ON ŁUKASIEWICZIAN LOGICS WITH SEVERAL DESIGNATED VALUES

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A commonly held view about the many-valued logics of Łukasiewicz is that the non-classical character of these systems depends on the choice of designated values. For example, some writers have claimed that if 1 and $^1/_2$ (T and I) are taken as designated in Łukasiewicz's 3-valued logic, then the system that results is the same as classical logic. (see [1], p. 3) However, in *Many Valued Logic,* Rescher reports a counterexample to this claim (due to Turquette); the sentence $\neg(p \supset \neg p) \vee \neg(\neg p \supset p)$ is a classical tautology, but assumes the value 0 in $Ł_3$ if p has the value $^1/_2$.

In view of this counterexample, it is important to try to obtain a more precise picture of the non-classical core of the Łukasiewiczian logics. This paper describes the results of my investigation of this problem. Speaking generally, my results bear out the idea that the truly «non-classical» component is the Łukasiewiczian tables for the conditional and the biconditional. In Section 1, I examine fragments of Łukasiewiczian logics which do not contain the conditional or the biconditional. I show that these fragments can be rendered classical by suitable choices of designated values. In Section 2, I investigate fragments containing the conditional. The strongly non-classical nature of this connective is evidenced by the fact that *no* choice of designated values can render fragments containing the conditional classical. In Section 3, I examine several fragments containing the biconditional. The situation here appears to be a hybrid between the cases previously examined. The pure biconditional fragment of any Łukasiewiczian logic and the negation-biconditional fragment of even-valued systems can be rendered classical by suitable choices of designated values. On the other hand, the negation-biconditional fragment of odd-valued systems and the disjunction-biconditional fragment of any system is strongly non-classical in the sense that there is no choice of designated values which will render this fragment classical.

Before proceeding, it is advisable to fix the form of representing Łukasiewiczian logics. Throughout the discussion, I confine my attention to logics with a finite number of values. For the n-valued logic $Ł_n$, let us suppose that the values are 1, n—2/n—1,..., 1/n—1, 0. The basic Łukasiewiczian connectives are evaluated according to the following rules:

1. $v(\neg p) = 1 - v(p)$;
2. $v(p \,\&\, q) = \min(v(p), v(q))$;
3. $v(p \vee q) = \max(v(p), v(q))$;
4. $v(p \supset q) = \begin{cases} 1 \text{ if } v(p) \leqq v(q); \\ 1 - v(p) + v(q), \text{ otherwise;} \end{cases}$
5. $v(p \equiv q) = 1 - |v(p) - v(q)|$.

A wff B is said to be valid in $Ł_n$ relative to the set D of designated values iff for every valuation v on $Ł_n$, v(B) is in D.

1. *Fragments not containing* $\supset$ *or* $\equiv$

In some sense, the Łukasiewiczian tables for negation, conjunction, and disjunction are more classical and natural than the tables for the conditional and the biconditional. This idea manifests itself in several ways. For example, it is possible to give a straightforward semantic account, based on 2-valued logic, for the $\neg$, &, and $\vee$ tables for $Ł_3$, whereas no such account seems available for the $\supset$ and $\equiv$ tables.

The classical character of Łukasiewiczian $\neg$, &, and $\vee$ also manifests itself in the matter of designated values. For, as I shall show, fragments of $Ł_n$ containing only $\neg$, &, and $\vee$ are classical for certain natural choices of designated values. I shall discuss only the $\neg\&\vee$-fragment of $Ł_n$, but the claims made also hold for more limited fragments. The basic theorem here is:

THEOREM 1. The $\neg\&\vee$-fragment of $Ł_n$ is classical relative to the set D of designated values iff (1) there is a value u of $Ł_n$ that is not in D and (2) for every value u of $Ł_n$, max(u, 1 — u) is in D.

(⇒) If condition (1) isn't satisfied, everything is valid. If condition (2) isn't satisfied, there is a value u in $Ł_n$ such that max (u, 1 — u) is not in D. But then $p \vee \neg p$ receives an undesignated value when v(p) = u.
(⇐) Suppose that conditions (1) and (2) are satisfied. Then two claims must be proved:

a. If A is not a classical tautology, then A is not valid in $Ł_n$ relative to D.
b. If A is a classical tautology, then A is valid in $Ł_n$ relative to D.

I consider (a) first. By condition (1) there is a value u of Ł which is not in D. By condition (2), $u < 1—u$. Consequently, if we restrict attention to the sub-tables for ¬, &, and ∨ involving only u and 1—u, the result is the following:

p	¬ p
1—u	u
u	1—u

p & q	1—u	u
1—u	1—u	u
u	u	u

p ∨ q	1—u	u
1—u	1—u	1—u
u	1—u	u

These tables are isomorphic to the classical truth-tables with 1—u for T and u for F.

Now, suppose that there is a classical valuation v such that v(A) = F. Let $p_1, \ldots, p_c$ be the propositional variables in A. Let v* be a valuation on $Ł_n$ such that if $v(p_i)$ is T, then $v^*(p_i)$ is 1—u and if $v(p_i)$ is F, then $v^*(p_i)$ is u. By the isomorphism just mentioned, v*(A) is u. So, A isn't valid relative to D.

Claim (b) remains to be established. To prove this, I employ conjunctive normal forms in a crucial way. So, an intermediate lemma is needed.

LEMMA 1. If A is a wff in the $\neg\&\vee$-fragment of $Ł_n$, then there is a wff B in the $\neg\&\vee$-fragment such that B is in conjunctive normal form (CNF) and B has the same $Ł_n$ truth-table as A.

Note first that a wff A containing only $\neg$, &, and $\vee$ can be transformed into CNF using just (i) double negation, (ii) DeMorgan's Laws, and (iii) the $\&\vee$ distribution laws. To prove Lemma 1, it therefore suffices to show that operations (i)-(iii) have the property of transforming a wff C into a wff D which has the same $Ł_n$ truth-table as C.

case (i). For any v, $v(p) = 1-(1-v(p)) = v(\neg\neg p)$.
case (ii). I show that for any v, $v(p \& q) = v(\neg(\neg p \vee \neg q))$.

Other cases are similar. Suppose first that $v(p) = \min(v(p), v(q))$. Then $1-v(p)$ is $\max(1-v(p), 1-v(q))$. So, $v(p) = 1-\max(1-v(p), 1-v(q))$, as required. If $v(q) = \min(v(p), v(q))$, the argument is parallel.

case (iii). I show, by the following table, that for any v, $v(p\&(qvr)) = v((p\&q) \vee (p\&r))$: (here I use $+(p^*, q^*)$ for $\max(v(p), v(q))$ and $-(p^*, q^*)$ for $\min(v(p), v(q))$.)

$-(p^*, q^*)$	$-(p^*, r^*)$	$-(q^*, r^*)$	$-(p^*, +(q^*, r^*))$	$+(-(p^*, q^*), -(p^*, r^*))$
p	p	q	p	p
p	p	r	p	p
q	p	q	p	p
q	r	q	r	r
p	r	r	p	p
q	r	r	q	q

Since the last two columns are identical, the claim is proved. Proof of the other distribution law is similar.

Given Lemma 1, we proceed to prove claim (b). Suppose that A is a classical tautology. Then the CNF B proved to exist in Lemma 1 is classically equivalent to A. Thus, B is a classical tautology. Since B is in CNF, every conjunct of B is a disjunction which contains some sentence letter and its negation. Let E be a conjunct in B. Suppose that p_e is such that both p_e and $\neg p_e$ occur in E. Let v be a valuation on $Ł_n$, and suppose $v(p_e)$ is u. Then $v(E) \geqq \max(u, 1—u)$. But this implies that v(E) is a designated value. For if v(E) is not in D, then it follows by condition (1) of the theorem that $1—v(E) > v(E) \geqq \max(u, 1—u)$, which is impossible. Consequently each disjunct of B has a designated value on every valuation in $Ł_n$. Since the value of B is the least of these values, B receives a designated value in every case. By Lemma 1, A has the same $Ł_n$ truth table as B. So, A is valid in $Ł_n$ relative to D (1).

Several comments about the theorem are in order. For $Ł_2$, the only D which meets the conditions of the theorem is the set {1}. For $Ł_3$, the only stasfactory D is the set {1, $^1/_2$}. For $n > 3$, there will be several sets of values which determine classical logic. Among them, there is a minimal set D* which contains only max(u, 1—u) for each u. Any superset of D* which is not

(1) One consequence of Theorem 1 is that if D meets conditions (1) and (2) and both $\neg A \vee B$ and A are valid relative to D, then B is valid relative to D. That is, the set of wffs valid relative to D is closed under a *modus ponens* principle formulated in terms of negation and disjunction.

In [2], Page 70, Rescher attempts to argue for this closure principle directly. Noting that he uses $/p \rightarrow q/$, where I use $v(\neg p \vee q)$, his argument is this, for the case of L_3:

> ...for assume that p and $/p \rightarrow q/$ are both tautologies, but that q is not. Now this can only be when for some assignment of values in p and q it results that $/p/ = I$ and $/p \rightarrow q/ = I$ and $/q/ = F$. But then under the modified assignment obtained by replacing each I in the initial one by T, we shall have it that: $/q/ = F$, and either $/p/ = F$ or $/p \rightarrow q/ = F$, contrary to assumption.

What this argument fails to provide is a reason for believing that $/q/ = F$, on the modified assignment. That $/q/ = F$ on this assignment must depend somehow on the fact that $/A \rightarrow B/ = I$ when $/A/ = /B/ = I$. For if $/A \rightarrow B/$ were T in this case, then Rescher's claim could be falsified. What is apparently needed here is some sort of inductive argument on the structure of q. But without something like Lemma 1 of the paper, it is unclear how such an argument could proceed.

identical with the set of all values will also determine classical logic. For example, in $Ł_6$, the set {1, $^4/_5$, $^3/_5$, $^1/_5$, 0} determine classical logic.

2. *Fragments containing the Conditional*

In contrast to the tables for negation, conjunction, and disjunction, the Łukasiewiczian table for the conditional seems distinctly unnatural. For example, it has proved to be very difficult to find any plausible account of the choice of values for the conditional in $Ł_3$. The unusual character of the Łukasiewiczian conditional also manifests itself in the matter of designated values, as the following theorem shows:

THEOREM 2. If Ł' is a fragment of $Ł_n$ containing the conditional, then there is no set D of values in $Ł_n$ such that Ł' is classical relative to D.

If D is the set of all values, then everything is valid. If D is not universal, then there is a value u such that u is not in D. If u is 1, nothing is valid. To complete the proof, it suffices to show that for every u in $Ł_n$ such that $u \neq 1$, there is a classical tautology A and a valuation v on $Ł_n$, for which v(A) = u.

To construct the required formulas, let M be the sentence $((p \supset q) \supset p) \supset p$. Define J_i, for $i < n$, inductively, as follows:

$$J_1 = (M \supset q),$$
$$J_{k+1} = (M \supset J_k).$$

Now, let N_i be $J_i \supset q$.

Observe first that for every i, N_i is a classical tautology. Suppose that v is a valuation on $Ł_n$ such that v(q) = F. It must be shown that $v(J_i) = F$. But M is a classical tautology, and hence is uniformly T. So, clearly, $v(J_1) = F$. But if, by induction, $v(J_k) = F$, then, by the same argument, $v(J_{k+1}) = F$, as required.

I complete the proof by showing that if v(p) = n—2/n—1 and

$v(q) = O$, then $v(N_k) = n—1—k/n—1$. To establish this, first prove that for every $i < n$, $v(J_i) = i/n—1$. Now, computation shows that $v(M)$ is $n—2/n—1$. So, $v(M \supset q)$ is $1 — n—2/n—1$, which is $1/n—1$, as required. Suppose then that $v(J_k)$ is $k/n—1$. If $k + 1$ is defined, $k \leqq n—2$. If $k = n—2$, then $v(M \supset J_k)$ is 1, as required; if $k < n—2$, then $v(M \supset J_k) = 1 — n—2/n—1 + k/n—1$. By computation, this equals $k+1/n—1$, as required. Now, observe that $v(N_k) = 1 — k/n—1 = n—1—k/n—1$.

3. *Fragments containing the Biconditional*

As mentioned at the outset, the Łukasiewiczian biconditional seems to occupy a position intermediate between the «classical» connectives — $\neg$, &, and $\vee$ — and the strongly «non-classical» conditional connective. This intermediate position is evidenced by the fact that, whereas the pure $\equiv$ -fragment of $Ł_n$ can be rendered classical by a suitable choice of designated values, there is no choice of designated values which will render the $\vee \equiv$ -fragment classical. The $\neg \equiv$ -fragment is a mixed case, being like the pure $\equiv$ -case if n is even and like the $\vee \equiv$ -case if n is odd.

I begin my discussion of biconditional fragments by considering the pure $\equiv$ -fragment. Before the main theorem can be stated, a bit of terminology must be introduced. Let us say that a value u in $Ł_n$ is *positive* iff u has the form:

$$n—1—2s/n—1, \text{ for some s such that } 0 \leqq s \leqq q,$$

where q is $n—1/2$ or $n—2/2$ depending on whether n is odd or even. A value u in $Ł_n$ is *negative* iff u is not positive. So, for example, in $Ł_3$, 1 and 0 are positive values and $^1/_2$ is negative. In $Ł_6$, all of 1, $^3/_5$, and $^1/_5$ are positive, whereas $^4/_5$, $^2/_5$, and 0 are negative. With this distinction in mind, the main theorem about the $\equiv$ -fragment can now be stated.

THEOREM 3. The $\equiv$ -fragment of $Ł_n$ is classical relative to D iff (1) there is a value u such that u is not in D and (2) D contains all positive values.

($\Rightarrow$) If D is the set of all values, then everything is valid. Suppose then that there is a positive value u such that u is not in D. If u is 1, then nothing is valid. To complete the proof, it will suffice to show that for every positive value u ($\neq 1$), there is a classical tautology C and a valuation v on $Ł_n$ such that $v(C) = u$.

Consider then the sentence $(p \equiv (q \equiv r)) \equiv ((p \equiv q) \equiv r)$. This is a classical tautology, of course. I show that this sentence can receive any positive value. Suppose u is a positive value of the form n—1—2s/n—1, where $1 \leqq s \leqq q$, with q = n—1/2 or q = n—2/2 depending on whether n is odd or even. Let v be the valuation on $Ł_n$ such that v(p) = n—1—s/n—1, v(q) = s/n—1, and v(r) = 0. Then, by computation, $v(q) \equiv r)$ = n—1—s/n—1, and so, $v(p \equiv (q \equiv r)) \equiv 1$. On the other hand, $v(p \equiv q)$ = 1 —((n—1—s/n—1)—(s/n—1)). So, $v(((p \equiv q) \equiv r) = 1 - v(p \equiv q)$, which is n—1—2s/n—1. Consequently, v assigns the sentence $(p \equiv (q \equiv r)) \equiv ((p \equiv q) \equiv r)$ the value n—1—2s/n—1.

($\Leftarrow$) Suppose that conditions (1) and (2) are satisfied. Then two claims must be established:

a. If A is not a classical tautology, then A isn't valid in $Ł_n$ relative to D.
b. If A is a classical tautology, then A is valid in $Ł_n$ relative to D.

I begin with (a). Let u be a value which is not designed. Observe that the part of the $Ł_n$ biconditional table involving only 1 and u has the form:

	1	u
1	1	u
u	u	1

This is isomorphic to the classical table with 1 as T and u as F.

Let A be a wff which isn't a classical tautology, and let v be a classical valuation such that v(A) = F. Suppose that $p_1, \ldots, p_e$ are the propositional variables in A. Now, let v* be a valuation in L_n such that if $v(p_i)=T$, then $v^*(p_i)=1$ and if $v(p_i)=F$, then $v^*(p_i)=u$. By the isomorphism just mentioned, $v^*(A)=u$.

Claim (b) remains to be established. To prove this, I introduce the notion of equivalent normal form (ENF). Let A be a pure ≡—wff whose propositional variables are $p_{k_1}, \ldots, p_{k_m}$. Then A is said to be in ENF iff it has the form:

$$A_1 \equiv (A_2 \equiv \ldots \equiv (A_{n-1} \equiv A_m) \ldots),$$

where A_i is a biconditional containing all the occurrences of p_{k_i} in A. The key fact connecting the logic $Ł_n$ with the notion of ENF is this:

LEMMA 2. For every pure ≡ -wff A, there is a pure ≡ -wff B such that B is in ENF and the $Ł_n$ truth table for B has positive values at exactly the same places as the $Ł_n$ truth table for A.

Observe first that a pure ≡ -wff A can be transformed into a wff in ENF using just (i) commutativity of the biconditional and (ii) associativity of the biconditional. So, to prove the lemma, it will be sufficient to show that the operations described in (i) and (ii) have the property of transforming a wff C into a wff D that has positive values in its $Ł_n$ truth-table at exactly the same places as C.

case (i). This is clear, since v(A ≡ B) = v(B ≡ A), on every $Ł_n$ valuation.

case (ii). To prove this, note the following facts:

1. If v(p) and v(q) are both positive, so is v(p ≡ q).
2. If v(p) and v(q) are both negative, then v(p ≡ q) is positive.
3. If v(p) is positive and v(q) is negative, or vice versa, then v(p) ≡ q) is negative.

Proof of (1)-(3) is straightforward. Take (2), for example. Let v(p) be n—1—(2t+1)/n—1 and let v(q) be n—1—(2u+1)/n—1. Suppose, by symmetry, that $t \leqq u$. Then, $v(p \equiv q) = 1$ —((n—1—(2t+1)/n—1) — (n—1—(2u+1)/n—1). But this equals n—1—2(u—t)/n—1, which is a positive value.

Given Facts (1)-(3), the following table shows that associativity preserves positive and negative values.

p	q	r	$p \equiv q$	$q \equiv r$	$(p \equiv q) \equiv r$	$p \equiv (q \equiv r)$
P	P	P	P	P	P	P
P	P	N	P	N	N	N
P	N	P	N	N	N	N
P	N	N	N	P	P	P
N	P	P	N	P	N	N
N	P	N	N	N	P	P
N	N	P	P	N	P	P
N	N	N	P	P	N	N

The equivalence of the last two columns proves the desired claim and completes the proof of Lemma 2.

Given Lemma 2, I return to claim (b). Suppose then that A is a classical tautology. Then, since the operations used in constructing the ENF in Lemma 2 are classically valid, it follows that the ENF B given for A by Lemma 2 is also a classical tautology. Consequently, every variable in B occurs an even number of times. So, B has the form:

$$A_1 \equiv (A_2 \equiv \ldots \equiv (A_{m-1} \equiv A_m)\ldots),$$

where p_{k_i} occurs an even number of times A_i. Note now that for every valuation v on L_n, $v(A_i)$ is positive. This is obvious if $v(p_{k_i})$ is positive. If $v(p_{k_i})$ is negative, then A_i contains an odd number of applications of the biconditional operation to a negative value, and this will always yield a positive value.

Since each A_i is positive on every valuation, it follows that B has a positive value on every valuation. By Lemma 2, A has positive values exactly where B does. Therefore A is valid relative to D in $Ł_n$. This completes the proof of Theorem 3.

I now turn to the consideration of the $\neg \equiv$ -fragment of $Ł_n$. I begin by showing that the $\neg \equiv$ -fragment of odd-valued Łukasiewiczian systems is strongly non-classical.

THEOREM 4. There is no set D of values in L_{2n+1} such that the $\neg \equiv$ -fragment of $Ł_{2n+1}$ is classical relative to D (²).

If a set D is to determine classical logic, then the value 0 must be in D. For $\neg(p \equiv \neg p)$ is a classical tautology, and $v(\neg(p \equiv \neg p)) = 0$ when $v(p) = 1/2$. ($1/2$ is a value in any odd-valued system.) But if 0 is in D, then $\neg(p \equiv p)$ is valid, since it always takes the value 0.

Theorem 3 establishes that the pure biconditional fragment of $Ł_n$ can be rendered classical by suitable choices of designated values. I now show that, for even-valued systems, this result can be extended to the $\neg \equiv$ -fragment.

THEOREM 5. The $\neg \in$ -fragment of $Ł_{2n}$ is classical relative to D iff (1) the value 0 is not in D (2) D contains all positive values.

($\Rightarrow$) Proof is parallel to the ($\Rightarrow$) part of Theorem 3.
($\Leftarrow$) Assume conditions (1) and (2) hold. Two claims must be established.

a. If A isn't a classical tautology, then A isn't valid in $Ł_{2n}$ relative to D.
b. If A is a classical tautology, then A is valid in $Ł_{2n}$ relative to D.

(²) I mention in passing the following odd fact about the L_{2n+1} $\neg \equiv$-fragment. Let D be the set of all positive values. Then the set $\neg \equiv$-wffs valid relative to D is the union of the set of classical tautologies and the set of classical contradictions.

In view of the fact that 0 is not designated, claim (a) is obvious. To prove claim (b), I introduce the notion of negation-equivalential normal form (NENF). Let A be a $\neg\equiv$ -wff whose variables are $p_{k_1},\ldots, p_{k_m}$. Then A is said to be in NENF iff (1) all negation signs in A are confined to propositional variables and (2) A has the form:

$$A_1 \equiv (A_2 \equiv \ldots \equiv (A_{m-1} \equiv A_m)\ldots),$$

where A_i has the form $(A_i^a \equiv A_i^n)$. Here A_i^a is a biconditional containing all the occurrences in A of p_k that are not preceded by a negation sign, and A_i^n is a biconditional containing all the occurrences of $\neg p_{k_i}$ in A. If either A_i^a or A_i^n is null, then A_i is to be identified with the other constituent. Unless specified by the preceding definition, assume that parentheses are grouped to the left.

The key connection between the logic $Ł_{2n}$ and the notion of an NENF is given by the following lemma:

LEMMA 3. For every $\neg\equiv$ -wff A, there is a $\neg\equiv$ -wff B such that B is in NENF and the $Ł_{2n}$ truth-table for B has positive values at exactly the same places as the $Ł_{2n}$ truth-tables for A.

Observe first that a $\neg\equiv$ -wff can be transformed into a wff in NENF using just these four operations: (i) double negation, (ii) the biconditional confinement principle permitting the replacement of $\neg(F \equiv G)$ by $F \equiv \neg G$, (iii) commutativity of the biconditional, and (iv) associativity of the biconditional. So, to prove the lemma, it will be sufficient to show that operations (i)-(iv) preserve positive and negative values. Cases (iii) and (iv) were discussed above.

case i. $v(p) = v(\neg\neg p)$, for all valuations v.

case ii. In an even-valued Łukasiewiczian logic, the negation of a positive value is negative value, and the negation of a negative value is a positive value. Using this information and Facts (1)-(3) established in connection with Lemma 2, we construct the following table to show that the confinement principle has the desired property:

p	q	$\neg q$	$p \equiv q$	$\neg(p \equiv q)$	$p \equiv \neg q$
P	P	N	P	N	N
P	N	P	N	P	P
N	P	N	N	P	P
N	N	P	P	N	N

This completes the proof of Lemma 3.

Before attempting to prove claim (b), it is necessary to bring out one further fact about wffs in NENF. Let B be a wff in NENF whose variables are $p_{k_1}, \ldots, p_{k_m}$. Let $p_{k_i}*$ be the smallest subformula of B that contains all the occurrences of p_{k_i} in B. By the definition NENF, $p_{k_i}*$ has the form $(A_i^a \equiv A_i^n)$, where A_i^a contains all the unnegated occurrences of p_{k_i} in B and A_i contains all the negated ones. Let us say that a subformula $p_{k_i}*$ is *tautologous* iff p_{k_i} occurs an even number of times in both A_i and A_i. A subformula $p_{k_i}*$ is said to be *contradictory* iff p_{k_i} occurs an odd number of times in both A_i^a and A_i^n. The remaining about NENF's needed to complete the proof of (b) is this:

LEMMA 4. Suppose B is a wff in NENF. Then B is a classical tautology iff (1) all subformulas $p_{k_i}*$ in B are tautologous or contradictory and (2) there are an even number of contradictory subformulas.

This claim is proved easily.

Recall that our aim is to show that if A is a classical tautology, then A is valid in $Ł_{2n}$ relative to D. Suppose then that A is a classical tautology. Since the operations used in obtaining the NENF in Lemma 3 are classically valid, the NENF B given for A by Lemma 3 is also a classical tautology. By Lemma 4, all subformulas $p_{k_i}*$ of B are tautologous or contradictory and there are an even number of contradictory subformula. Now, rewrite B in the form B′:

$$(p_{t_1}* \equiv (p_{t_2}* \equiv \dots \equiv (p_{t_{j-1}}* \equiv p_{t_j}*)\dots) \equiv (p_{c_1}* \equiv (p_{c_2}* \equiv \dots \equiv (p_{c_{r-1}}* \equiv p_{c_r})\dots),$$

where the $p_{t_i}*$ are all the tautologous subformulas of B and the $p_{c_i}*$ are the contradictory subformulas of B. Since B′ is obtained from B using commutativity and associativity of the biconditional, the $Ł_{2n}$ truth-table for B′ has positive values at exactly the same places as the $Ł_{2n}$ truth-table for B.

I now show that for every valuation v on $Ł_{2n}$, v(B′) is positive. First, consider the left-hand constituent of B′. On every valuation v, $v(p_{t_i}*)$ is positive. For suppose that $v(p_{t_i})$ is positive. Then the left-hand constituent A_{t_i} is clearly positive. The right-hand constituent is also positive, since (1) p_{t_i} occurs an even number of times in $A^n_{t_i}$, (2) the negation of a positive value is a negative value, and (3) an odd number of applications of the biconditional operation to a negative value is a positive value. The argument is parallel if $v(p_{t_i})$ is negative. Given that each $p_{t_i}*$ is positive in every case, it follows that the left-hand constituent of B′ is positive in every case.

Now, consider the right-hand constituent of B′. On every valuation v, $v(p_{c_i}*)$ is negative. For suppose $v(p_{c_i})$ is positive. Then the left-hand constituent $A^a_{c_i}$ of $p_{c_i}*$ is positive. The right-hand constituent $A^n_{c_i}$ is negative, since (1) p_{c_i} occurs an odd number of times in $A^n_{c_i}$, (2) the negation of a positive value is negative, and (3) an even number of applications of the biconditional operation to a negative value is negative. The argument is parallel if $v(p_{c_i})$ is negative. Given that each $p_{c_i}*$ is negative on every valuation and that there are, by Lemma 4, an even number of contradictory subformulas in B', it follows that the right-hand constituent of B' is positive on every valuation.

Since both the left and right constituents of B' are positive on every $Ł_{2n}$ valuation, it follows that B' itself is always positive. But now A has positive values at exactly the same places as B'; so A always receives a positive value. By the condition of the theorem, all positive values are designated. So, A is valid in $Ł_{2n}$ relative to D, thus concluding the proof of Theorem 6.

To complete the investigation of biconditional fragments, I shall show that the $\vee\equiv$ -fragment of any Lukasiewiczian logic is strongly non-classical.

THEOREM 7. There is no set D of values of L_n such that the $\vee\equiv$ -fragment of L_n is classical relative to D.

Clearly, the value 1 must be designated and there must be some undesignated value. It will therefore be sufficient to show that for every s such that $1 \leqq s \leqq n—1$, there is a classical tautology C and a valuation v on $Ł_n$ for which $v(C) = n—1—s/n—1$.

Let W be the sentence $((p \equiv q) \vee (q \equiv r)) \vee (p \equiv r)$. For $i \leqq n—1$, define Y_i inductively by:

$$Y_1 = (W \equiv p')$$
$$Y_{k+1} = (W \equiv Y_k)$$

Now, let Z_i be $Y_i \equiv p'$.

First note that Z_i is a classical tautology for every i. Since W is a classical tautology, Y_i will always receive the same value as p' on a classical valuation.

Let v be a valuation on $Ł_n$ that makes the assignment: $v(p) = 1$, $v(q) = n—2/n—1$, $v(r) = n—3/n—1$, $v(p') = 0$. Such values exist, because $n \geqq 3$. I show that for any s such that $1 \leqq s \leqq n—1$, $v(Z_s) = n—1—s/n—1$. To prove this, first establish that

for any s such that $1 \leqq s \leqq n—1$, $v(Y_s) = s/n—1$. By computation, $v(W) = n—2/n—1$. So, $v(Y_1) = 1/n—1$. For the inductive case, suppose that $v(Y_k) = k/n—1$. If $k + 1$ is defined, $k \leqq n—2$. If $k = n—2$, then $v(Y_k)$ is 1, as required. If $k < n—2$, then $v(Y_{k+1}) = 1 — ((n—2/n—1)—(k/n—1)) = k+1/n—1$.
Since $v(Y_s) = s/n—1$, $v(Z_s) = n—1—s/n—1$, as promised.

In conclusion, let me indicate briefly how the results obtained system $L_{\aleph}$ whose values are all the rationals in the interval [0,1]. As might be expected, Theorems 1 and 2 carry over to $L_{\aleph}$. Interestingly, however, the biconditional fragment of $L_{\aleph}$ is strongly non-classical.

THEOREM 8. There is no set of values D in $Ł_{\aleph}$ such that the pure $\equiv$ -fragment is classical relative to D.

The first point to note is that the biconditional table for $Ł_n$ occurs as a sub-table of the biconditional table for $Ł_{\aleph}$. From the proof of Theorem 3, we know that there is a classical tautology which can assume every value that is positive in $Ł_n$. To prove the theorem, it will suffice to show that every rational in [0,1] is positive in some $Ł_n$. But this is evident. For if u is negative in $Ł_n$, then u has the form $n—1—(2s+1)/n—1$. In that case, u is also equal to $2(n—1)—2(2s+1)/2(n—1)$, which is a positive value in $Ł_{2n-1}$. Thus, the biconditional emerges as strongly non-classical when the limitation to finitely many values is removed.

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