An S5 DIODOREAN MODAL SYSTEM

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As is now well known, the alethic modalities were normally conceived in temporal terms by the ancients (1). In particular the Megarian logician Diodorus Cronos defined a possible proposition as one that either is now or will be true, an impossible proposition as one that is now false and will always be false, a necessary proposition as one that is now true and will always be true, and a nonnecessary proposition as one that either is now false or will be false (2). The research—both historical and logical—of Arthur Prior has proven especially fruitful in the contemporary analytical study of «Diodorean modalities».

In the John Locke Lectures for 1956 (later incorporated into his book *Time and Modality* [12]) Prior presents an analysis of the Diodorean concepts of possibility and necessity in terms of infinite matrices or rows of truth values. As Hughes and Cresswell aptly remark. «Prior was thinking of propositions as things which could change their truth-values (could *become* true or *become* false) with the passage of time». ([6], p. 262.) Propositions, in other words, are conceived as *temporally indeterminate*. Prior's infinite matrices (rows) of 1's and 0's represent the truth and falsity, respectively, of a proposition at successive times (moving from left to right). Consequently, a given valuation is an assignment of such an infinite matrix (rather than a single 1 or O) to each propositional variable. Each valuation then proceeds to assign matrices to complex propositional wff's in the expected recursive way. For example, a valuation V_i will assign to a wff of the form '$(\varphi \vee \psi)$' a matrix that has 1 at all points where the matrix assigned by V_i to φ

(1) See the discussion in Hintikka [5], especially Chs. VIII and IX.

(2) As reported in Boethius, *in Librium Aristotelis De Interpretatione, Editio secunda* III, in *Patrologiae Cursus Completus*, Vol. 64, ed. J. P. Migne (Paris, 1847), p. 511.

has a 1 or where the matrix assigned by V_i to ψ has a 1. Most importantly, V_i will assign to a wff of the form '$M\varphi$' a matrix that has a 1 at just those points where the matrix assigned by V_i to φ has a 1 *either* at the corresponding point *or* at any point to the right of the corresponding point. V_i will assign to a wff of the form '$L\varphi$' a matrix that has a 1 at just those points where the matrix assigned by V_i to φ has a 1 *both* at the corresponding point *and* at all points to the right of the corresponding point. A wff is then held to be Diodorean thesis or theorem iff all valuations (all possibles assignments of infinite sequences to its propositional variables) result in the wff's being assigned the sequence with 1's throughout.

The modal PC whose theorems are just those that can be shown to be Diodorean theses by means of the matrix method has been designated D. In *Time and Modality* Prior speculated that D is equivalent to the Lewis system S4. However, it was soon found that a system stronger than S4 is needed to axiomatize the Diodorean system D. In 1957 Prior noted that a formula investigated by P. T. Geach (3),

1. $MLp \supset LMp$,

is not an S4 theorem but is a Diodorean thesis. Dummett and Lemmon [4] named the system formed by adding 1 to the axioms of S4 S4.2. Also in 1957, Lemmon discovered another proof that D is stronger than S4: the wff

2. $L(Lp \supset Lq) \vee L(Lq \supset Lp)$.

is a D theorem but not a theorem of S4 or S4.2 (4). The system formed by adding 2 to the axioms of S4 Dummett and Lemmon [4] named S4.3. This system they showed to contain S4.2. At roughly the same time Hintikka discovered another Diodorean thesis that is not derivable in S4:

3. $(Mp \,.\, Mq) \supset (M(p \,.\, Mq) \vee M(q \,.\, Mp))$.

(3) In [12], pp. 25ff. There Prior uses 3 in place of 0.
(4) The discovery is reported by Prior in [13], p. 24.

This wff was later proved equivalent to Lemmon's 2 (5).

Finally, Dummett discovered another wff that is a Diodorean theorem (i.e., is verified by Prior's matrix method) but can be proved not to be derivable in S4.3:

4. $L(L(p \supset Lp) \supset Lp) \supset (MLp \supset Lp)$.

Prior reports in *Past, Present and Future* ([13], p. 29) that this formula was shortened by Geach to

5. $L(L(p \supset Lp) \supset p) \supset (MLp \supset p)$.

In turn, Prior ([13], p. 29) has shown 5 to be equivalent to the following wff:

6. $(MLp \,.\, L({\sim}p \supset M(p \,.\, M{\sim}p))) \supset p$.

Prior then proceeded to show that the reason 6 is verified by his matrix method but is not an S4.3 theorem is that the matrix method does not allow for the possibility that time might be dense but the axioms of S4.3, in effect, leave open the question of the density or discreteness of time. Kripke and Bull [1] (independently) settled the issue of the axiomatization of D by showing that S4.3 axiomatizes the Diodorean modalities (in the sense *merely* of possibility as presentness-or-futurity) if the assumption of discrete time is not made, while S4.3 plus 6 axiomatize the Diodorean modalities if such an assumption is made—i.e., axiomatize D or just those theses verifiable by Prior's matrix method (6).

(5) Prior reports in [13], p. 25, that a variant of 3 was given by Hintikka in the latter's review of *Time and Modality* in the *Philosophical Review*, 67 (1958), pp. 401-404. The equivalence of 2 and 3 was proved by Prior in «K1, K2 and Related Modal Systems», *Notre Dame Journal of Formal Logic*, 5:4 (October, 1964), pp. 299-304.

(6) More complete accounts of the various systems between S4 and S5, together with further bibliographical information, can be found in Prior [13], pp. 23-31, and Hughes and Cresswell [6], pp. 260-267. The relations among the various modal systems between S4 and S5 are spelled out in Sobocinski [15].

Although either the system D or the weaker S4.3 captures the fundamental Diodorean concepts of possibility as presentness-or-futurity and necessity as presentness-and-permanent-futurity, neither system really expresses the fatalism or logical determinism that ancient sources ascribe to Diodorus. This may seem a peculiar comment, since most contemporary analyses of Diodorus' system have regarded it as fatalistic (7). However, the principal reason why most contemporary students (including myself) (8) have so regarded the Diodorean systems stems from the consideration of *temporally determinate* propositions, i.e., propositions conceived as being forever tied to a given time. Diodorus would seem to have no reason not to regard such propositions (e.g., «A sea battle occurs on May 22, 1805») as being eternally true if true and eternally false if false. But, according to Diodorus' temporal account of the modalities, such a temporally determinate proposition would then be necessary if true, impossible if false.

To ascribe some form of logical determinism to Diodorus on this basis, however, is really to refuse to take seriously the concept of a proposition as a temporally indeterminate entity. It is analogous to regarding a contemporary modal system with a possible-worlds semantic interpretation as entailing fatalism because such a system entails that '*world-determinate*' or 'world-indexed' (9) propositions (propositions 'bound' to a particular possible world ,e.g., «Carter is elected President in

(7) See, for example, the discussion in J. Hintikka, «Aristotle and the 'Master Argument' of Diodorus», *American Philosophical Quarterly,* 1:2 (April, 1964), especially p. 110. This article, in a modified form, appears as Ch. IX in Hintikka [5]. See also P. M. Schuhl, *Le Dominateur et les Possibles* (Paris, 1960).

(8) In an article «Diodorus' 'Master' Argument: A Semantic Interpretation» (forthcoming in *Erkenntnis*). Also in a paper «Aristotle and Temporally Relative Modalities,» (forthcoming in *Analysis*).

(9) Alvin Plantinga makes elaborate use of «world-indexed properties» in *The Nature of Necessity* (Oxford, 1974). I believe that implicit in Hintikka's discussion of Aristotle's famous sea-battle problem ([5], Ch. VIII) is a point concerning the relation between temporally determinate propositions and a temporal account of the alethic modalities similar to the point I make in this paragraph.

1976 in world α») are necessary (true at all worlds) if true and impossible (true in no world) if false.

If the ascription of fatalism to a Diodorean modal system such as D involves no more that the tacit importation of the semantical entities (times, in the case of D) employed in the evaluation of wffs of the system *into those very wffs,* analogous arguments can be constructed demonstrating the 'fatalistic consequences' of other modal systems that are not usually regarded as fatalistic. How, then, might the determinism espoused by Diodorus be captured *within* a modal system without 'collapsing' that system, i.e., without destroying the distinction between φ and $L\varphi$ and between φ and $M\varphi$ for all propositions φ ?

Cicero reports that Diodorus held that «whatever will happen necessarily will happen.» [10] Represented as a potential Diodorem thesis, this claim becomes

7. $Fp \supset LFp$.

Analyzing 'L' in the Diodorean manner, we find that the sense of 7 is the following: if it (now) *will* be the case that p, then it is (now) and always will be true that it *will* be the case that p. The claim has little *prima facie* plausibility and clearly is not verified by Prior's matrix method, i.e., is not a theorem of D. However, it is fairly obvious that the assumption of «eternal recurrence» or cyclical time would verify the formula. Given this assumption, if it is true that a proposition *will become* true, it is now and forever hereafter true that it *will become* true. Although it is not known whether Diodorus himself subscribed to the concept of cyclical time, there is abundant evidence that the notion of eternal recurrence was common in antiquity and was tied, from Aristotle's *De Generatione et Corruptione* on, to «necessary coming-to-be» or, in other words, to the necessary occurrence of events [11]. In particular, the doctrine of cyclical time was ascribed to the determinist Stoic

[10] *De Fato* 7. 13.

[11] *De Gen. et Corr.* 2. 11. See also the Peripatetic *Problemata* 17. 3.

inheritors of the Megarian logical tradition ([12]). Thus, I propose a deterministic Diodorean system D' that will yield 7 as a theorem.

Prior's infinite matrix method can be adapted for the purpose of representing theoremhood in this strengthened Diodorean system. Consider the subset M' of the set M of infinite matrices such that each $m \in M'$ is constituted of eternally recurrent finite subsections or 'runs'. It can be seen that each $m \in M'$ verifies 7 and that, in general, such matrices capture the idea of eternal recurrence. Thus I propose that a wff is a theorem of D' if and only if it is verified (assigned a matrix with 1's throughout) by each $m \in M'$. Since M' is a subset of M, each theorem of D (wff verified by each $m \in M$) will be a theorem of D'.

Fortunately, the axiomatization of D' seems considerably simpler than the axiomatization of D proved to be. The most intuitive way is to adjoin to the classical PC axioms for a tense logic for circular time (with operators 'P' [it will, at least once, be the case that] and 'G' [it will always be the case that]) and to define 'L' and 'M' in terms of these operators. Thus the following seems a workable axiomatization of D':

Rules:

R1. From $\vdash p$, infer $\vdash Gp$

Definitions:

D1. $Fp \equiv \sim G \sim p$
D2. $Mp \equiv p \vee Fp$
D3. $Lp \equiv p \, . \, Gp$

Axioms:

A1. $G(p \supset q) \supset (Gp \supset Gq)$
A2. $Gp \supset p$ [or $p \supset Fp$]

([12]) ORIGEN, *Contra Celsum* 4. 12, 4. 68, and 5. 20. Also Lactantius, *Divinae Institutiones* 7. 23.

A3. $Gp \supset GGp$ [or $FFp \supset Fp$]
A4. $p \supset G{\sim}G{\sim}p$ [or $p \supset GFp$] (13)

Adapting a standard natural deduction proof technique to our new system (any uniform substitution instance of an axiom may occur as a line in a proof; if a wff φ in a proof depends on no assumptions, $G\varphi$ may occur as a line in the proof), it can be shown that 7 is provable. First, 7 is by definition equivalent to

7'. $Fp \supset Fp \,.\, GFp$.

then,

1.	Fp	Ass.
2.	$Fp \supset GFFp$	A4(Fp/p), AxI
3.	$GFFp$	1, 2, $\supset$E
4.	$FFp \supset Fp$	A3, AxI
5.	$G(FFp \supset Fp)$	4, GI
6.	$G(FFp \supset Fp) \supset (GFFp \supset GFp)$	A1(FFp/p, Fp/q), AxI
7.	$GFFp \supset GFp$	5, 6, $\supset$E
8.	GFp	3, 7, $\supset$E
9.	$Fp \,.\, GFp$	1, 8, ·I
10.	$Fp \supset Fp \,.\, GFp$	1-9, $\supset$I

The question arises as to the relation of the 'pure modal' fragment of D' to traditional modal systems. Since our postulates for circular time insure that the anteriority-posteriority relation among times ($t < t'$) is reflexive, transitive, and symmetrical, the relation $\leqq$ ($t \leqq t'$ iff $t < t' \vee t = t'$) also possesses these characteristics. It is the latter relation, of course, that serves as the 'accessibility relation' in semantically interpreting 'L' and 'M' in a Diodorean modal logic. Since the accessibility relation among possible worlds in the possible-world interpretation of the Lewis S5 system is also reflexive, tran-

(13) See Prior [13], pp. 176-178, for alternative axiomatizations of circular time.

sitive, and symmetrical, one would expect the 'pure modal' fragment to D' to coincide with S5 ([14]) In fact, a characteristic axiom for S5,

8. $Mp \supset LMp$,

can be proved in D'. 8 is definitionally equivalent to

8'. $p \vee Fp \supset (p \vee Fp) \cdot G(p \vee Fp)$.

The proof of the latter thesis in D' is rather lengthy but simple:

1.	$p \vee Fp$	**Ass.**
12.	p	Ass.
3.	$p \supset Fp$	A2, AxI
4.	Fp	2, 3, $\supset$E
5.	$Fp \supset GFFp$	A4(Fp/p), AxI
6.	$GFFp$	4, 5, $\supset$E
7-10.	GFp	[as in steps 5-8 of previous proof]
11.	Fp	Ass.
12.	$p \vee Fp$	11, $\vee$I
13.	$Fp \supset p \vee Fp$	11-12, $\supset$I
14.	$G(Fp \supset p \vee Fp)$	13, GI
15.	$G(Fp \supset p \vee Fp) \supset (GFp \supset G(p \vee Fp))$	A1(Fp/p, p $\vee$ Fp/q), AxI
16.	$GFp \supset G(p \vee Fp)$	14, 15, $\supset$E
17.	$GFp \supset G(p \vee Fp)$	16, Reit
18.	$G(p \vee Fp)$	10, 17, $\supset$E
19.	$p \vee Fp$	2, $\vee$I
20.	$(p \vee Fp) \cdot G(p \vee Fp)$	18, 19, $\cdot$I
21.	$p \supset (p \vee Fp) \cdot G(p \vee Fp)$	2-20, $\supset$I
22-39.	$Fp \supset (p \vee Fp) \cdot G(p \vee Fp)$	[as in steps 4-21]
40.	$(p \vee Fp) \cdot G(p \vee Fp)$	1, 21, 39, $\vee$E
41.	$p \vee Fp \supset (p \vee Fp) \cdot G(p \vee Fp)$	1-40, $\supset$I

([14]) Prior notes that the «simplest way to axiomatize circular time is to define G as H, or both as L, and use known postulates for S5». ([13], p. 64.)

The system D' thus represents a deterministic but non-trivial Diodorean modal system; it, in other words, supplies a Diodorean account of 'necessary becoming', the necessary occurrence of events. While 7 and

9. $\sim Fp \supset \sim MFp$

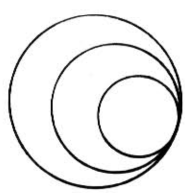
are theorems of the system, the following formulae are not:

*10. $p \supset Lp$
*11. $\sim p \supset \sim Mp$

The truth of 10 or 11 would indeed result in the collapse of the Diodorean system and, in effect, yield a 'static universe'. It is interesting to note that Aristotle, in *Metaphysics* Θ, 3, claims that the earlier Megarians' refusal to distinguish the modalities of necessity and possibility from the modality of actuality has precisely this effect. The Diodorean system D' may thus be interpreted as an answer to Aristotle, i.e., a way to preserve a form of fatalism without 'destroying becoming' within the context of a temporal account of the alethic modalities.

POSTSCRIPT

A question of mild interest is whether there exists an S5 Diodorean modal system that does not entail the fatalistic consequences of D'. The answer seems affirmative. Consider a 'multi-looped' system of time in which all the loops intersect in at least one point, e.g., Figure 1 below.



Rescher and Urquhart ([14], p. 133) also note the relation between circular time and S5 when the alethic modalities are defined temporally.

Here, too, the $\leqq$ relation will be reflexive, transitive, and symmetrical. Hence 'M' and 'L', semantically interpreted in terms of this relation, will behave according to the S5 modal postulates. However, the set T of times standing in this relation can be thought of as the set of *possible times*: $M\varphi$ is true now iff φ is true now or at some possible future time (in some loop); $L\varphi$ is true now iff φ is true now and at all possible future times (throughout all loops). *Actual* time might be specified in terms of the loop one is 'now traversing' (minus the point joining its 'beginning' and 'end'). That is, actual time might be conceived as non-cyclical and linear. Consequently, the truth of Fp (with axioms for the tense operator 'F' given for *actual* time) need not entail the truth of LFp, and the truth of $\sim$Fp need not entail the truth of $\sim$MFp (15).

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(15) Cf. the discussion of the Kripke S4 model for necessity as presentness-and-permanent-futurity, possibility as presentness-or-futurity and of Lemmon's S4.2 modification of this model in Prior [13], pp. 27-28.

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