A RESEARCH IN MODAL LOGICS

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1. *The basic idea* — This paper is the development of Porte 1958, which was only an abstract.

The idea underlying this work is that «necessity» is in some way a stronger thing than simple «acceptability» (1). It could be interesting to carry out this idea by using *two* formal systems, a stronger one which expresses «necessity», and a weaker one which expresses «acceptability». But tradition — and perhaps conveniency as well — leads us to use only one formal system, in which «necessity», and the dual notion of «possibility», are represented by unary connectives (see for instance Lewis-Langford 1932, Gödel 1933, Feys-Dopp 1965). This traditional method will be used in the present work.

In Porte 1958, the basic idea was carried out in an algebraic way. Considering a matrix, the algebra of it is boolean, the «acceptable» elements form a filter (or «sum-ideal»), known as the set of designated elements, $\mathscr{D}$, while the «necessary» elements form another filter, $\mathscr{D}'$ such as $\mathscr{D}' \subset \mathscr{D}$, and the «necessity» is represented by a unary function which maps $\mathscr{D}'$ into $\mathscr{D}$.

The relationship between connective systems and matrices being by no means simple (see, for instances, Church 1953, Harrop 1958, Łoś-Suszko 1958 or Porte 1965), an exclusively algebraic point of view leads often to obscurity... Here, the point of view is chiefly logistic, algebraic considerations being relegated to the last paragraph (§ 11).

The same basic idea is carried out in a quite different form in the so-called «Ł-system» (see Łukasiewicz 1953 or Porte 1979). That system will not be studied here.

(1) It is customary, in mathematical logic, to use the word «truth» only with a semantical meaning.

The present work begins by defining very weak modal logics, and goes on using certain «natural» means to strengthen the systems. Twelve different modal systems are generated in this way, and are eventually compared to the classical systems (Lewis' and related systems).

2. *The weakiest system* — I begin by defining a weak modal system, S_a, the theses of which consist of the tautologies, the formulas of the form N*t*, where *t* is a tautology, and as few other formulas that it is possible for a logistic system which keeps the detachment rule (or «modus ponens»).

More precisely, the system S_a is defined as follows:

— Alphabet: An infinite denumerable set of atoms (or «propositional variables»), p_1, p_2, ...; three connectives: $\Rightarrow$ («implication» — binary), $\neg$ («negation» — unary), Necessity» — unary); the usual parentheses.

— Formulas: constructed as usual in connective (or «propositional») systems.

— Theses: from the following set of postulates (axiom schemas and rules). The letters «x, y, z» denote arbitrary formulas. The conventional names of the postulates are on the left (the meaning of the letter v will be explained later: see § 5).

$$
S_a \begin{cases}
P_1: & \vdash x \Rightarrow (y \Rightarrow x) \\
P_2: & \vdash (x \Rightarrow (y \Rightarrow z)) \Rightarrow ((x \Rightarrow y) \Rightarrow (x \Rightarrow z)) \\
P_3: & \vdash (\neg x \Rightarrow \neg y) \Rightarrow (y \Rightarrow x) \\
D: & x,\ x \Rightarrow y \vdash y \qquad \textit{(detachment rule)} \\
vP_1: & \vdash N\,(x \Rightarrow (y \Rightarrow x)) \\
vP_2: & \vdash N\,((\neg x \Rightarrow \neg y) \Rightarrow (y \Rightarrow x)) \Rightarrow (x \Rightarrow z))) \\
vP_3: & \vdash N\,((\neg x \Rightarrow \neg y) \Rightarrow (y \Rightarrow x)) \\
vD: & Nx,\ N\,(x \Rightarrow y) \vdash Ny \\
W: & Nx \vdash x \qquad \textit{(weakening rule)}
\end{cases}
$$

The other propositional connectives and the possibility (P) will be defined in the usual way:

$$x \vee y = (x \Rightarrow y) \Rightarrow y$$

$x \wedge y = \neg(\neg x \vee \neg y)$
$x \Leftrightarrow y = (x \Rightarrow y) \wedge (y \Rightarrow x)$
$Px = \neg N \neg x$

It is clear that every tautology (²) is provable from P_1, P_2, P_3, D, and that every formula of the form Nt, where t is a tautology is provable from νP_1, νP_2, νP_3, νD. In addition a few theses, such as $\vdash p_1 \Rightarrow N(P_2 \Rightarrow p_2)$ are obtained from the preceding ones by rule D.

It is obvious that P_1, P_2, P_3 are redundant, and can be suppressed. In what follows, it will be assumed that this suppression has been made.

But we might simplify the preceding set of postulates by keeping P_1, P_2, and P_3, and suppressing W. It can be proved that the resulting system would have the same theses as S_a, but a weaker deductibility: For instance, in S_a, we have $Np_1 \vdash p_1$, and this would not be true if W was suppressed. It is for this reason that rule W will be kept in S_a. The «basic idea» (§ 1) is in a certain way «imbedded» into W. — Indeed the motivation to keep W in order to carry out the basic idea can be explained better by semantical considerations (see § 11).

A few further results can be proved about S_a.

Theorem 2.1 — If

$x_1, \ldots x_n \vdash y$ in the propositional calculus (PC) then $Nx_1, \ldots Nx_n \vdash Ny$ in S_a,

Proof: just use νP_1, νP_2, νP_3, νD in order to imitate a deduction in PC (by P_1, P_2, P_3, D).

Theorem 2.2 — S_a is consistent.

Proof: let τ be a translation of S_a into PC defined by suppressing every occurrence of N. Then the axioms of S_a are trans-

(²) It is clear that the set of theses is invariant by substitution in S_a — as well as in the other systems defined in this paper. By a «tautology» is meant every formula resulting from a thesis of the classical propositional calculus by substituting formulas of S_a for the atoms — for instance $Np_1 \Rightarrow (p_2 \Rightarrow Np_1)$ is a tautology.

lated into theses of PC, and each rule of S_a is translated as a derived rule of PC. Thus: if $\underset{S_a}{\vdash} x$, then $\underset{PC}{\vdash} \tau(x)$. It follows that (for instance) $\neg (p_1 \Rightarrow p_1)$ is not a thesis of S_a, because it is invariant by τ, and it is not a thesis of PC — This proof will be used later (§8) to prove the consistency of other systems.

Theorem 2.3 — $\underset{S_a}{\vdash} Nx$ if and only if $\underset{PC}{\vdash} x$.

Proof: it follows from Theorem 2.1 that if $\underset{PC}{\vdash} x$ then $\underset{S_a}{\vdash} Nx$ — Now let us suppose that there is a formula x such as $\underset{S_a}{\vdash} Nx$ and not — $\underset{PC}{\vdash} x$. Then there would be a formula, x′, deriving from x by a substitution, such as $\underset{PC}{\vdash} \neg x'$, whence $\underset{S_a}{\vdash} \neg x'$. By the same substitution, we would have $\underset{S_a}{\vdash} Nx'$ whence, by W, $\underset{S_a}{\vdash} x'$ and S_a would be inconsistent, contrary to Theorem 2.2.

Theorem 2.4 — (decision procedure for S_a) — The theses of S_a are the consequences by PC of the formulas Nt where t is a tautology.

Proof: By Theorem 2.3 every thesis of S_a is provable from P_1, P_2, P_3, D, νP_1, νP_2, νP_3 νD — without W.

Let us construct the theses of S_a as follows:

(1) Start with νP_1, νP_2, νP_3 and use νD.
(2) Add P_1, P_2, P_3 and apply D.
(3) Apply νD to the preceding theses.
(4) Apply D to the preceding theses.
(5) Apply νD, etc...

Then, step (1) gives the formulas Nt where t is a tautology. Step (2) gives the consequences of these formulas — Now, step (3) does not give any new thesis, for the rule νD can only be applied to premisses of the form Nx; but, after Theorem 2.3, the theses of this form had already been obtained in step (1). Then the construction stops at step (2). Whence Theorem 2.4.

It follows as a corollary that no thesis of S_a can have the forms $\neg$ Nx; specially, for every formula x, not— $\vdash_{S_a}$ Px.

Nothing is «possible» in this modal logic! S_a is not a «good» modal system — but experience shows that it is a good starting point to build interesting systems by certain natural ways of strengthening, which will be defined in §§ 3, 4 and 5.

3. *Compatibility* — A way to strenthen S_a consists in adding to it a rule of «compatibility».

Thus the system S_b is defined by the postulates of S_a plus the *rule of weak compatibility*:

$$\text{WC: } N(x \Leftrightarrow y) \vdash Nx \Leftrightarrow Ny$$

The system S_c is defined by the postulates of S_a plus the *rule of strong compatibility*:

$$\text{C: } N(x \Leftrightarrow y) \vdash N(Nx \Leftrightarrow Ny)$$

It is clear that S_b is at least as strong as S_a, and S_c is at least as strong as S_c. It will be proved later (§ 9) that S_b is strictly stronger than S_a, and S_c strictly stronger than S_b (there are theses of S_c which are not theses of S_b, and theses of S_b which are not theses of S_a).

These rules — as well as the word «compatibility» — will be better justified by semantical considerations (§ 11).

On the syntactical plan, C is the key to the so-called «replacement of strict equivalents».

Let us write

$$x \; eq \; y \quad \textit{if and only if} \quad \vdash N(x \Leftrightarrow y)$$

In S_a (and in every stronger system) *eq* is an equivalence relation.

Theorem 3.1 — In S_c, if the formula v differs from u by the replacement of an occurrence of the sub-formula x by y, and if x *eq* y then u *eq* v.

Proof. We have

	$x \Leftrightarrow y \vdash \neg x \Leftrightarrow \neg y$	in PC
then	$N(x \Leftrightarrow y) \vdash N(\neg x \Leftrightarrow \neg y)$	in S_a and S_c
Similarly	$N(x \Leftrightarrow y) \vdash N((x \Rightarrow z) \Leftrightarrow (y \Rightarrow z))$	
	$N(x \Leftrightarrow y) \vdash N((z \Rightarrow x) \Leftrightarrow (z \Rightarrow y))$	
and	$N(x \Leftrightarrow y) \vdash N(Nx \Leftrightarrow Ny)$	is C

It follows, by construction of the formulas u and v

$$N(x \Leftrightarrow y) \vdash N(u \Leftrightarrow v)$$

Particularly, if $\vdash N(x \Leftrightarrow y)$ then $\vdash N(u \Leftrightarrow v)$ which is Theorem 3.1.

Remark. It is more usual to define the «strict equivalents» by

$$x\ eq'\ y \quad \text{if and only if} \quad \vdash N(x \Rightarrow y) \wedge N(y \Rightarrow x)$$

But, let a, b be arbitrary formulas, then we have

	$a \wedge b \vdash a$	in PC
whence	$N(a \wedge b) \vdash Na$	in S_a
similarly	$N(a \wedge b) \vdash Nb$	in S_a
But	$Na, Nb \vdash Na \wedge Nb$	in PC, then in S_a
whence	$N(a \wedge b) \vdash Na \wedge Nb$	in S_a

On the other hand

	$a, b \vdash a \wedge b$	in PC
whence	$Na, Nb \vdash N(a \wedge b)$	in S_a
But	$Na \wedge Nb \vdash Na$	in PC, then in S_a
similarly	$Na \wedge Nb \vdash Nb$	in PC, then in S_a
whence	$Na \wedge Nb \vdash N(a \wedge b)$	in S_a

It follows

$$N(x \Leftrightarrow y) \vdash N(x \Rightarrow y) \wedge N(y \Rightarrow x)$$

and

$$N(x \Rightarrow y) \wedge N(y \Rightarrow x) \vdash N(x \Leftrightarrow y)$$

in S_a and every stronger system.

Then *eq'* could replace *eq* in Theorem 3.1.

4. *Reinforcement* — The operation called here «reinforcement» is a particular way of strengthening a modal system.

In S_a the so-called «deduction theorem» does not hold: for instance we have

$$N(p_1 \Rightarrow p_2),\ Np_1 \vdash Np_2 \qquad \text{by } v\,D$$
$$Np_1 \vdash p_1 \qquad \text{by } W$$

but, after Theorem 2.4, neither $N(p_1 \Rightarrow p_2) \Rightarrow (Np_1 \Rightarrow Np_2)$ nor $Np_1 \Rightarrow p_1$ are theses of S_a. In S_b and S_c, the deduction theorem does not hold either, as will be proved later (3).

Definition 4.1 — The *reinforcement of a logistic system* S, ϱS, is the weakiest of the systems that are stronger than S and admit the deduction theorem.

For the modal systems S (at least as strong as S_a), the existence of ϱS will be proved by Theorem 4.1.

Definition 4.2 — *The reinforcement of a rule*

$$R: \quad x_1, x_2, \dots x_n \vdash y$$

is the axiom schema

$$\varrho R: \quad \vdash x_1 \Rightarrow (x_2 \Rightarrow \dots (x_n \Rightarrow y) \dots)$$

Theorem 4.1 — For a system S at least as strong as S_a, defined by axiom schemas and rules, one of these being D, a logistic system for ϱS consists of:

(3) This will follow from $S_b \neq \varrho S_b$ $(= \varrho S_a)$ and $S_c \neq \varrho S_c$ (§ 9).

1° The axiom schemas of S;
2° The reinforcements of the rules of S other than D;
3° Rule D.

Proof — This system is stronger than S, since R results from ϱR and D. Every system stronger than S and admitting the deduction theorem must have ϱR as a thesis if S has R as a rule. The new system admits the deduction theorem, since D is its sole rule and P_1 and P_2 are among its theses.

The reason for not postulating ϱD is that ϱD is $\vdash x \Rightarrow ((x \Rightarrow y) \Rightarrow y)$, which would be redundant as a tautology, already provable from the axiom schemas of S and D.

Remark — The reinforcement of a rule

$$R = x_1, x_2 \vdash y \qquad \text{(for instance)}$$

could be

$$\varrho_1 R: \ \vdash \ x_1 \Rightarrow (x_2 \Rightarrow y)$$

or

$$\varrho_2 R: \ \vdash \ x_2 \Rightarrow (x_1 \Rightarrow y)$$

This is unimportant, since $\varrho R_1 \Leftrightarrow \varrho R_2$ is a tautology. The choice between them is a matter of convenience.

Definition 4.3. — A *canonical system* is a system admitting D as a rule (postulated or derived) and the deduction theorem.

Definition 4.4 — We write «$S \subset S'$» if S' is at least as strong as S, i.e. if every statement of deducibility, $x_1, \dots x_n \vdash y$, which holds in S, holds as well in S' — It follows that every thesis of S is a thesis of S'.

The following result is trivial, but will be used extensively in what follows.

Theorem 4.2. — For all systems, S and S':

1° $S \subset \varrho S$;
2° if $S \subset S'$ then $\varrho S \subset \varrho S'$;
3° $\varrho\varrho S = \varrho S$;
4° $\varrho S = S$ if and only if S is canonical.

Remark 1 — In Theorem 4.2, 3° and 4°, the equality sign between, for instance, ϱS and S, means that these systems have the same deducibility. Thus, two logistic systems with different postulate systems but the same deductibility are not considered different. It could happen, for instance, that a system S be canonical while its postulate set contains a rule other than D (this rule being redundant if D is postulated and not derived).

Remark 2 — When it is spoken of a «derived rule» (as in Definition 4.3, or in Remark 1), it is intended as a «deductively acceptable» rule (4), i.e. a rule the addition of which does not change the deducibility — not a «thetically acceptable» rule (4), i.e. a rule the addition of which does not change the set of theses. The deductively acceptable rules are also thetically acceptable, but the converse is false. For instance, in the system $\nu\varrho\nu S_a$ defined later (§ 8) the rule

$$\text{RN: } x \vdash Nx$$

is thetically acceptable but not deductively acceptable: we have:

if $\vdash x$ then $\vdash Nx$
but not $p_1 \vdash Np_1$ (for instance)

Such a rule is not considered here as a «derived rule».

5. *Normalization* — After McKinsey-Tarski 1948 we put:

(4) In Porte 1965, deductively acceptable rules are called «D-acceptable», thetically acceptable rules are called «T-acceptable».

Definition 5.1 — A modal system is normal when:

if $\vdash x$ then $\vdash Nx$

S_a is not normal: after Theorem 2.4 we have $\vdash N(p_1 \Rightarrow p_1)$ and not — $\vdash NN(p_1 \Rightarrow p_1)$. It will be proved that S_b and S_c are not normal either (5).

«Normalization» will be an operation which transforms a modal system into a normal one.

Indeed there are an infinity of such operations. We could simply add the rule

$$RN:\ x \vdash Nx$$

But an ulterior reinforcement would give

$$\varrho RN:\ \vdash x \Rightarrow Nx$$

while we have, in ϱS_a and every stronger system

$$\varrho W:\ \vdash Nx \Rightarrow x$$

whence

$$\vdash x \Leftrightarrow Nx$$

(for whatever formula x)

— a thesis that most logicians prefer not to get...

The normalization described below seems to be the simplest operation that achieves the desired result without postulating the preceding rule RN — This operation can however be better motivated by semantical considerations (see § 11).

Definition 5.2.

1° *The normalization of an axiom schema*

(5) This will follow from $S_b \neq \nu S_b = \nu S_c$ and $S_c \neq \nu S_c$)(§ 9).

$$A: \vdash x$$

is the schema

$$\nu A: \vdash Nx$$

2° *The normalization of a rule*

$$R: x_1, \dots x_n \vdash y$$

is the rule

$$\nu R: Nx_1, \dots Nx_n \vdash Ny$$

Remark — Definition 5.2 explains the notations νP_1, νP_2, νP_3, νD used in the definition of S_a (§ 2).

Definition 5.3 — *The normalization of a logistic system* S, defined by axiom schemas A_1, A_2, ... A_m and rules R_1, ... R_n will be the system νS defined by

> axiom schemas: A_1, ... A_m; νA_1, ... νA_m rules: R_1, ... R_n,νR_1, ... νR_n, and I: $Nx \vdash NNx$ (*iteration rule*) (6)

Remark 1 — If the system S is at least as strong as S_a, so that W is one of its rules (postulated or derived), the postulates of νS can be simplified in the following way:

1° if νA is a set of theses (axiom schema or deduced theses) in S, $\nu\nu A$ would be a redundant axiom schema in νS (use I);
2° if νR is a rule (postulated or derived of S, νR is redundant in νS (by I and W).

This remark will be important when applied to S_a, which has the postulates νP_1, νP_2, νP_3 and the rule νD.

Remark 2 — In the same cases (system S at least as strong as S_a), the axiom schemas A_1, ... A_m are redundant in νS (use

(6) Iteration rule was called «règle de normalisation» in PORTE 1958.

$\nu A_1, \ldots \nu A_m$ and W)· — But this remark does not apply to axiom schema ϱW: we must keep it in νS in order that W be a derived rule.

The following result is parallel to Theorem 4.2.

Theorem 5.1

1° $S \subset \nu S$;
2° if $S \subset S'$ then $\nu S \subset \nu S'$
3° $\nu\nu S = \nu S$
4° if $\nu S = S$ then S is normal (7)

6. *Generating seven systems* — Let us start with a modal system S, at least as strong as S_a, the set of its axiom schemas being $\mathscr{Y}$ and the set of its rules (other than W, D, and νD) being $\mathscr{Z}$. We will call «$\nu\mathscr{Y}$» the set $\{\nu A;\ A \in \mathscr{Y}\}$ — and similarly for $\nu\mathscr{Z}$, $\varrho\mathscr{Z}$, etc.

We will generate new modal systems by applying alternatively reinforcement and normalization to S. Taking into account the results of §§ 4 and 5 (particularly the remarks which follow Definition 5.3) we can write the postulates of the new systems as in table 1 (the names of the rules are underlined).

Table 1

S :	$\mathscr{Y}, \underline{\mathscr{Z}}, \underline{W}, \underline{\nu D}, \underline{D}$
ϱS :	$\mathscr{Y}, \varrho\mathscr{Z}, \varrho W, \varrho\nu D, \underline{D}$
νS :	$\nu\mathscr{Y}, \underline{\mathscr{Z}}, \underline{\nu\mathscr{Z}}, \underline{W}, \underline{\nu D}, \underline{I}, \underline{D}$
$\varrho\nu S$:	$\nu\mathscr{Y}, \varrho\mathscr{Z}, \varrho\nu\mathscr{Z}, \varrho W, \varrho\nu D, \varrho I, \underline{D}$
$\nu\varrho S$:	$\nu\mathscr{Y}, \nu\varrho\mathscr{Z}, \nu\varrho W, \varrho W, \nu\varrho\nu D, \underline{I}, \underline{D}$
$\varrho\nu\varrho S$:	$\nu\mathscr{Y}, \nu\varrho\mathscr{Z}, \nu\varrho W, \varrho W, \nu\varrho\nu D, \varrho I, \underline{D}$
$\nu\varrho\nu S$:	$\nu\mathscr{Y}, \nu\varrho\mathscr{Z}, \nu\varrho\nu\mathscr{Z}, \nu\varrho W, \varrho W, \nu\varrho\nu D, \nu\varrho I, \underline{I}, \underline{D}$
$\nu\varrho\nu\varrho S$:	$\nu\mathscr{Y}, \nu\varrho\mathscr{Z}, \nu\varrho W, \varrho W, \nu\varrho\nu D, \nu\varrho I, \underline{I}, \underline{D}$

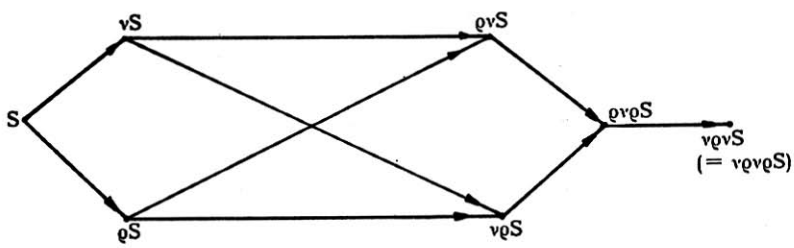
(7) But the converse is false. This seems to contradict «Prop. 10» in PORTE 1958; but in that text «une espèce normale» was defined in such a way that it was a set of matrices where I is strongly valid (see § 11).

But in $\nu\varrho\nu S$ and $\nu\varrho\nu\varrho S$ the rule I is redundant: we have W from ϱW and D, ϱI from $\nu\varrho I$ and W, and I from ϱI and D. When we have suppressed I from the postulates of $\nu\varrho\nu\varrho S$ and $\nu\varrho\nu S$, there remains only one rule, namely D. Then, these systems are canonical (by Theorem 4.2, for reinforcement gives the same systems of postulates); they are normal as well (Theorem 5.1); and the generating process stops here.

Now in table 1 every postulate of $\nu\varrho\nu\varrho S$ is a postulate of $\nu\varrho\nu S$; whence $\nu\varrho\nu\varrho S \subset \nu\varrho\nu S$. But $\nu\varrho\nu S \subset \nu\varrho\nu\varrho S$ from Theorems 4.2 and 5.1 Then $\nu\varrho\nu S = \nu\varrho\nu\varrho S$.

Eventually we have produced seven systems, which can be represented in figure 1.

Figure 1



The arrows represent relative strength (for instance $S \subset \nu S$). Every arrow is provable by Theorems 4.2 and 5.1 — except $\varrho\nu\varrho S \subset \nu\varrho\nu S$, which is apparent in list 1: every postulate of $\varrho\nu\varrho S$ is deducible from the postulates of $\nu\varrho\nu S$.

These seven systems are not distinct for every S (see the case of S_c in § 7). But they are distinct if $S = S_a$ (see § 9).

7. *The case of S_b and S_c* — If we apply the generating process to S_b and to S_c, several reductions appear.

Theorem 7.1 — $\varrho S_b = \varrho S_a$

Proof: we have successively in ϱS_a:

(1)	$N(x \Rightarrow y) \vdash Nx \Rightarrow Ny$	by $\varrho\nu$D, D
(2)	$N(y \Rightarrow x) \vdash Ny \Rightarrow Nx$	similar to (1)
(3)	$x \Leftrightarrow y \vdash x \Rightarrow y$	by PC
(4)	$N(x \Leftrightarrow y) \vdash N(x \Rightarrow y)$	by (3) in S_a
(5)	$N(x \Leftrightarrow y) \vdash N(y \Rightarrow x)$	similar to (2)
(6)	$N(x \Leftrightarrow y) \vdash Nx \Rightarrow Ny$	by (4), (1)
(7)	$N(x \Leftrightarrow y) \vdash Ny \Rightarrow Nx$	by (5), (2)
(8)	$Nx \Rightarrow Ny, Ny \Rightarrow Nx \vdash Nx \Leftrightarrow Ny$	by PC
(9)	$N(x \Leftrightarrow y) \vdash Nx \Leftrightarrow Ny$	by (6), (7), (8)

and (9) is WC.

Theorem 7.2 — $\nu S_b = \nu S_c$

Proof: we have in νS_b

(1)	$N(x \Leftrightarrow y) \vdash Nx \Leftrightarrow Ny$	WC
(2)	$NN(x \Leftrightarrow y) \vdash N(Nx \Leftrightarrow Ny)$	νWC
(3)	$N(x \Leftrightarrow y) \vdash NN(x \Leftrightarrow y)$	I
(4)	$N(x \Leftrightarrow y) \vdash N(Nx \Leftrightarrow Ny)$	by (2) and (3)

and (4) is C. But now WC is deducible from C and W, and is redundant. In νS_c, νC and νWC can also be proved from C, I and W; so the postulates of νS_c reduce to: νP_1, νP_2, νP_3, W, νD, I, D, C.

Thus S_b does not generate new systems, except those generated by S_a or S_c.

Theorem 7.3 — When C holds, rule I holds if and only if $\vdash NNt$ where t is tautology.

(i) Let us suppose we have rule I, then

(1)	$\vdash t$	by PC
(2)	$\vdash Nt$	by (1), S_a
(3)	$\vdash NNt$	by (2), I

(ii) Let us suppose we have C and $\vdash NNt$, then

(1)	$\vdash x \Leftrightarrow (x \Leftrightarrow t)$	by PC

(2) $\vdash N(x \Leftrightarrow (x \Leftrightarrow t))$	by (1), S_a
(3) $\vdash N(Nx \Leftrightarrow N(x \Leftrightarrow t))$	by (2), C
(4) $\vdash Nx \Leftrightarrow N(x \Leftrightarrow t)$	by (3), W
(5) $Nx \vdash N(x \Leftrightarrow t)$	by (4), PC
(6) $Nx \vdash N(Nx \Leftrightarrow Nt)$	by (5), C
(7) $Nx \vdash N(NNx \Leftrightarrow NNt)$	by (6), C
(8) $Nx \vdash NNx \Leftrightarrow NNt$	by (7), W
(9) $NNt,\ NNx \Leftrightarrow NNt \vdash NNx$	by PC
(10) $\vdash NNt$	hypothesis
(11) $NNx \Leftrightarrow NNt \vdash NNx$	by (9), (10)
(12) $Nx \vdash NNx$	by (8), (11)

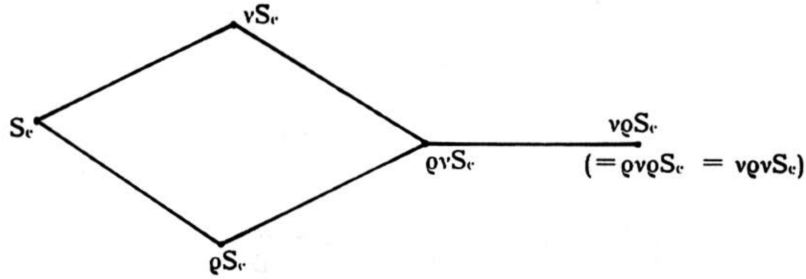
The impact of Theorem 7.3 is that in any system at least as strong as S_c, the rule I can be replaced by an axiom schema such as $\vdash NN(x \Leftrightarrow x)$.

Returning to table 1, we see that if we replace I by an axiom schema in the list of postulates of $\nu\varrho S_c$, the only remaining rule is D. So $\nu\varrho S_c$ is canonical. But it is normal too (Theorem 5.1). Since I holds in $\nu\varrho S_c$, ϱI and $\varrho\nu$I hold as well. But we see in table 1 that the only difference between the postulates of $\nu\varrho S_c$, $\varrho\nu\varrho S_c$, $\nu\varrho\nu S_c$ is that there is I in the first, ϱI in the second, and $\nu\varrho$I in third. Then:

Theorem 7.4 — $\nu\varrho S_c = \varrho\nu\varrho S_c = \nu\varrho\nu S_c$

Then, in the case of $S = S_c$, figure 1 reduces to figure 2.

Figure 2



Theorem 7.5 — Rule C holds in $\nu\varrho S_a$

Proof: in $\nu\varrho S_a$

(1)	$\vdash N(N(x \Rightarrow y) \Rightarrow (Nx \Rightarrow Ny))$	which is $\nu\varrho\nu D$
(2)	$N(x \Rightarrow y) \vdash Nx \Rightarrow Ny$	by (1), W, D
(3)	$\vdash NN(x \Rightarrow y) \Rightarrow N(Ny \Rightarrow Ny)$	by (2) applied to (1)
(4)	$NN(x \Rightarrow y) \vdash N(Nx \Rightarrow Ny)$	by (3), D
(5)	$N(x \Rightarrow y) \vdash NN(x \Rightarrow y)$	by 1
(6)	$N(x \Rightarrow y) \vdash N(Nx \Rightarrow Ny)$	by (4), (5)
(7)	$N(y \Rightarrow x) \vdash N(Ny \Rightarrow Nx)$	similar to (6)
(8)	$x \Leftrightarrow y \vdash x \Rightarrow y$	in PC
(9)	$N(x \Leftrightarrow y) \vdash N(x \Rightarrow y)$	like (8) in S_a
(10)	$N(x \Leftrightarrow y) \vdash N(y \Rightarrow x)$	similar to (9)
(11)	$u \Rightarrow v,\ v \Rightarrow u \vdash u \Leftrightarrow v$	in PC
(12)	$N(u \Rightarrow v),\ N(v \Rightarrow u) \vdash N(u \Leftrightarrow v)$	like (11) in S_a
(13)	$N(Nx \Rightarrow Ny),\ N(Ny \Rightarrow Nx) \vdash N(Nx \Leftrightarrow Ny)$	by (12)
(14)	$N(x \Leftrightarrow y) \vdash N(Nx \Leftrightarrow Ny)$	by (9), (10), (6), (7), (13)

and (14) is C.

Remarks

1° (6) is the so-called «Becker's rule», which holds in $\nu\varrho S_a$.
2° The preceding proofs (of C and of Becker's rule) make use only of $\nu\varrho\nu D$ and I in addition to S_a; $\nu\varrho W$ and ϱW are not necessary for the proof.

Theorem 7.6 — $\nu\varrho S_c = \nu\varrho\nu S_a$

Let us return to table 1 and write down the postulates for $\nu\varrho S_c$ and $\nu\varrho\nu S_a$

$\nu\varrho S_c$: νP_1, νP_2, νP_3, $\nu\varrho C$, $\nu\varrho W$, ϱW, $\nu\varrho\nu D$, I, D
$\nu\varrho\nu S_a$: νP_1, νP_2, νP_3, $\nu\varrho W$, ϱW, $\nu\varrho\nu D$, $\nu\varrho I$, D

But:

1° By Theorem 7.4 $\nu\varrho I$ holds in $\nu\varrho S_c$ and can replace I (I is provable by $\nu\varrho I$, ϱW and D).

2° By Theorem 7.5, rule C holds in $\nu\varrho S_a$, then as well in $\varrho\nu\varrho S_a$ and $\nu\varrho\nu S_a$. But $\nu\varrho\nu S_a$ is normal and canonical (§ 6), so that $\nu\varrho C$ holds in $\nu\varrho\nu S_a$ as being a schema of theses, and might be added to the postulates without changing the system.

It results that the two systems have the same lists of (modified) postulates, and are therefore identical.

Theorem 7.7 — $\varrho\nu S_c \subset \varrho\nu\varrho S_a$

Let us return to Table 1 and write down the postulates of $\varrho\nu S_c$ and $\varrho\nu\varrho S_a$

$\varrho\nu S_c$: νP_1, νP_2, νP_3, $\varrho\nu C$, ϱW, $\varrho\nu D$, ϱI, D
$\varrho\nu\varrho S_a$: νP_1, νP_2, νP_3, $\nu\varrho W$, ϱW, $\nu\varrho\nu D$, ϱI, D

But:

1° νC would be $NN(x \Leftrightarrow y) \vdash NN(Nx \Leftrightarrow Ny)$ which is provable by C, W and I and could be replaced by C in νS_c; $\varrho\nu C$ can therefore be replaced by ϱC in $\varrho\nu S_c$ (8).

2° Rule C holds in $\nu\varrho S_a$ (Theorem 7.5), then as well in $\varrho\nu\varrho S_a$. This last system being canonical ϱC holds in it.

Then every postulate of the (modified) list for $\varrho\nu S_c$ is provable from the postulates of $\varrho\nu\varrho S_a$.

8. *The twelve systems* — Starting from S_a, S_b and S_c we have produced twelve systems.

We can summarize the results of §§ 6 and 7 in table 2, table 3 and figure 3.

(8) We may alternatively derive νC from ϱC, W, and I, and conclude that $\varrho\nu C$ is provable in the canonical system $\varrho\nu S_c$, if ϱC is among its postulates — and ϱC holds in $\varrho\nu S_c$ since C holds and the system is canonical.

Table 2 is a list of postulates (9).

Table 2

νP_1: $\vdash N(x \Rightarrow (y \Rightarrow x))$
νP_2: $\vdash N((x \Rightarrow (y \Rightarrow z)) \Rightarrow ((x \Rightarrow (x \Rightarrow z)))$
νP_3: $\vdash N((\neg x \Rightarrow \neg y) \Rightarrow (y \Rightarrow x))$
D: $x \Rightarrow y,\ x \vdash y$
νD: $N(x \Rightarrow y),\ Nx \vdash Ny$
$\varrho\nu$D: $\vdash N(x \Rightarrow y) \Rightarrow (Nx \Rightarrow Ny)$
$\nu\varrho\nu$D: $\vdash N(N(x \Rightarrow y) \Rightarrow (Nx \Rightarrow Ny))$
WC: $N(x \Leftrightarrow y) \vdash Nx \Leftrightarrow Ny$
C: $N(x \Leftrightarrow y) \vdash N(Nx \Leftrightarrow Ny)$
ϱC: $\vdash N(x \Leftrightarrow y) \Rightarrow N(Nx \Leftrightarrow Ny)$
W: $Nx \vdash x$
ϱW: $\vdash Nx \Rightarrow x$
$\nu\varrho$W: $\vdash N(Nx \Rightarrow x)$
I: $Nx \vdash NNx$
ϱI: $\vdash Nx \Rightarrow NNx$
$\nu\varrho$I: $\vdash N(Nx \Rightarrow NNx)$

Table 3 is the list of the postulates of each of the twelve systems — νP_1, νP_2, νP_3, D, common to all the systems, have not been repeated.

Table 3

S_a: νD, W
S_b: νD, W, WC
S_c: νD, W, C
νS_a: νD, W, I
$\nu S_c (= \nu S_b)$: νD, W, I, C
$\varrho S_a (= \varrho S_b)$: $\varrho\nu$D, ϱW
ϱS_c: $\varrho\nu$D, ϱW, ϱC

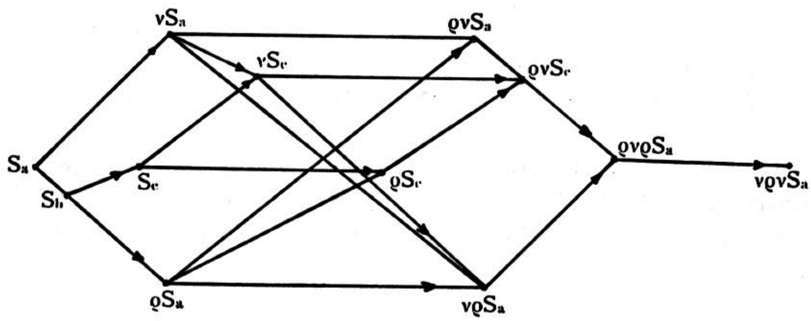
(9) Key to PORTE 1958 for the names of the postulates

this paper	WC	C	W	I
PORTE 1958	Cf	CF	A	Rv

$\varrho v S_a$: $\varrho v D$, ϱW, ϱI
$\varrho v S_c$: $\varrho v D$, ϱW, ϱI, ϱC
$v \varrho S_a$: $v \varrho v D$, $v \varrho W$, ϱW, I
$\varrho v \varrho S_a$: $v \varrho v D$, $v \varrho W$, ϱW, ϱI
$v \varrho v S_a$: $v \varrho v D$, $v \varrho W$, ϱW, $v \varrho I$
$(= v \varrho v \varrho S_a$
$= v \varrho S_c$
$= \varrho v \varrho S_c$
$= v \varrho v S_c)$

Figure 3 summarizes the relationships between the systems as in figures 1 or 2.

Figure 3



It will be proved (§ 9) that all these systems are different.

The independance of the postulates of Table 3 has not been examined — but of course a few cases of independence will result from the fact that the sysems are different: for instance $v \varrho I$ is independent in the postulates of $v \varrho v S_a$, since it is the only postulate by which $v \varrho v S_a$ differs from $\varrho v \varrho S_a$.

All these systems are consistent (same proof as for the consistence of S_a: Theorem 2.2).

It is often easy to prove an apparently complicated proposition about one of these systems using a few postulates, general properties of S_a, canonicity (Theorem 4.2), and normality (Theorem 5.1).

For instance, here is a proof that

$$\vdash N(N(x \Rightarrow y) \Rightarrow N(Nx \Rightarrow Ny)) \text{ in } \nu\varrho\nu S_a$$

— i.e. in S_4 (see § 10)

(1)	$x \Rightarrow y,\ x \vdash y$	by PC
(2)	$N(x \Rightarrow y),\ Nx \vdash Ny$	like (1) in S_a
(3)	$N(x \Rightarrow y) \vdash Nx \Rightarrow Ny$	by (2), $\varrho S = S$
(4)	$\vdash N(x \Rightarrow y) \Rightarrow (Nx \Rightarrow Ny)$	like (3)
(5)	$\vdash N(Nx \Rightarrow y) \Rightarrow (Nx \Rightarrow Ny))$	by (4), $\nu S = S$
(6)	$\vdash NN(x \Rightarrow y) \Rightarrow N(Nx \Rightarrow Ny)$	by (3), (5)
(7)	$NN(x \Rightarrow y) \vdash N(Nx \Rightarrow Ny)$	by (6), D
(8)	$N(x \Rightarrow y) \vdash NN(x \Rightarrow y)$	by I
(9)	$N(x \Rightarrow y) \vdash N(Nx \Rightarrow Ny)$	by (7), (8)
(10)	$\vdash N(x \Rightarrow y) \Rightarrow N(Nx \Rightarrow Ny)$	by (7), $\varrho S = S$
(11)	$\vdash N(Nx \Rightarrow y) \Rightarrow N(Nx \Rightarrow Ny))$	by (10), $\nu S = S$

If we examine the proof in details, we see that all the propositions until (4) hold in ϱS_a (i.e. in S0.5, see § 10), until (9) they hold in $\nu\varrho S_a$ (i.e. in T, see § 10), and until (10) they hold in $\varrho\nu\varrho S_a$.

9. *Comparing the systems* — It will now be proved that the 12 systems are different of each other. Moreover there is no case of comparable strength other than those which are represented by arrows in figure 3 or those which follow by transitivity — for instance, we have neither $\nu\varrho S_a \subset \varrho\nu S_c$ nor $\varrho\nu S_c \subset \nu\varrho S_a$.

This can be proved from the following propositions: for all systems S and S', we have

(1) $\varrho\varrho S = \varrho S$
(2) $\nu\nu S = \nu S$
(3) $S \subset \varrho S$
(4) $S \subset \nu S$

(5) if $S \subset S'$ then $\varrho S \subset \varrho S'$
(6) if $S \subset S'$ then $\nu S \subset \nu S'$
(7) $\varrho S_b = \varrho S_a$
(8) $\nu S_b = \nu S_c$
(9) $\nu\varrho\nu S_a = \varrho\nu\varrho\nu S_a$
(10) $\nu\varrho\nu S_a = \nu\varrho\nu\varrho S_a$
(11) $\nu\varrho S_c = \nu\varrho\nu S_c$
(12) $\nu\varrho S_c = \varrho\nu\varrho S_c$
(13) $\nu\varrho S_c = \nu\varrho\nu S_a$
(14) transitivity of inclusion
(15) $\nu\varrho S_c \not\subset \varrho\nu\varrho S_a$
(16) $\varrho\nu\varrho S_a \not\subset \varrho\nu S_c$
(17) $\varrho S_c \not\subset \varrho\nu S_a$

The proofs are long and tedious, but trivial. For instance let us suppose

$$\nu\varrho S_a \subset \varrho\nu S_c$$

it would follow $\varrho\nu\varrho S_a \subset \varrho\varrho\nu S_c$ by (5)
$\varrho\nu\varrho S_a \subset \varrho\nu S_c$ by (1)

which would contradict (15); let us suppose now

$$\varrho\nu S_c \subset \nu\varrho S_a$$

it would follow $\nu\varrho\nu S_c \subset \nu\varrho S_a$ by (6)
$\nu\varrho S_c \subset \nu\varrho S_a$ by (11)

which would contradict (15) and (16) taking into account that

$$\nu\varrho S_a \subset \varrho\nu\varrho S_a \subset \nu\varrho S_c$$

(see fig. 3); and so on.

Propositions (1) to (13) have already been proved; (15), (16), and (17) will now be proved by means of three matrices.

These matrices have all certain common features, namely:

— the basic set is the set of all subset of a particular finite set such as $\{a, b, c\}$;

— the representatives of $\neg$ and $\Rightarrow$ are the usual set-theoretic functions, namely complement (**C**) and $(\alpha\ \beta) \mapsto \mathbf{C}\, \alpha \cup \beta$.

It will then be sufficient to give the tables of the representative functions of N, and the sets of the designated elements ($\mathscr{D}$).

In order to spare room a set like $\{a, b\}$ will be denoted simply by «ab».

Here are the matrices:

M1

α	Ø	b	c	bc	a	ac	ab	abc
Nα	Ø	Ø	Ø	Ø	Ø	c	b	abc

$\mathscr{D} = \{a, ac, ab, abc\}$

It is a model of $\varrho\nu\varrho S_a$, but $\nu\varrho I$ is not valid; whence (15).

M2

α	Ø	b	c	bc	a	ac	ab	abc
Nα	Ø	Ø	Ø	Ø	a	ac	ac	abc

$\mathscr{D} = \{ab, abc\}$

It is a model of $\varrho\nu S_c$, but $\nu\varrho W$ is not valid; whence (16).

M3

α	Ø	b	c	d	bc	bd	cd	bcd	a	ab	ac	ad	abc	abd	acd	abcd
Nα	Ø	Ø	Ø	Ø	Ø	Ø	Ø	Ø	a	ad	ac	acd	abc	ad	acd	abcd

$\mathscr{D} = \{ab, abd, abc, abcd\}$

It is a model of $\varrho\nu S_a$, but ϱC is not valid; whence (17).

Now $S \not\subset S'$ means that there are cases of deducibility in S which do not hold in S′. But if S is not canonical, it might happen that every thesis of S be a thesis of S′.

To complete the proof that when $S \not\subset S'$, we have also (theses of S) $\not\subset$ (theses of S′) it is sufficient to prove:

(18) $\nu\varrho S_a \not\subset \varrho\nu S_c$ in that stronger sense
(19) $\nu S_c \not\subset \varrho S_c$ ″
(20) $S_c \not\subset \varrho\nu S_a$ ″
(21) $S_b \not\subset \nu S_a$ ″

All these propositions are proved by means of matrices. (18) has already been proved, since M2 is a model of $\nu\varrho S_a$ in which $\nu\varrho W$ is not valid. We need three new matrices. Here they are:

M4

α	Ø	a	b	ab
Nα	Ø	Ø	Ø	a

$\mathscr{D} = \{a, ab\}$

is a model of S_c in which the formula $NN(x \Leftrightarrow x)$ (a thesis of S_c) is not valid; whence (19). Indeed it is «group I» of Lewis-Langford 1932, i.e. a model of S3 not of S4; it proves as well $\varrho\nu\varrho S_a \not\subset S3$ and $\nu\varrho S_a \not\subset S3$.

For the last two matrices it will be necessary to consider a homomorphic counter-image of a boolean algebra rather than a proper boolean algebra. This aim will be achieved by «splitting» each member of the set $E = \{\varnothing, a, b, ab\}$ in two, — the counter-images of Ø, by instance, being Ø and Ø′, and so on. The tables for the representatives of $\neg$ and $\Rightarrow$ will be obtained by ignoring the difference between «dashed» and «non-dashed» elements in the usual set-theoretic functions in E. So:

α	Ø	Ø′	b	b′	a	a′	ab	(ab)′
¬ α	ab	ab	a	a	b	b	Ø	Ø

and similarly ⇒. It will then be sufficient to give the tables for N and the sets of designated elements in order to define the matrices.

M5

α	Ø	Ø′	b	b′	a	a′	ab	(ab)′
Nα	Ø	Ø	Ø	b	Ø	Ø	ab	(ab)′

$\mathscr{D} = \{a, a', ab, (ab)'\}$

It is a model of $\varrho\nu S_a$ in which the formula $N(Nx \Leftrightarrow N\neg\neg x)$, a a thesis of S_c, is not valid. Whence (20).

M6

α	Ø	Ø′	b	b′	a	a′	ab	(ab)′
Nα	Ø	Ø	Ø	Ø	a	a′	a	(ab)′

$\mathscr{D} = \{a, a', ab, (ab)'\}$

It is a model of vS_a in which the formula $Nx \Leftrightarrow N \neg\neg x$, a thesis of S_b, is not valid. Whence (21).

10. *The place of the classical systems* — At once:

(i) *ϱS_a is Lemmon's S0.5* (Lemmon 1957, p. 181), — a system also considered by Pollock (Pollock 1967, p. 362, axioms and rules A1 to R) without particular name.

(ii) *$v\varrho S_a$ is Feys' T*, i.e. von Wright's M — a system also used in Gödel 1933 without particular name.

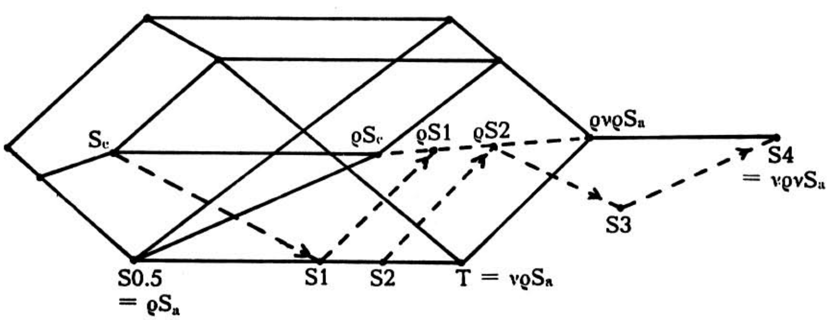
(iii) *$v\varrho S_c = v\varrho v S_a$ is Lewis' S4* (see especially the axiomatization of S4 in Gödel 1933).

The «*Basic Modal Logic*» (BSM) of Pollock 1967 is not of the same nature since its formulas do not contain superpositions of modalities. But Theorems 3 and 4 of Pollock can be summarized in:

Theorem 10.1 — In every modal system at least as strong ϱS_a and a most as strong as Lewis S5, the set of theses not involving superpositions of modalities is just the set of valid formulas of the BSM.

The other classical systems, in particular Lewis' S1, S2, and S3 may be put into relationship with the 12 construced systems (see fig. 4), but have no simple connection with them.

Figure 4



In addition, we have

$$\varrho S3 = S3 \text{ (S3 is canonical)}$$
$$\nu S1 = \nu S2 = T$$
$$\nu S3 = S4$$

It is well known that S3 is not comparable to T. Matrix M4 shows that $\varrho\nu\varrho S_a \not\subset S3$. Matrix M1 shows that $S3 \not\subset \varrho\nu\varrho S_a$, for formula $N(N(x \Rightarrow y) \Rightarrow N(Nx \Rightarrow Ny))$, a thesis of S3, is not valid in M1.

The following matrix

M7

α	Ø	b	c	bc	a	ac	ab	abc
Nα	Ø	Ø	Ø	Ø	Ø	Ø	a	ab

$D = \{ab, abc\}$

is a model of S2 and of S1 in which ϱC is not valid. Whence $\varrho S2 \not\subset S2$ and $\varrho S1 \not\subset S1$

The systems that are not at least as strong as ϱS_a seem too weak to be considered as «good» modal systems. They are useful to build the other systems. On the contrary, systems ϱS_c, $\varrho\nu S_a$, $\varrho\nu S_c$ and $\varrho\nu\varrho S_a$ may be worthy of some study.

It must be pointed out that the construction starting with S_a by means of ϱ and ν was not (at least not consciously !) intended to reach T and S4, and anyway I did not know Lemmon's S0.5 when I constructed ϱS_a (the paper Porte 1958 has been written in 1956).

S5 cannot be constructed by the means studied here, since everything stops at the first system which is both canonical and normal, that is S4.

The so-called Ł-system (see Łukasiewicz 1953 or Porte 1979) differs in its very nature from the systems studied here.

11. *On regular models* — We return now to the point of view of the paper Porte 1958 — a point of view that may be called «algebraic» on «semantical», i.e. the study of certain matrices.

Let us first recall a few classical definitions, in order to state the notations and the vocabulary.

Definition 11·1 — A *matrix* for a modal system is a structure $\mathscr{M} = \langle \mathscr{B}, \Leftrightarrow^*, \neg^*, N^*, \mathscr{D} \rangle$, where $\mathscr{B}$ is the basic set or set of values, $\Rightarrow^*$ an application $\mathscr{B} \times \mathscr{B} \rightarrow \mathscr{B}$, $\neg^*$ and N^* two applications $\mathscr{B} \rightarrow \mathscr{B}$ and $\mathscr{D}$ a subset of $\mathscr{B}$.

$\mathscr{D}$ is the set of *designated* elements (or designated values).

Definition 11.2 — An *assignment* (of values) into a matrix for a formal system is an application $\tau : \mathscr{F} \rightarrow \mathscr{B}$ ($\mathscr{F}$ being the set of the formulas of the system), such as

$$\tau(x \Rightarrow y) = \tau(x) \Rightarrow^* \tau(y)$$
$$\tau(\neg x) = \neg^* \tau(x)$$
$$\tau(Nx) = N^* \tau(x)$$

for every $x, y \in \mathscr{F}$. It follows that an assignment is completely defined by the values assigned to the atoms.

Definition 11.3 — A *formula* x is *valid* in a matrix $\mathscr{M}$ if $\tau(x) \in \mathscr{D}$ for every assignment into $\mathscr{M}$.

Definition 11.4 — A *rule* such as

$$R: f_1(x, y),\ f_2(x, y) \vdash g(x, y)$$

is *weakly valid* in $\mathscr{M}$ if:

if $f_1(x, y)$ is valid and $f_2(x,y)$ is valid, then $g(x,y)$ is valid

Definition 11.5 — A *rule* such as R (above) is *strongly valid* in $\mathscr{M}$ if:

if $f_1{}^*(\alpha, \beta) \in \mathscr{D}$ and $f_2{}^*(\alpha, \beta) \in \mathscr{D}$, then
$g^*(\alpha, \beta) \in \mathscr{D}$ for every $\alpha, \beta \in \mathscr{B}$

Definitions 11.4 and 11.5 are taken from Harrop 1958 — see also Łos-Suszko 1958 and Porte 1965 ([10]).

A strongly valid rule is also weakly valid but the converse proposition is false. A matrix for PC is often said «regular» when D is strongly valid, but there are non-regular matrices in which D is weakly valid (see for instance Church 1953).

Definition 11.6 — A matrix $\mathcal{M}$ is a *model* of the formal system S, if every thesis of S is valid in $\mathcal{M}$.

By a natural generalization of the classical notion of a «regular matrix», (Church 1953), we state:

Definition 11.7 — A matrix $\mathcal{M}$ is a *regular model* of the logistic system S, if every postulated rule of S is strongly valid in $\mathcal{M}$.

II would be equivalent to say that every deductively acceptable rule (see Remark 2 after Theorem 4.2) is strongly valid in a regular model.

In what follows we will consider only modal systems, S, which are at least as strong as S_a; rule D is postulate (or deductively acceptable) in such an S·

Theorem 11.1 — In a regular model of a modal system, the algebra $\langle \mathcal{B}, \Rightarrow^*, \neg^* \rangle$ is boolean ([11]).

Proof: See Church 1953, pp. 44-45.

Theorem 11.2 — In a regular model of a modal system, the set of the designated values, $\mathcal{D}$, is a filter (or «sum-ideal») of the algebra — i.e. we have, for every $\alpha, \beta \in \mathcal{B}$:

([10]) In Porte 1965 weakly valid rules are said «T-vaides», and strongly valid rules are said «D-valides». In a regular model (Definition 11..7) every D-acceptable rule is D-valid.

([11]) Strictly speaking, it is the algebra $\langle \mathcal{B}, \vee^*, \wedge^*, \neg^* \rangle$ which is boolean, with

$$\alpha \vee^* \beta = (\alpha \Rightarrow^* \beta) \Rightarrow^* \beta$$
$$\alpha \wedge^* \beta = \neg^* (\neg^* \alpha \vee^* \neg^* \beta)$$

1° if $\alpha \in \mathscr{D}$ and $\beta \in \mathscr{D}$, then $\alpha \wedge^* \beta \in \mathscr{D}$;
2° if $\alpha \in \mathscr{D}$, then $\alpha \vee^* \beta \in \mathscr{D}$
3° $\alpha \Rightarrow^* \alpha \in \mathscr{D}$;
4° if $\alpha \in \mathscr{D}$ and $\alpha \Rightarrow^* \beta \in \mathscr{D}$, then $\beta \in \mathscr{D}$ ([12])

This follows from the fact that

$\vdash x \Rightarrow (y \Leftrightarrow (x \wedge y))$
$\vdash x \Rightarrow (y \vee y)$
$\vdash x \Rightarrow x$
$\vdash x \Rightarrow ((x \Leftrightarrow y) \Rightarrow y)$

(for every formula x, y) in PC, and rule D is strongly valid in a regular model.

Let us define a new notion.

Definition 11.8 — The elements α of a matrix, $\mathscr{M}$, such as $N^*\alpha \in \mathscr{D}$ are the *ν-designated* elements.

Theorem 11.3 — In a regular model of a modal system, the set, $\mathscr{D}'$, of the ν-designated elements is a filter.

Proof. Just like the proof of Theorem 11.2, but using rule νD and the theses of S_a of the form N*t* where *t* is a tautology.

We can now see the semantical motivation of the definitions used in §§ 2, 3, 4, and 5.

Motivation of rule W. The ν-designated elements represent the idea of «necessary propositions» just as the designated elements represent the idea of «acceptable propositions». In order to carry out the basic idea (§ 1) we must then assume $\mathscr{D}' \subset \mathscr{D}$ that is:

if $N^*\alpha \in \mathscr{D}$, then $\alpha \in \mathscr{D}$

([12]) Condition 4° is not a part of the definition of a filter. It is used here as a means to be sure (in the case where the algebra is a homomorphic counter image of a boolean algebra, rather than a true boolean algebra) that $\mathscr{D}$ is a union of classes determined by the homomorphism equivalence.

But this expresses the strong validity of W.

If we did not postulate the rule W in S_a, we could consider models like the following matrix

M9

α	$\emptyset$	b	a	ab
$N^*\alpha$	$\emptyset$	a	b	ab

$$\mathscr{D} = \{a, ab\}$$

whence $\mathscr{D}' = \{b, ab\}$. All the theses of S_a are valid (among them, P_1, P_2, P_3, νP_1, νP_2, νP_3), and the rules D and νD are strongly valid. But W is not strongly valid.

It is clear that the basic idea is not carried out in M9 — In other models, we could have $\mathscr{D} \subset \mathscr{D}'$ and $\mathscr{D} \neq \mathscr{D}'$ as in the following matrix

M10

α	$\emptyset$	b	a	ab
$N^*\alpha$	$\emptyset$	$\emptyset$	ab	ab

$$\mathscr{D} \neq \{ab\}$$

— whence $\mathscr{D}' \neq \{a, ab\}$

Compatibility — Let us define two relations on $\mathscr{B}$, E and E′, by

$\alpha E \beta$ if and only if $\alpha \Leftrightarrow^* \beta \in \mathscr{D}$
$\alpha E' \beta$ if and only if $\alpha \Leftrightarrow^* \beta \in \mathscr{D}'$
(or $N^*((\alpha \Leftrightarrow^* \beta) \in \mathscr{D})$ (13)

If we consider the filters $\mathscr{D}$ and $\mathscr{D}'$ as ideals of a boolean ring (14). E and E′ are respectively the congruence modulo $\mathscr{D}$

(13) In PORTE 1958 the relation E′ was called E; E was not explicitely studied.

(14) The operation of these rings are

— for $\mathscr{D}$: multiplication: $\bigvee^*$
addition: $\Leftrightarrow^*$

— for $\mathscr{D}'$: multiplication: $\alpha, \beta \mapsto N^*(\alpha \bigvee^* \beta)$
addition: $\alpha, \beta \mapsto N^*(\alpha \Leftrightarrow^* \beta)$

and the congruence modulo $\mathcal{D}'$, and $\mathcal{D}$ and $\mathcal{D}'$ are respectively an equivalence class of E and an equivalence class of E'.

When W is strongly valid, $\mathcal{D}' \subset \mathcal{D}$, so that each equivalence class of E is a union of equivalence classes of E'.

A function is compatible with an equivalence relation if it maps each equivalence class into an equivalence class. To say that N* is compatible with E' is to say

$$\text{if } \alpha E'\beta \quad \text{then} \quad N^*\alpha\, E'\, N^*\beta$$
$$\text{or: if } N^*(\alpha \Leftrightarrow^* \beta) \in \mathcal{D} \quad \text{then} \quad N^*(N^*\alpha \Leftrightarrow^* N^*\beta)$$

— which means that the rule C is strongly valid.

To say that the rule WC is strongly valid is to state the condition —weaker that true compatibility — that Nα maps each equivalence class of E' into an equivalence class of E, which generalizes the fact that N* maps $\mathcal{D}'$ into $\mathcal{D}$.

Normalization — The choice that has been made in § 5 to define νS can only be justified by the neat result of §§ 6, 7 and 8. Other choices could have produced a normal system from any modal system S. For instance let us consider the following system (S having $\mathcal{Y}$ as axiom schemas and $\mathcal{Z}$ as rules — see § 6):

$$\nu_2 S\colon\ \nu\nu\mathcal{Y},\ \mathcal{Z},\ \nu\mathcal{Z},\ \nu\nu\mathcal{Z},\ W,\ I_2,\ D$$

where the rule I_2 is: NNx $\vdash$ NNNx
Then ν_2S is normal, but the rule I is not, in general, deductively acceptable in ν_2S...

From the semantical point of view, let us consider ν-*models,* defined by ν-*matrices*

$$\mathcal{N} = \langle \mathcal{B}, \Rightarrow^*, \neg^*, N^*, \mathcal{D}, \mathcal{D}' \rangle$$

where $\mathcal{D}' = \{\alpha \in \mathcal{B};\ N^*\alpha \in \mathcal{D}\}$; $\mathcal{D}$ is the set of designated elements; ν-models and regular ν-models are defined as in Definitions 11.6 and 11.7.

Let us consider the regular ν-models of S such as $\mathcal{N}' = \langle \mathcal{B}, \Rightarrow^*, \neg^*, N^*, \mathcal{D}', \mathcal{D}' \rangle$ is also a regular ν-model of S, and let us call them «normal regular ν-models». Then νS is the formal system whose regular ν-models are exactly the normal regular ν-models of S. This process may be considered as a way of carrying out the idea of a normal system by algebraic notions: a regular ν-model, $\mathcal{N}$, is normal if its $\mathcal{D}$ (which represents the «acceptable propositions») can be restricted to $\mathcal{D}'$ (which represents the «necessary propositions») without altering the fact that $\mathcal{N}'$ is a regular ν-model. But it seems somewhat arbitrary to keep $\mathcal{D}$ unchanged in $\mathcal{N}'$ (one may only remark that $\mathcal{D}'$ could not be restricted when it is a singleton, i.e., when E′ is the identity) [15].

[15] in PORTE 1958, certain notations may be explained by the following example:

— S4′ (= νϱνS$_a$) is the set of the regular models of S4

— (S4)$_m$ is the set of the models of S4 in which rule D is strongly valid («regular matrices» in the sense of CHURCH 1953, i.e. regular models of PC).

What was called «espèces normales d'algèbres» were the underlying matrices of the normal regular ν-models of some formal systems. Rule I was strongly valid in them.

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