_{T(\alpha)}\}) \text{ and } P_{=}(\gamma, \{\sim a, W_{T(\alpha)}\}, \gamma, \{\varepsilon, \alpha, W_{T(\alpha)}\}).$$

II.1. $P_{<}(\gamma, \{\sim\alpha, W_{T(\alpha)}\}, \gamma, \{\alpha, W_{T(\alpha)}\})$

2. For every wff ε of QTL, if $\varepsilon \in W_{T(\alpha), T(\gamma)}$), then:

2.1. If $P_{<}(\gamma, \{\sim\alpha, W_{T(\alpha)}\}, \gamma, \{\varepsilon, \sim\alpha, W_{T(\alpha)}\})$, then: $P_{<}(\gamma \{\alpha, W_{T(\alpha)}\}, \gamma, \{\varepsilon, \alpha, W_{T(\alpha)}\})$: and $P_{\leqq}(\varepsilon, \{\sim\alpha, W_{T(\alpha)}\}, \alpha, W_{T(\alpha)}\}$,

2.2. If $P_{\leqq}(\gamma, \{\sim\alpha, W_{T(\alpha)}\}, \gamma \{\varepsilon, \sim\alpha, W_{T(\alpha)}\})$, then: $P_{\leqq}(\gamma, \{\alpha, W_{T(\alpha)}\}, \gamma, \{\varepsilon, \alpha, W_{T(\alpha)}\})$, and $P_{\leqq}(\gamma, \{\varepsilon, \sim\alpha, W_{T(\alpha)}\}, \gamma, \{\varepsilon, \alpha, W_{T(\alpha)}\}$.

2.3. If $P_{\leqq}(\gamma, \{\sim\alpha, W_{T(\alpha)}\}, \gamma, \{\varepsilon, \sim\alpha, W_{T(\alpha)}\})$, then: $P_{\leqq}(\gamma, \{\varepsilon, \alpha, W_{T(\alpha)}\}, \gamma, \{\varepsilon, \sim\alpha, W_{T(\alpha)}\})$.

2.4. If $P_{<}(\gamma, \{\alpha, W_{T(\alpha)}\}, \gamma, \{\varepsilon, \alpha, W_{T(\alpha)}\})$, then: $P_{<}(\varepsilon, \{\sim\alpha, W_{T(\alpha)}\}, \varepsilon, \{\alpha, W_{T(\alpha)}\})$.

Now, to $V(\alpha > \beta)$.

1. $V(\alpha > \beta)$ is defined iff: $\alpha > \beta$ is a wff of QTL; $V(\alpha) = 0$; $V(T(\alpha) \leqq T(\beta) = 1$; $\{L, \alpha, W_{T(\alpha)}\}$ is TQC-consistent; $\alpha, \beta \notin L$.
2. If $V(\alpha > \beta)$ is defined, then: $V(\alpha > \beta) = 1$ iff $\{\alpha, W_{T(\alpha)}, L, CIP(\alpha, T(\beta)), \sim\beta\}$ is QTL-*in*-consistent.

Brandeis University, Igal KVART