FREE TOPOLOGICAL LOGIC

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0. *Introduction*

Topological logics are systems which contain an indexed operator T, with the reading 'A is the case at (or 'as of') x' for 'TxA'. So far, such systems have been propositional, in the sense that while quantification is introduced over the indices, they have lacked predicate letters, and individual terms. It turns out that full-fledged systems of topological predicate calculus with identity and descriptions may be constructed, and shown sound and complete. The semantics for these systems which is most faithful to ordinary usage, is axiomitized by extending free description theory [10].

1. *Syntax*

The syntax for such systems is easily defined. We simply add 'T' to the usual notation for predicate logic with identity and descriptions, and define the set *Wff* of formulas to be the smallest set which has the formulas of the predicate calculus with identity and descriptions as a subset, and satisfies the condition that if $A \in Wff$ and n is a term, then $TnA \in Wff$.

We might consider developping a 2-sorted system, one sort for individuals, and the other for indices. But this is not essential for expressive adequacy, and so we will begin by discussing single sorted systems, leaving a few remarks about 2-sorted systems to the final section.

We also want to add the symbol '$\cong$' for quasi-(or intensional) identity, as we reserve '$=$' for strict identity; so we will count formulas having the form $n \cong n'$ as members of *Wff* as well.

2. *Classical Semantics: TQ≅—Satisfiability*

We will now define a simple semantics for such a system. We let a TQ≅—model U be a pair <D, u>, where D is a non-empty set, which we may imagine to be the union of the set of individuals with the set of contexts (times, places, possible worlds or the like or sequences of these), and where u is an *interpretation function,* which assigns to each term n, a function u(n) from D into D, assigns to each j-ary predicate letter P^j a function $u(P^j)$ from D into the power set of D_j, and assigns to each formula of *Wff* a function u(A) from D into the set {T, ⊥ } of truth values, and satisfies the following conditions for all d∈D:

=. u(n=n')(d) is T iff u(n) is u(n')
≅. u(n≅n')(d) is T iff u(n)(d) is u(n')(d)
P_j. $u(P^j n_1 \ldots n_j)(d)$ is T iff $\langle u(n_1)(d), \ldots, u(n_j)(d) \rangle \in u(P^j)(d)$
~. u(~A)(d) is T iff u(A)(d) is not T
⊃. u(A⊃B))(d) is T iff u(A)(d) is not T or u(B)(d) is **T**
T. u(TnA)(d) is T iff u(A)(u(n)(d)) is T
∃. u(∃xØx)(d) is T iff there is a variable y such that u(Øy)(d) is T

In ∃., Øx is any formula, and Øy is the result of replacing all bound occurrences of x in Øx by the first variable not in Øx, and replacing every free occurrence of x in the result with y. In the above conditions we have assumed a system with the symbols =, ≅, ~, ⊃, T and ∃ as primitives, and take the other logical symbols to be defined. A set of formulas Γ is is TQ≅—satisfiable just in case there is a syntax with Γ⊆Wff and a TQ≅-model U=<D, u> such that u(A)(d) for some d ∈ D, for all A ∈ Γ ([11]).

We assume that our language contains both definite terms («rigid designators», such as '1969' and 'R. M. Nixon'), and indefinite terms («non-rigid designators», such as 'four years ago' and 'the president of the U.S.A.') ([1]). The latter are taken to refer

([1]) An indefinite term is one whose denotation is a function of the context of its use. A definite term is a term which is not indefinite.

to functions from D into D, so that 'four years ago' may refer to the function λxx-4, on times, and 'the president of the U.S.A.' to the function f such that f(d) is the individual who is president of the U.S.A. at time d [2]. The definite terms may be taken to refer to constant functions so that '1969' refers to the constant function with 1969 as value, and 'R. M. Nixon' to the function which has Richard Nixon as value. This is the standard strategy for handling non-rigid designators in modal logics, but it is particularly appropriate in topological logics. The treatment of formulas and predicate letters continues this basic strategy [3].

The condition ≅. reflects an intensional treatment of identity. For the truth of n≅n' at d we do not require that u(n) be u(n'), but only that they agree at the argument d. This corresponds to the intuition that a sentence such as 'R. M. Nixon is the president of the U.S.A.' is true in spite of the fact that 'R. M. Nixon' and 'the president of the U.S.A.' do not refer to the same functions. In clause T., we give the truth conditions for a formula of the form TnA, saying that 'TnA' is true in context d just in case 'A' is true in the context which results from applying the function referred to by 'n' to the context d. So we say 'A is the case (as of) four years ago' is true in 1973 just in case 'A' is true in the year which reasults from applying the function λxx-4 to 1973 (i.e. in 1969). In ∃., we adopt the substitution interpretation of the quantifiers. This is mostly for simplicity. A standard interpretation of the quantifiers may be given which is equivalent to this [4].

(2) We have chosen examples where our contexts are times, but of course our systems have much wider interpretations.

(3) In [1] and [4] we defined u as a binary function which assigns a subset of Dj to each pair <d, Pj>, and similarly for formulas. The present definition is a trivial reformulation of that one.

(4) When we adopt the standard interpretation of the quantifiers, then a TQ-model is a triple <D, u Θ>, where D is not empty, Θ is a subset of the set of functions from D into D, and u(n), u(Pj) and u(p), are as before, for atomic formulas p. The truth value of a formula A at d (u(A)(d)) is the defined recursively by clauses =., ≅., Pj., ~., ⊃., T., and f∃.: u(∃xØx)(d) is ⊤ iff u^f/x(Øx)(d) is ⊤ for some f∈Θ, where u^f/x is the function which agrees with u save that u(x)=f. For a free semantics with the standard

Our semantics has one ackward consequence. We do not distinguish between contexts and individuals at the start, as D contains both. So it might turn out that a term 'n' appearing in a formula of the form 'TnA' refers to a function which ranges over a set of individuals. This would force us to assign a truth value to sentence A at the value u(n)(d), which requires that u(A) be defined for arguments which turn out to be individuals. This is a notion which has promise in other uses (for instance when A is an open sentence, and TnA, the result of replacing some variable in A with n), but it is merely puzzling here. Further difficulties arise when there are more than one type of context. Of course one way out of this is to introduce sorts in our syntax, but I think it is important to the flexibility of these systems that syntactic complications be kept to a minimum. If that is our aim we may point to one of several good reasons for using free logic as a foundation for topological logic. For in semantics for free logic, the extension of a term is a *partial* function, and we may extend this idea to the extension of a predicate letter and sentence. So we may count the value of u(four years ago) at Richard Nixon as undefined, and we may do the same for a formula.

But there is another way out, if we are willing to change the definition of a model a bit. We define a C-model to be a pair $<D, u>$, where D is not empty and where u assigns to terms, predicate letters, and formulas, functions of the appropriate kind which are defined on a non-empty subset C of D. We then require that u satisfy conditions for an interpretation function for members $d \in C$, and require that whenever a term n appears in the context TnA, then $u(n)(d) \in C$ for all $d \in C$.

interpretation, we need not introduce Θ, since we may take the definition of a TF-model, let u be defined for terms, predicate letters, and atomic formulas (including formulas of the form $Pjn_1 \ldots n_j$, but not including those of the forms n=n' and n≅n'), and define the truth value of a formula A

at d on u by clauses =., F≅., F~., F⊃. FT., F$\dot{\exists}$.: u(∃xA)(d) is T iff $u^{u(y)}/x(A)(d)$ is T for some y such that u(y) is defined at d., and FI. If e is the only member of D such that for some y, u(y)(d) is e and $u^{u(y)}/x(A)$ is T, then u(IxA)(d) is e. Otherwise u(IxA) is not defined at d. It is easy to see that the semantics just given is a simple reformulation of TF-sematics.

Actually, this repair is not necessary in a formal sense, for it is not difficult to show that notions of TQ=satisfiability and C-satisfiability are equivalent (5).

3. *The system TQ≅.*

The system TQ≅, which captures the concept of TQ≅satisfiability is a simple extension of the system TQ [1] consisting of the principles of quantificational logic with identity plus the following rule and axiom schemata:

(R) If $\vdash$ A, then $\vdash$ TnA (6)
(A~) $(\sim TnA \supset Tn \sim A)$
(~A) $(Tn \sim A \supset \sim TnA)$
(A⊃) $(Tn(A \supset B) \supset (TnA \supset TnB))$
(AQ) $(Tn \exists yA \supset \exists yTnA)$, for $y \neq n$.

To finish our description of TQ≅, we provide axioms for intensional identity. We want

(A≅) $n \cong n'$,

but we do not want an axiom of unrestricted substitution for ≅, for substitution behind a T-operator is not licit generally. So we add the axiom schemata

(AS) $(n \cong n' \supset (P(n) \supset P(n')))$, where P(n) has one of the forms $P^j n_1 \ldots n_j$, $n_1 \cong n_2$, and P(n') results from replacing n' for one or more occurrences of n in P(n) (7).

(5) We prove that Γ is TQ≅-satisfiable iff Γ is C-satisfiable. Suppose Γ is TQ≅-satisfiable. It follows trivially that Γ is C-satisfiable, since a TQ≅model is a species of C-model with C=D. Now suppose Γ is C-satisfiable. It is a simple matter to show that if Γ is C-satisfiable then Γ is TQ≅-consistent, by the usual methods for soundness proofs. By the completeness proof of section 8, Γ is TQ≅-consistent entails that Γ is TQ≅-satisfiable, so Γ is TQ≅-satisfiable.

(6) '$\vdash$ A' means there is a proof of A in the system under discussion.

(7) Notice that (AS) does not have as an instance $n \cong n' \supset (n=n \supset n=n')$; nor should it, since this formula is not valid.

and

(AT) $(n \cong n' \supset (TnA \supset Tn'A))$.

We also need a rule to govern the relationship between '$=$' and '$\cong$'

($R\cong$) If $\vdash (A \supset Tyn \cong n')$, and y is foreign to A, n, and n', then $\vdash (A \supset n = n')$.

$TQ\cong$ is TQ plus ($A\cong$), (AS), (AT), and ($R\cong$). We prove it complete in section 8. The soundness is a simple exercise left to the reader.

4. *Contextual Quantifiers*

In the semantic given so far, we have not tried to capture an important idea of quantified modal logic: that the domain of quantification might shift from context to context (possible world to possible world). This idea is easily motivated when we take our contexts to be times, and wish to give the semantics for a present tense quantifier $\dot{\exists}$, read 'there is *now* an x such that'. For example, we might count the sentence '$\dot{\exists}$ x taught Aristotle.' false, on the grounds that while Plato taught Aristotle, he does not exist now, nor does anyone else who taught Aristotle. On this interpretation 'Plato taught Aristotle' is counted true, and so it is clear that $\dot{\exists}$ does not obey the usual rule of existential generalization. In fact, as we will see, the logical structure of this quantifier may be given by strenghtening a system of free logic [10].

For the moment, however, we will use more conventional methods to define the present tense (or more generally, contextual) quantifier. We simply introduce a one-place predicate constant 'E', which is read '——exists'. We needn't consider that the introduction of this letter commits us to the position that 'existence is a predicate', for it is not existence *per se* (whatever that might be), but existence-in-a-context which we mean to capture. When our contexts are times, 'E' is read in the present tense, when they are positions in space, it takes on the reading '——exists here' or '——is present'. Now it is quite standard to count an objects position in space, as representable by a predicate, so there shouldn't be any difficulty when our contexts are spatial, or spatio-temporal points. If they are times, we can support our use of 'E' by pointing out that for persons, and artifacts, at least, the notion captured by 'E' may be defined in terms of perfectly standard predicates such as '——is born', '——is dead', '——is manufactured', '——destroyed' and the like.

With 'E' safely in hand, we may provide the obvious definition of the contextual quantifier:

$$\text{Def}_{\dot{\exists}} \qquad \dot{\exists}xA =_{df} \exists x(Ex\&A).$$

Since '$\dot{\exists}$' is defined, we need only provide semantics for 'E' to give a semantics for this system. To do so we expand the definition of a TQ≅-model. An ETQ=-model U is a triple <D, u, Θ>, where D is as before, and where Θ (since it is to be the extension of a unary predicate letter 'E') is a function from D into the power set of D. The definition of u is the same as before, save that we require that u(E) is Θ.

It is a simple matter to show that no new axioms are necessary to completely axiomitize this semantics. The completeness proof is given in section 8.

5. *Free Topological Logic*

In this section, we formulate semantics and axiomatics for a system which takes the contextual quantifier as primitive. We assume that we have '—', '⊃', '$\dot{\exists}$', '=', '≅', and 'T' in our morphology but not '∃' and not 'E'. Then we define a semantics for the system by letting a $\dot{\exists}$T-model U be a triple <D, u, Θ> where D, u, and Θ are as described before, save that u is defined so that

$\Theta\dot{\exists}$. u ($\dot{\exists}$x∅x)(d) = T iff there is a variable y such that u(y)(d) ∈ Θ(d) and u(∅y)(d) = T.

This condition is simply the result of copying the truth condition for '∃x(Ex&∅y)', on the semantics of the previous section.

It turns out that we may capture this semantics with a system of free topological logic (TF≅) which is described below. The completeness and soundness of TF≅, which we prove in section 8, allows us to characterize the theorems of TF≅ in a new way. Let us assume that TQ≅ is built on a morphology with the symbols '—', '⊃', 'T', '=', '≅' '⊃', 'E', '∃' and '$\dot{\exists}$', and and that '$\dot{\exists}$' is defined by $Def_{\dot{\exists}}$. Then the set of theorems of TF≅ is the set of theorems of TQ≅ which are written in the morphology of TF≅ [8].

We turn now to an alternative semantics for the contextual quantifier that retains more of the spirit of free logic, in that it offers a Strawsonian account of the denotation of a term. Up until now, we have assumed that the function to which a term refers is defined in every context. We have already seen a reason to think that this should not be so. Consider 'the president of the U.S.A.'. Now there are times in the past and (presumably) times in the future when this expression fails to denote. So we will want to be able to leave 'the president of the U.S.A.' undefined except for those times when the president of the U.S.A. exists.

So far we have treated the description operator as defined in the usual way. Of course this has the drawback that descriptions are not completely «termlike», in that universal instantiation to a description is not licit unless uniqueness conditions are met. This problem is simply dealt with in mathematics by requiring of the syntax that no description be introduced unless the uniqueness condition has been proven. But that makes the

(8) Proof. Suppose $\vdash_{TF}$A. Then clearly $\vdash_{TQ\cong}$ A. Suppose $\vdash_{TQ\cong}$ A, and A fails to contain '∃' or 'E'. Then by ETQ-soundness of TQ≅, A is ETQ-valid. But then A must be ∃T-valid, and by the completeness of TF, $\vdash_{TF}$ A.

syntax depend on the proof theory of a system, and it is a remedy which doesn't apply to ordinary language where the underlying theory is not specified.

But now that we have adopted a free semantics of the partial function sort, a natural treatment of descriptions may be developed without fiddling with the syntax. Descriptions are just terms, an so refer to partial functions from D into D, and they are defined at context d just in case their uniqueness condition is satisfied at d. But what uniqueness condition ought to be satisfied ? We have at least two choices. The first is that IxØx denotes at d just in case there is exactly one member in the extension of Ø at d, and the other that it is defined just in case there is exactly one member in the domain of quantification Θ(d) for d in the extension of Ø at d. But it is not clear to me either of these choices is completely satisfactory.

It is often pointed out that 'The cat is on the mat.' is resistant to the usual treatment of descriptions because it does not imply that there is exactly one cat, any more than that there is exactly one mat. In fact the vast majority of sentences of ordinary language which contain descriptive phrazes fail to entail their corresponding uniqueness conditions. We might find a way to categorize descriptive phrazes into syntactic categories depending on what «uniqueness» condition is relevent to each. But apart from the complication of the syntax involved, it is doubtful that such a classification could be carried out with any degree of accuracy. The reason is, I believe, that the kind of condition which is relevent to fixing the reference of a descriptive phraze depends on the context of the use of that expression. For instance in one use of 'the cat', the speaker understands what I am referring to because I *own* exactly one cat, and in another, because we are both in the presence of a cat, and in another because we were just talking about *his* cat etc... In none of these cases do we require that there be only one cat in the context of our utterence (at least when spatio-temporal contexts are at issue). Perhaps the second sort of uniqueness condition then is more relevent here, for we might maintain that in all these examples that the domain of quantification, or domain of our discussion for the context of our utterence included exactly one cat. If we must make a choice between the two treatments of descriptions, then, the second is more faithful to ordinary language. Description theory of both kinds are discussed in [7], and the second is developed in [9] for modal languages. We will develop a theory of the second kind for topological languages below, but hesitantly.

Consider the sentence

(1) In 1942, the cat wasn't even born yet.

This we count true just in case

(2) 'The cat wasn't even born yet.' is true in 1942;

but we do not fix the reference of 'the cat' in (1) by the uniqueness condition

(3) There is exactly one cat in the domain of quantification of 1942,

for when (3) is satisfied, the unique cat in the domain of quantification of 1942 is exactly not the cat we meant to refer to since the one we wanted to refer to wasn't even born yet.

So if we want to do justice to the way ordinary language works, we need to allow the possibility that some descriptions behave even more like terms than we thought, in that their reference is not fixed by the satisfaction of a uniqueness condition. Of course we will still expect the referent of 'the cat' to be a cat, and so we will want $\exists x x = Ix\emptyset x \supset \emptyset(Ix\emptyset x)$; it is just that we do not want $\exists x x = Ix\emptyset x \supset \exists!'\emptyset'$, or even $\exists x x = Ix\emptyset x \supset \dot{\exists}!'\emptyset x$.

Actually there is another way to solve the problem just presented by (1). If we extended our formation rules so that the T-operator formed a new predicate Tn[P] from a predicate P, then we might represent (1) in the form: The cat T1942 [wasn't even born yet]. Then we can no longer claim that (1) iff (2) but rather that (1) iff the referent of 'the cat' falls into the extension of the predicate T1942[wasn't even born yet]. Then *our* domain of quantification, not the one for 1942, could help fix the reference of 'the cat'. This modification has considerable promise, and we develop it further in [3], but given the present more usual symbolic resources, a weak description theory seems indicated (9).

6. *Free Semantics: TF-Satisfiability*

We present a free semantics where descriptions have their reference fixed by a context dependent domain of quantification. A TF-model is a pair $\langle D, u \rangle$, where D is a non-empty set, and u is a *partial interpretation,* which assigns to each term, a partial function from D into D, to each j-ary predicate letter, a partial function from D into the power set of D^j, and to each formula, a partial function from D into $\{T, \bot\}$, and satisfies the following conditions:

I. If e is the only member of D such that for some y, u(y)(d) is e and u(Øy)(d) is T, then u(IxØx)(d) is e. Otherwise u(IxØx) is not defined at d.

≅. u(n=n')(d) is T, if u(n) is u(n'). Otherwise u(n=n') (d) is ⊥.

F≅. u(n≅n') is defined at d iff u(n) or u(n') is defined at d. When defined, u(n≅n')(d) is T iff u(n)(d) is u(n') (d).

FP^j. If $u(n_1)$, ..., $u(n_j)$, $u(P^j)$, are all defined at d, then $u(P^j n_1 \ldots n_j)$ is defined at d, and $u(P^j n_1 \ldots n_j)(d)$ is T iff $\langle u(n_1)(d), \ldots, u(n_j)(d) \rangle \in u(P^j)(d)$.

F∼. u(∼A) is defined at d iff u(A) is, and when u(A) is defined u(∼A)(d) is T iff u(A)(d) is ⊥.

(9) A system for this weakest sort of description theory is given by deleting (I!) and (!I) from TF (See section 6.) and adding $\exists xx = Iy\text{Ø}y \supset \text{Ø}(Iy\text{Ø}y)$. The semantics for this system results from replacing clause I. in the definition of a partial interpretation by:If u(IxØx) is defined at d, then u(Ø(IxØx))(d) is T, and replacing wI. in the definition of a f.i.u. by u(IxØx)(d) is T for all d∈D. The completeness of this system is easily proven using the strategy of section 8 and simple revisions for clauses involving descriptions.

F$\supset$. u((A$\supset$B)) is defined at d iff u(A) and u(B) are defined at d, and when u((A$\supset$B)) is defined u((A$\supset$B))(d) is T iff u(A)(d) is $\perp$ or u(B)(d) is T.

FT. u(TnA) is defined at d iff u(n) is defined at d and u(A) is defined at u(n)(d), and when u(TnA) is defined, u(TnA)(d) is u(A)(u(n(d)).

F$\dot{\exists}$. u($\dot{\exists}$x∅x) is defined at d iff u(∅y) is defined at d for some variable y, and when u($\dot{\exists}$x∅x) is defined, u($\exists$x∅x)(d) is T iff there is a variable y such that u(y) is defined at d, and u(∅y) is T [10]

In FP., we do not require that u($P^j n_1 \ldots n_j$) be undefined if one of u(n_1), ..., u(n_j) is undefined, because we make true and false statements about entities which do not exist. (Example: Pegasus is mythical.) But in F$\cong$., we have required that u(n$\cong$n') be undefined if one of u(n), u(n') is undefined, for otherwise n=n ceases to be valid. This definition has the advantage of causing n=n to be valid without resorting to the *ad hoc* conditions placed on identity used in [10].

To complete the definition of satisfability, we define a *full interpretation for u* (f.i.u.). A function w is a f.i.u. iff w assigns to each term n a function w(n) from D into D, which agrees with u(n), whenever u(n) is defined, to each j-ary predicate letter P^j, a function w(P^j) from D into the power set of D, which agrees with u(P^j) wherever u(P^j) is defined, and to each formula A a function w(A) from D into {T}, which agrees with u(A) whenever u(A) is defined, and satisfies the conditions =., $\cong$., P^j., ~., $\supset$., T·, plus

wI. If e is the only member of D such that for some y, u(y)(d) is e and w(∅y)(d) is T, then u(Ix∅x)(d) is e.

[10] Thomason [8] p. 135 complains of a vicious circularity in the definition of satisfaction when the substitution interpretation is used wiih descriptions. He is right if one interprets the quantifiers so that their truth value of $\exists$x∅x depends on the truth values of the formulas of the shape ∅n, where n is any *term*. But by restricting the interpretation of the quantifiers to the *variables*, we avoid the difficulty he mentions.

and

> w$\dot{\exists}$. w($\dot{\exists}$xØx)(d) is **T** iff there is a variable y such that u(y) is defined at d and w(Øy)(d) is **T**.

A set of formulas Γ is TF-satisfiable just in case there is a syntax which includes the members of Γ, and a TF-model U=<D, u> and a f.i.u. w such that for some d$\in$D and all A$\in\Gamma$, w(A)(d) is **T** (11).

7. *The System TF.*

TF is the system of topological logic which axiomitizes the semantics just given. The soundness of the following principles is easily verified. TF consists of the principles of propositional logic, axioms (A$\sim$), ($\sim$A), (A$\supset$), axioms for identity, (A$\cong$), (AS), (AT), the rules (R), (R$\cong$), and the following principles governing quantifiers and descriptions:

($\dot{\exists}$G) (($\dot{\exists}$xx=n & Øn) $\supset$ $\dot{\exists}$xØx)

($\dot{\exists}$=) $\dot{\forall}$y$\dot{\exists}$xx=y

(RQ) (A$\supset$tØy) is a theorem of TF, then so is (A$\supset$t$\dot{\forall}$xØx).

(I!) ($\dot{\exists}$xx$\cong$IyØy $\supset$ $\dot{\exists}$xØ!x)

(!I) $\dot{\forall}$x(Ø!x$\supset$x$\cong$IyØy)

In these principles '$\dot{\forall}$x' abbreviates '$\sim\dot{\exists}$x$\sim$', and 'Ø!x' abbreviates 'Øx&$\dot{\forall}$y(Øy$\supset$y$\cong$x)', and 't' is any sequence (possibly null) $Tn_1 \ldots Tn_j$ of T-operators, and y is a variable which does not appear in A or t.

The axioms ($\dot{\exists}$G), and ($\dot{\exists}$=) are familiar from free logic, [10],

(11) We mention the syntax in this definition to insure that the resulting notion of satisfiability is compact.

[11]. In [10] the axiom $\forall x(\emptyset!x \equiv x \cong Iy\emptyset y)$ is given, but this is not sound on our semantics. Thomason [9] gives (I!) and (!I) where '$\emptyset!x$' abbreviates '$\forall y(\emptyset y \cong y \cong x)$'; but this formulation is also too strong, and only works with a semantics where the variables are rigid designators. The rule (RQ) is a strengthening of the usual rule of universal generalization, and was already developped for what I called t-formulations of topological logic ([12]). It has some similarities with rules R4 and R5 of [9]. Notice that the axiom (AQ) (see section 3) is missing from TF. (AQ) is the topological analogue of the Barcan formula and is not valid on our semantics because the domain of quantification shifts from context to context ([13]).

8. *Completeness Proofs*

We prove the completeness of TF, and derive completeness results for the other systems as corrolaries. We begin by proving the Lindenbaum Lemma for our system, but for a somewhat modified notion of a saturated set. A set of formulas m is a TF-model set just in case m is maximally consistent ([14]) and $t\dot{\exists}x\emptyset x \in m$ if there is a variable y not in t such that $t(\dot{\exists}xx = y \& \emptyset y) \in m$, and if $\sim n = n' \in m$, then $Ty \sim n \cong n' \in m$ for some variable y.

Theorem 1. If Γ is a consistent set of formulas and there are infinitely many variables foreign to Γ, then there is a TF-model set Δ, such that $\Gamma \subseteq \Delta$.

Proof. We construct Δ from Γ as follows. We order the set of formulas, and then produce a sequence $A_1, \ldots, A_i, \ldots$ such that

([12]) So called t-formulations for topological logic appear in sections 2.3 and 3.3 of [2].

([13]) See [6].

([14]) We say $\Gamma \vdash A$ iff there is a conjunction G of members of Γ such that $\vdash G \supset A$. Γ is consistent iff not $\Gamma \vdash \bot$, where $\bot$ is an arbitrarily selected contradiction of propositional logic. Γ is maximally consistent iff Γ is consistent and if $A \notin \Gamma$, then $\Gamma \cup \{A\}$ is not consistent.

if A has the form $t\dot{\exists}x\emptyset x$, then A_{j+1} is $t(\dot{\exists}zz=y\&\emptyset y)$, and if A_j has the form $\sim n=n'$, then A_{j+1} is $Ty\sim n=n'$, where y is a variable which fails to appear in t or any of $A_1, \ldots, A_j$. We then define Δ_j recursively by $\Delta_0=\Gamma$, and $\Delta_j=\Delta_{j-1} \cup \{A_j\}$ if $\Delta_{j-1} \cup \{A_j\}$ is consistent, and $\Delta_j=\Delta_{j-1}$ otherwise. Then $\Delta= \bigcup_{j=0}\Delta_j$. It is a simple matter to show that Δ so constructed is a maximally consistent set. To finish the demonstration that Δ is a TF-model set, we prove the following two lemmas.

Lemma 1. $t\dot{\exists}x\emptyset x\in\Delta$ iff there is a variable y not in t such that $t(\dot{\exists}xx=y\&\emptyset y)\in\Delta$.

Proof. Suppose that $\dot{\exists}x\emptyset x\in\Delta$. Then for some value of j, A_j is $t\exists x\emptyset x$ and $\Delta_{j-1} \cup \{A_j\}$ is consistent. So it follows that $\Delta_j= \Delta_{j-1} \cup \{A_j\}$. Now consider $\Delta_j \cup \{A_{j+1}\}$. By the construction of the sequence this is $\Delta_j \cup \{t(\dot{\exists}zz=y\&\emptyset y)\}$, where y is a variable chosen so that it fails to appear in t or free in any of $A_1, \ldots, A_j$. Now we prove that this last mentioned set is consistent by *reductio*. Suppose $\Delta_j \cup \{t(\dot{\exists}zz=y\&\emptyset y)\}$ is not consistent. Then $\Delta_j \vdash \sim t(\dot{\exists}zz=y\&\emptyset y)$. By many applications of axiom $(A\sim)$, we have $\Delta_j \vdash t\sim(\dot{\exists}zz=y\&\emptyset y)$. Since y does not appear in t nor free in any formula in Δ_j, we may apply (RQ) to show that $\Delta_j \vdash t\dot{\forall}y\sim(\dot{\exists}zz=y\&\emptyset y)$. It follows then that $\Delta_j \vdash t\dot{\forall}y(\dot{\exists}zz=y\supset\sim\emptyset y)$, and so $\Delta_j \vdash (t\dot{\forall}y\dot{\exists}zz=y \supset t\dot{\forall}y\sim\emptyset y)$. But $\dot{\forall}y\exists zz=y$ is provable by $(F=)$, and so, by many applications of (R), $t\dot{\forall}\dot{\exists}zz=y$ is provable. So $\Delta_j \vdash t\dot{\forall}y\sim\emptyset y$, and so $\Delta_j \vdash \sim t\dot{\exists}\emptyset x$ which contradicts the consistency of Δ_j. We conclude from the *reductio* that $\Delta_j \cup \{t(\dot{\exists}zz= y\&\emptyset y)\}$ is consistent, and so $t(\dot{\exists}zz=y\&\emptyset y) \in \Delta$.

Now suppose that $t(\exists zz=y\&\emptyset y) \in \Delta$ for some variable y not in t. Then by $(\dot{\exists}G)$, (R), and $(A\supset)$, $t(\dot{\exists}zz=y\&\emptyset y) \supset t\dot{\exists}x\emptyset x$ is provable and so it follows that $t\dot{\exists}x\emptyset x \in \Delta$.

Lemma 2. If $\sim n=n' \in \Delta$, then $Ty\sim n\cong n \in \Delta$, for some variable y.

Proof. Suppose $\sim n=n' \in \Delta$. $\sim n=n'$ is A_j for some j, and $\Delta_{j-1} \cup \{A_j\}$ is consistent. A_{j+1} is $Ty\sim n=n'$ for y foreign to $A_1, \ldots, A_j$. Consider $\Delta_j \cup \{A_{j+1}\}$. Suppose for *reductio* that this set is not consistent. Then $\Delta_j \vdash Tyn\cong n'$ by (A$\sim$) and (R). By (R$\cong$) it follows that $\Delta_j \vdash n=n'$, which contradicts the consistency of Δ. We conclude that $\Delta_j \cup \{A_{j+1}\}$ is consistent, hence Δ_{j+1} is $\Delta_j \cup \{A_{j+1}\}$, and $Ty\sim n\cong n' \in \Delta$.

We turn now to the second stage of the completeness proof.

Theorem 2. If Δ is a TF-model, there is a TF-model $U_\Delta=\langle D, u\rangle$, such that $\Delta \in D$, and there is a f.i.u. w such that $w(A)(\Delta)$ is T for all $A \in \Delta$.

Proof. Let us construct $U_\Delta=\langle D, u\rangle$ and w from Δ as follows. Let M be defined so that $m\in M$ iff for some string t of T-operators m is $\{A: tA\in\Delta\}$. For each term n, let s_n be the function defined on M such that $s_n(m)$ is $\{n' : n'$ is a term and $n'\cong n\in m\}$ for $m\in M$. Let S be $\{s_n(m)$: n is a term and $m\in M\}$. Let D be MxS ([15]). We define u and w simultaneously. Let w be the function that assigns to each term n the function w(n) defined so that $w(n)(\langle m, s\rangle)$ is $\langle\{A: TnA\in m\}, s_n(m)\rangle$, for $m\in M$ lnd $s\in S$, and let u assign to each term the function u(n) which is defined at $\langle m, s\rangle$ iff $\exists xx=n\in m$, and is w(n) when defined. Let w assign to each j-ary predicate letter P^j the function $w(P^j)$ defined so that $w(P^j)(\langle m, s\rangle)$ is $\{\langle d_1, \ldots, d_j\rangle$: there are terms $n_1, \ldots, n_j$ such that $u(n_i)(\langle m, s\rangle)$ is d_i for $1\leqslant i\leqslant j$, and $P^jn_1\ldots n_j\in m\}$. Let u assign $w(P^j)$ to P^j. Finally w assigns to each formula A the function w(A) defined so that $w(A)(\langle m, s\rangle)$ is T when $A\in m$, and $w(A)(\langle m, s\rangle)$ is $\perp$

([15]) This choice of D is a bit cumbersome, but the use of ordered pairs is necessary. If we define D to be the set of all model sets, and let u(n)(d) be $\{A: TnA\in d\}$, as we do in [1] and [4], the resulting model fails to satisfy condition $\cong$.

when $A \notin m$. We let u assign A the function u(A) which agrees with w(A) whenever u(A) is defined, and we specify where u(A) is defined, recursively as follows. If A has the form $P^j n_1 \ldots n_j$, then u(A) is defined everywhere on D. If A has the form $n = n'$, then u(A) is defined everywhere on D. If A has the form $n \cong n'$, then u(A) is defined at $\langle m, s \rangle$ just in case both u(n) and u(n') are defined at $\langle m, s \rangle$. If A has the form $\sim B$, then $u(\sim B)$ is defined at d iff u(B) is defined at d. If A has the form $(B \supset C)$, then u(A) is definded at d just in case both u(A) and u(B) are defined at d. If A has the form TnB, then u(TnA) is defined at d just in case both u(n) is defined at d and u(B) is defined at u(n)(d). If A has the form $\dot{\exists} x \emptyset x$, then u(A) is defined at d iff $u(\emptyset y)$ is defined at d, for some variable y.

Now we must show that u and w are a partial interpretation and a f.i.u. respectively. First, we show w is a f.i.u. Clearly w(n) is a function from D into D (for $\{A\colon tA \in m\} \in M$, and $s_n(m) \in S$, for all $m \in M$), and it agrees with u(n) when defined. Also $w(P^j)$ is a function from D into the power set of D^j, which agrees with $u(P^j)$ when defined; and u(A) is a function from D into the set $\{\mathsf{T}, \perp\}$, which agrees with u(A) when the latter is defined. So we need only show that w(A) satisfies the conditions $=$., $\cong$., P^j., $\sim$., $\supset$., T., wI., and w$\dot{\exists}$.. To do that we need a few lemmas.

Lemma 3. If $m \in M$, then m is a TF-model set.

Proof. Suppose $m \in M$; then m is $\{A\colon tA \in \Delta\}$ for some string t. In [4], p. 113, we showed that $\{A\colon tA \in \Delta\}$ is maximally consistent when Δ is maximally consistent, so we need only show that $t'\dot{\exists} x \emptyset x \in m$ iff there is a variable y not in t' such that $t'(\dot{\exists} z z = y \ \& \ \emptyset y) \in m$, and if $\sim n = n' \in m$, then $\mathsf{T} y \sim n \cong n' \in m$ for some variable y. Suppose $t'\dot{\exists} x \emptyset x \in m$; then $t'\dot{\exists} x \emptyset x \in \{A\colon tA \in \Delta\}$, and $tt'\dot{\exists} x \emptyset x \in \Delta$. Since Δ is a TF-model, there is a variable y not in tt' such that $tt'(\dot{\exists} z z = y \ \& \ \emptyset y)\ \Delta$, and so there is a variable y not in t' such that $t'(\exists zz{=}y \,\&\, \emptyset y) \in \{A\colon tA \in \Delta\}$. Now suppose there is a variable y not in t' such that $t'(\exists zz{=}y \,\&\, \emptyset y) \in \{A\colon tA \in \Delta\}$. So $tt'(\exists zz{=}y \,\&\, \emptyset y) \in \Delta$. But $tt'(\exists zz{=}y \,\&\, \emptyset y) \supset tt'\exists x \emptyset x$ is provable in TF using $(\exists G)$, $(A\supset)$, and (R). So by standard properties of a maximally consistent set $tt'\exists x \emptyset x \in \Delta$, hence $t'\exists x \emptyset x \in m$.

Suppose $\sim n{=}n' \in m$. Then for some conjunction D of members of m, $\vdash D \supset \sim n{=}n'$. By $(R\cong)$, $\vdash D \supset Tyn \cong n'$ for any variable y foreign to D, n, and n'. By (R) and $(A\supset)$, $\vdash tD \supset tTy \sim n \cong n'$. Since m is maximal $D \in m$; hence $tD \in \Delta$. But then $\Delta \vdash tTy \sim n \cong n'$, so $tTy \cong n' \in \Delta$, and $Ty \sim n{=}n' \in m$.

Lemma 4. $n{=}n' \in \Delta$ if $n{=}n' \in m$, for all terms n, n' and any $m \in M$

Proof. By (AS), $\vdash n{=}n' \supset n \cong n'$. By (R) and $(A\supset)$ $\vdash Tn''n{=}n' \supset Tn''n \cong n'$ for any term n''. Using $(R\cong)$, we conclude that $Tn''n{=}n' \supset n{=}n'$. Given (R) and $(A\supset)$, it is a simple matter to show by induction that for $\vdash tn{=}n' \supset n{=}n'$ for any string t. As a result, if $n{=}n' \in \{A{:}tA \in \Delta\}$, then $n{=}n' \in \Delta$ for any string t, hence if $n{=}n' \in m$, then $n{=}n' \in \Delta$ for $m \in M$.

Lemma 5. $w(n)(\langle m, s\rangle)$ is $w(n')(\langle m, s\rangle)$ iff $n \cong n' \in m$, for all $s \in S$.

Proof. We know that $w(n)(\langle m, s\rangle)$ is $\langle\{A\colon TnA \in m\}, \{n''\colon n'' \cong n \in m\}\rangle$, and similarly for n', so $w(n)(\langle m, s\rangle)$ is $w(n')(\langle m, s\rangle)$ iff $\{A{:}TnA \in m\}$ is $\{A\colon Tn'A \in m\}$ and $\{n''{:}n'' \cong n \in m\}$ is $\{n''{:}n'' \cong n' \in m\}$ iff

(4) $TnA \in m$ iff $Tn'A \in m$ and $n'' \cong n \in m$ iff $n'' \cong n' \in m$ for all formulas A and terms n''.

Now suppose $n \cong n' \in m$. Then by (AS) and (AT), (4). Now suppose that (4). We have as a special case that $n \cong n \in m$ iff $n \cong n' \in m$, so by $(A\cong)$, $n{=}n' \in m$.

Now we will show that w(A) satisfies the conditions mentioned.

$=$. Suppose $w(n=n')(<m, s>)$ is T. Then $n=n' \in m$ and $n=n' \in \Delta$, by Lemma 4. But $\vdash n=n' \supset tn \cong n'$ for any string t, by axioms for identity, so $tn \cong n' \in \Delta$, for any t. It follows that $n \cong n' \in m$ for all $m \in M$. By Lemma 5, w(n) is w(n'). Suppose $w(n=n')(<m, s>)$ is $\perp$. Then $n=n' \notin m$. So $\sim n=n' \in m$ by Lemma 3. But then $Ty \sim n \cong n' \in m$ for some variable y, and $\sim n \cong n' \in \{A:TyA \in m\}$. But $m' = \{A:TyA \in m\}$ is a member of M. By Lemma 5, $w(n)\ (<m', s>)$ is not $w(n')(<m', s>)$ for any s, and so u(n) is not u(n').

P^j. $<w(n_1)(<m, s>), \ldots, w(n_j)(<m, s>)> \in w(P^j)(<m, s>)$ iff

(5) There are terms $n'_i, \ldots, n'_j$ such that $w(n_i)(<m, s>)$ is $w(n'_i)(<m, s>)$ for $1<i<j$, and $P^j n'_1 \ldots n'_j \in m$.

To complete this cause, we show that (5) iff $P^j n_1 \ldots n_j \in m$, for $w(P^j n_1 \ldots n_j)(<m, s>)$ is T iff $P^j n_1 \ldots n_j \in m$. Suppose $P^j n_1 \ldots n_j \in m$. Then trivially, (5). Now suppose (5). By many applications of (AS) $\vdash n_1 \cong n'_1 \& \ldots \& n_j \cong n'_j \supset (P^j n'_1 \ldots n'_j \supset P^j n_1 \ldots n_j)$. By Lemma 5, $n_i \cong n'_i \in m$ for $1<n<j$; so $P^j n_1 \ldots n_j \in m$.

$\cong$. Trivial, by Lemma 5 and $(n \cong n')(<m, s>)$ is T iff $n \cong n' \in m$.

$\sim$. Trivial by Lemma 3 and properties of a maximally consistent set.

$\supset$. Trivial by Lemma 3 and properties of a maximally consistent set.

T. $w(TnB)(<m, s>)$ is T iff $TnB \in m$ iff $B \in \{A:TnB \in m\}$ iff $w(B)(<\{A:TnB \in m\}, s_n(m)>)$ is T iff $w(B)(w(n)\ (<m, s>))$ is T.

wI. Suppose that e is the only member of D such that for some variable y, $u(y)(<m, s>)$ is e, and $w(\emptyset y)\ (<m, s>)$ is T. Let z be any variable and suppose that $\dot{\exists} x x=z \in m$. It follows that u(z) is defined at $<m, s>$, so let c be $u(z)(<m,$

s>). Now if $Øz \in m$, it follows that there is a member of c of D such that u(z)(<m, s>) is c and w(Øz)(<m, s>) is T. Since d is the only member of D that satisfies that condition, d is c. So u(z)(<m, s>) is u(y)(<m, s>) and by Lemma 5, $z \cong y \in m$. We have just shown that for any variable z, if $\dot{\exists}xx = z \in m$ and $Øz \in m$, then $z \cong y \in m$. By standard properties of a maximally consistent set $\dot{\exists}xx = z \supset (Øz \supset z \cong y) \in m$, for any z. Since m is a TF-model set $\dot{\forall}x(Øx \supset x \cong y) \in m$. We remember that $Øy \in m$, so $Ø!y \in m$. Since u(y)(<m, s>) is e, u(y) is defined at <m, s>, and $\dot{\exists}xx = y \in m$. By ($\dot{\exists}$G) and (!I), we may prove $\dot{\exists}xx = y \supset (Ø!y \supset y \cong IxØx)$, so $y \cong IxØx \in m$. By Lemma 5, u(y)(<m, s>) is u(IxØx)(<m, s>), and so the latter is e.

w$\dot{\exists}$. w($\dot{\exists}$xØx)(<m, s>) is T iff $\dot{\exists}xØx \in m$ iff there is a variable y such that $(\dot{\exists}zz = y \,\&\, Øy) \in m$ iff both $\dot{\exists}zz = y$ and Øy are in m. But $\dot{\exists}zz = y \in m$ iff u(y) is defined at <m, s>, and $Øy \in m$ iff w(Øy)(<m, s>) is T. So w($\dot{\exists}$xØx)(<m, s>) is T iff there is a variable y such that u(y) is defined at <m, s> and w(Øy)(<m, s>) is T.

We have completed the demonstration that w is a f.i.u., but we still need to show that u is a partial interpretation. But since u matches w where defined, it is clear that u(n), u(P^j) and u(A) are partial functions of the appropriate sort. It is also clear by the construction of u(A), and the fact that u(A), where defined, is w(A), that u satisfies conditions =., F≅., FP^j., F~., F⊃., FT., F$\dot{\exists}$.. Furthermore, u(IxØx)(d) must be e, when e is the only member of D such that for some y, u(y)(d) is e and u(Øy)(d) is T. So to show that u satisfies I., we need only show that otherwise u(IxØx) is not defined at d.

Suppose there is no unique $e \in D$ such that for some y, u(y)(<m, s>) is e and u(Øy)(<m, s>) is T. Suppose for *reductio* that u(IxØx) is defined at <m, s>. Then $\dot{\exists}zz = IxØx \in m$. By the axiom of substitution of identities and (RQ), $\dot{\exists}zz \cong IxØx \in$

m. Since m is a TF-model set, there is some y such that $\emptyset y \in m$, $\dot{\forall} x(\emptyset x \supset x \cong y) \in m$, and $\dot{\exists} xx = y \in m$. So $u(\emptyset y)(\langle m, s\rangle)$ is T. Now u(y) is defined at $\langle m, s\rangle$, so let $u(y)(\langle m, s\rangle)$ be e. Suppose there is some c in D such that for some variable z $u(z)(\langle m, s\rangle)$ is c and $u(\emptyset z)(\langle m, s\rangle)$ is T. Then $\dot{\exists} xx = z \in m$ and $\emptyset z \in m$. Since $\dot{\forall} x(\emptyset x \supset x \cong y) \in m$, it follows by ($\dot{\exists}$G) that $z \cong y \in m$, and by Lemma 5, $u(z)(\langle m, s\rangle)$ is $u(y)(\langle m, s\rangle)$. So c is e, and hence e is the only member of D such that for some y, $u(y)(\langle m, s\rangle)$ is e and $u(\emptyset y)(\langle m, s\rangle)$ is T. But that contradicts our initial premise, and we conclude that $u(\mathrm{I}x\emptyset x)$ is not defined at $\langle m, s\rangle$.

We have finished the proof of Theorem 2. From this result and Theorem 1 it follows that any consistent set Γ is TF-satisfiable, and so TF is complete.

Now that we have a completeness proof for TF, proofs for TQ≅, and ETQ≅ follow easily. Let us start with TQ≅. We may prove Theorem 1 for TQ≅, without any changes, other than deleting dots from the quantifiers. We may define U_Δ exactly as we did in the proof of Theorem 2. Since T≅ uses standard principles of quantification, $\exists xx = n$ is provable, and it follows that u(n) is defined everywhere on D. It follows that U_Δ is a TQ≅model. To show that ETQ≅ is complete, we use the completeness proof for TQ to generate a TQ≅-model $U_\Delta = \langle D, u\rangle$ for each model set Δ, and we let U'_Δ be $\langle D, u, Q\rangle$, where Q is u(E). U'_Δ is clearly an ETQ≅model.

9. *A 2-Sorted System*

We develop a 2-sorted system by distinguishing two sets of variables V^1, V^2, the first for contexts, and the second for individuals. The sets N^1 and N^2 of terms are the smallest sets such that $V^1 \subset N^1$, and $\mathrm{I}x\emptyset x \in N^1$ if $x \in V^1$ and $\emptyset x$ is a formula, and similarly for N^2. Then we may let a 2-model U be a triple $\langle C, I, u\rangle$, where C and I are non-empty sets (of contexts and individuals respectively), and where u assigns to each $n \in N^1$

a partial function from C into C, and to each term $n \in N^2$ a partial function from C into I, and to each predicate letter P^j a function from C into the power set of $(C \cup I)^j$, and to each formula a partial function from C into $\{T, \perp\}$, and u satisfies the conditions for a partial interpretation save that the condition for the quantifier reads.

> When $n \in N^i$, then $u(\exists x \emptyset x)$ is defined at d just in case $u(\emptyset y)$ is defined at d for some $y \in N^i$, and when defined $u(\exists \emptyset x)(d)$ is T, just in case there is a $y \in N^i$ such that $u(y)$ is defined at d, and $u(\emptyset y)(d)$ is T.

The definition of 2-satisfiability mirrors the one for TF-satisfiability.

This semantics is axiomitized by TF, save that a minor modification of $(\dot{\exists}G)$ must be made to block illicit inferences from one range of terms to the other. The completeness proof is obtained by trivial modifications of the proof for TF.

If we introduce a 2-sorted system, we open the possibility of providing separate quantificational principles for the two sets of variables, with the principles of quantificational logic and (AQ) governing $\exists x$ for $x \in V^1$, and $(\dot{\exists}G)$, $(\dot{\exists}=)$, and (RQ) governing $\exists x$ for $x \in V^2$. It makes sense to say that the set of (existing) individuals changes from one context to the other, but it seems odd to claim that the set of existing contexts changes from context to context. At least this seems odd when our contexts are times, places or spatio-temporal coordinates. The resulting system's proof theory, semantics and completeness proof may be pieced together using the results of this paper.

However, there are some ways to make sense of the idea of a context-relative domain of contexts. If our contexts are possible worlds, then we might identify the set $\Theta(d)$ of possible worlds, available at d to be $\{e{:}Rde\}$, where R is Kripke's famous accessibility relation. In fact we may define this relation in the object language of topological logic by $Rxy =_{df} Tx\dot{\exists}zz=y$ (16).

The theory of special relativity provides another, perhaps more spurious way to make sense of this idea when contexts are spatio-temporal coordinates. We might let $\Theta(s)$ be the set of points within the light cone with apex at s, counting the points not in $\Theta(s)$ as «unavailable» on grounds that persons at s have, in principle no knowledge of, or causal relationships with events at these points.

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(16) For more on the relationships between modal and topological logics see [5].