NATURAL DEDUCTION RULES FOR FREE LOGIC (*)

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1. Systems of natural deduction are often prized for the similarity they bear to intuitive, informal reasoning. This is particularly true of those using Fitch's method of subordinate proofs which allows for the perspicuous construction of what may be thought of as arguments for the sake of the argument, with representation of different types of arguments made possible by different types of subordinate proofs [1, 2]. For example, the concept of a general subordinate proof provides an extremely intuitive way of representing informal reasoning in which a term, such as «John Doe» or «x», that ordinarily has no referent (or even purported referent) stands in for terms that do.

In [3] Karel Lambert and Bas van Fraassen develop a system of Fitch-style natural deduction rules for a language whose statements contain the universal quantifier and the identity sign among others, but no free individual variables or individual constants. Quasistatements in which a variable x occurs free are restricted to subordinate proofs general (1) with respect to x; and vacuous quantifier elimination is allowed only within such general subordinate proofs. In case an assumption that the domain is non-empty is made, a special rule VQE, permitting unrestricted vacuous quantifier elimination, is provided; but the main rules are valid for the empty domain as well as non-empty ones. The system is later extended by simply adding individual constants to the intended language and modifying the rules for identity in an obvious way: a universally free logic results (2). The system, both before and after its

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(1) Instead of using Fitch's «a subordinate proof general with respect to x,» Lambert and van Fraassen use the terminology «a subordinate derivation general in x,» but there is no difference in meaning between the two.

(2) Following Karel Lambert, by a *free logic*, we mean a system of logic

extension, however, lacks the intuitive simplicity typical of those using Fitch's method. For instance, it is not clear: why the rules for quantifiers, although valid for the empty domain, are explained intuitively as if the domain were non-empty; or why for the empty domain vacuous quantifier elimination is allowed within general subordinate proofs; or why, once individual constants are added, general subordinate proofs are not formulated in their terms and the awkward use of quasi-statements — evident particularly in the formulation of the identity rules — thereby eliminated altogether.

In this paper, a system of Fitch-style rules for free logic is also proposed, but one which differs from Lambert and van Fraassen's in a number of ways. The language is richer, containing as singular terms both free individual variables and individual constants; but, as the syntactical notation used has the effect of treating both exactly alike, without rewriting the rules, either could be omitted in favor of the other. The function of general subordinate proofs is explained in a novel way and closely related to an understanding of the rules. And, in another departure from Lambert and van Fraassen, an interpretation of the quantifiers, which assumes neither that singular terms denote nor that the domain is non-empty, is used in addition to account for the rules. A system of rules for a logic valid for only nonempty domains with singular terms that are assumed to denote is proposed first. A free logic valid for only non-empty domains is then formed from these rules by simply placing a restriction on two rules, the rules of universal quantifier elimination and existential quantifier introduction. A universally free logic is formed by placing a somewhat stronger restriction on those same two rules.

2. The rules to be presented are intended for a language having the following characteristics. The primitive signs are: predicates; individual variables; individual constants; the usual

in which not all singular terms are assumed to denote; by a *universally free logic*, a logic which is both free and valid for empty as well as non-empty domains.

sentential connectives '$\sim$' and '$\supset$'; the quantifier letter '$\exists$'; the sign '='; and the two parentheses '(' and ')'. As syntactical notation, the letters 'x,' 'y,' and 'z' are used to stand for arbitrary individual variables; '*a*,' '*b*,' and '*c*' for arbitrary individual constants; 't,' 't_1,' 't_2,' ..., and 't_n' for arbitrary individual constants and variables alike. The letters '*A*,' '*B*,' and '*C*' are used to stand for what we shall call *sentences*. The sentences are: $Ft_1, t_2, \ldots, t_n$, where F is an n-place predicate; $t_1 = t_2$; $\sim A$; $(A \supset B)$; $(x)A$; and $(\exists x)A$. An occurrence of an individual variable x in A is bound in A if it occurs in a part $(\exists x)B$ or $(x)C$ of A; if an occurrence of an individual variable is not bound in A, it is free. All occurrences of individual constants are free. Notation like $(t_1/t_2)A$ is used to refer to a sentence exactly like A except that in it t_1 occurs free wherever t_2 occurs free in A, although there may be occurrences of t_1 at places where t_2 does not occur free in A. Notation like $A(t_1/t_2)$ is used to refer to a sentence exactly like A except that in it t_1 occurs free in just the places (and no others) that t_2 occurs free in A (3). Much simpler quantifier and identity rules may be formulated using this notation.

The method of subordinate proofs used is essentially Fitch's as are the rule of reiteration and the rules of introduction and elimination for each sentential connective in the language (see Appendix). The introduction and elimination rules for identity are the following:

Rule of identity introduction («id int»): $t = t$ may be entered as an item in any proof.

Rule of identity elimination («id elim»): A is a d.c. (direct) consequence) of $t_1 = t_2$ and $(t_1/t_2)A$. (Note: in $(t_1/t_2)A$ there may be occurrences of t_1 at places t_2 does not occur free in A; thus, all free occurrences of t_1 in $(t_1/t_2)A$ need no be replaced by occurrences of t_2 to get A.)

We understand that $t_1 = t_2$ is true if and only if t_1 and t_2 are

(3) In other words, $(t_1/t_2)A$ is the result of replacing all free occurrences of t_2 by t_1, where t_1 does not become bound anywhere in $(t_1/t_2)A$ that t_2 is free in A. $A(t_1/t_2)$ is $(t_1/t_2)A$, where t_1 does not occur free in $(t_1/t_2)A$ anywhere that t_2 does not occur free in A.

the same term or have either the same referent or the same purported referent.

3. There is also an introduction and elimination rule for each of the quantifiers. These rules are closely related to the following interpretation of universal and existential sentences: $(x)A$ is understood to be true if and only if

(1) for every singular term t that denotes, $(t/x)A$ is true;

and $(\exists x)A$ is understood to be true if and only if

(2) there is a singular term t that denotes such that $(t/x)A$ is true.

This interpretation assumes that everything in the domain has a name, but not that the domain is non-empty, nor that every singular term denotes ([4]). In an empty domain, of course, no singular term denotes; and thus, by our interpretation, all universally quantified sentences are true and all existentially quantified ones false.

General subordinate proofs are important in formulation of the rules for quantifiers. A subproof general with respect to t may be either hypothetical or categorical, and a sentence A may be reiterated into it from the proof to which it is directly subordinate so long as t is not free in A. The restriction on reiterates insures that any sentence outside of the subproof in which t is used has no bearing on t inside the subproof. Much like «John Doe» or «x» in informal reasoning, t in a subproof general with respect to t has no referent or even purported referent, yet because it stands in for any term that does, it is treated as if it does denote. The line of reasoning is then valid for any term that does. Thus, a categorical subproof general with respect to t which has A as an item holds if and only if for every singular term t_1 that denotes, $(t_1/t)A$ is derivable in the proof to which the subproof is directly subordinate. And

([4]) Subjunctive conditionals, it seems, are needed to provide an interpretation which does not make the assumption that everything in the domain has a name.

a subproof general with respect to t which has $A(t/x)$ as an hypothesis and B as an item, where t does not occur free in B, holds if and only if a subproof which has $(t_1/t)A(t/x)$ as an hypothesis and B as an item holds for every singular term t_1 that denotes.

The rules of universal quantifier introduction and existential quantifier elimination both make use of general subordinate proofs.

Rule of universal quantifier introduction («u q int»): $(x)A(x/t)$ is a d.c. of a categorical subproof general with respect to t that has A as an item.

Rule of existential quantifier elimination («e q elim»): B is a d.c. of $(\exists x)A$ and a subproof general with respect to t with $A(t/x)$ as its only hypothesis and B as an item, where t is not free in B.

The rule of universal quantifier introduction says, in effect, that $(x)A(x/t)$ may be entered as an item of a proof if it is shown previously that for every singular term t_1 that denotes, $(t_1/t)A$ is derivable in that proof. The rule is clearly valid, as $(x)A(x/t)$ is true if and only if for every singular term t_1 that denotes, $(t_1/x)A(x/t)$ is true and $(t_1/t)A$ is the same as $(t_1/x)A(x/t)$. The rule of existential quantifier elimination says, in effect, that B, where t is not free in B, may be entered as an item of a proof in which $(\exists x)A$ is a previous item if another previous item is a subproof which has $(t_1/t)A(t/x)$ as an hypothesis and B as an item and holds for every singular term t_1 that denotes. The rule is clearly valid, as $(\exists x)A$ is true if and only if there is a term t_1 that denotes such that $(t_1/x)A$ is true and $(t_1/t)A$ is the same as $(t_1/x)A(x/t)$.

4. The deduction rules presented so far are valid whether or not the domain is assumed non-empty and whether or not singular terms are assumed to denote; moreover, they could not be strengthened on either assumption. The situation is different for the rules of universal quanifier elimination and existential quantifier introduction.

The following are for a system intended for a language in which all singular terms are assumed to denote.

Rule of universal quantifier elimination («u q elim»):

$(t/x)A$ is a d.c. of $(x)A$.

Rule of existential quantifier introduction («e q int»):

$(\exists x)A$ is a d.c. of $(t/x)A$.

The rule of universal quantifier elimination says that $(t/x)A$ may be entered as an item of a proof in which $(x)A$ is an item. Where $(x)A$ is an item of main proof or a subproof not general with respect to t or subordinate to a subproof that is, the rule is justified, as $(x)A$ is true only if $(t/x)A$ is true, where t is a singular term that denotes, and terms are assumed to denote. Where $(x)A$ is an item of a subproof that is either general with respect to t or subordinate to a subproof that is, the rule is also justified, as, within a subproof general with respect to t, t is to be treated as if it refers. The rule of existential quantifier introduction says that $(\exists x)A$ may be entered as an item of a proof in which $(t/x)A$ is an item. Where $(t/x)A$ is an item of a main proof or a subproof not general with respect to t or subordinate to one that is, the rule is clearly justified, as $(\exists x)A$ is true if there is a term t that denotes such that $(t/x)A$ is true and all singular terms are assumed to denote. It can be readily seen that the rule is justified also where $(t/x)A$ is an item of a subproof general with respect to t or subordinate to one that is.

The two rules just presented are for a system intended for a language in which singular terms are assumed to denote and are, therefore, valid for only non-empty domains. If the assumption that singular terms denote is dropped, but the condition that the domain is non-empty retained, the rules of universal quantifier elimination and existential quantifier introduction are still valid in case the quantifiers are vacuous, in other words, x is not free in A. This is not difficult to see considering that: the domain is assumed non-empty (thus, at least one term denotes); $(t/x)A$ is the same as A regardless of whether t denotes or not; $(x)A$ is true if and only if for every singular term t that denotes, $(t/x)A$ is true; and $(\exists x)A$ is true if and only if there is a singular term t that denotes such that $(t/x)A$ is true. Indeed, if the domain is non-empty and if x is not free in A, than $(x)A$, $(\exists x)A$, and A should be derivable from each other. In case x is free in A, however, the rules are not valid: if t does not denote, $(x)A$ may be true, yet $(t/x)A$ false; and $(t/x)A$ true, yet $(\exists x)A$ false. In a subproof that is either general with respect to t or subordinate to one that is, t is always treated as if it denotes, so, within such a subproof, $(t/x)A$ should be derivable from $(x)A$ and $(\exists x)A$ from $(t/x)A$. Thus, for a free logic valid for only non-empty domains, the following restriction on the rules of universal quantifier elimination and existential quantifier introduction is needed.

Restriction I («re I»): x occurs free in A only if $(t/x)A$ is an item of a subproof that is either general with respect to t or subordinate to a subproof that is.

By the rule of universal quantifier elimination with restriction I («u q elim re I»), only within a subproof general with respect to t is $(t/x)A$ a direct consequence of $(x)A$, where x occurs free in A; and by the rule of existential quantifier introduction with restriction I («e q int re I»), only within a subproof general with respect to t is $(\exists x)A$ a direct consequence of $(t/x)A$, where x occurs free in A.

In a universally free logic, it is not assumed either that the domain is non-empty or that singular terms denote. Therefore, it does not follow that at least one term denotes. If no term denotes, even though x is not free in A (and, thus, $(t/x)A$ is the same as A), $(x)A$ may be true, yet $(t/x)A$ false; and $(t/x)A$ true, yet, $(\exists x)A$ false. Indeed, if the domain is empty, all universally quantified sentence are true and all existentially quantified ones false. But, within a subproof that is either general with respect to t or subordinate to a subproof that is, even though x is not free in A, $(t/x)A$ should be derivable from $(x)A$ and $(\exists x)A$ from $(t/x)A$, as within such a subproof the term t is treated as if it denotes and the domain, in effect, assumed non-empty. With this in mind, it is not difficult to see how any sentence could be derived in a categorical subproof general with respect to t if it is assumed that $\sim(\exists x)(x=x)$ (i.e., the domain is empty) (5). Thus, for a universally free logic, a somewhat stronger restriction on the rules of universal quantifier elimination and existential quantifier introduction is needed.

Restriction II («re II»): $(t/x)A$ is an item of a subproof that is either general with respect to t or subordinate to a subproof that is.

By the rule of universal quantifier elimination with restriction II («u q elim re II»), only within a subproof general with respect to t is $(t/x)A$ a direct consequence of $(x)A$, even if x does not occur free in A; and by the rule of existential quantifier introduction with restriction II («e q int re II»), only within a subproof general with respect to t is $(\exists x)A$ a direct consequence of $(t/x)A$, even if x does not occur free in A. In this system of universally free logic, however, the following may be obtained in only a few steps as derived rules.

DR1: $(t/x)A$ is a consequence of $(x)A$ and $(\exists x)$ $(x=t)$, where t is not x (6).

DR2: $(\exists x)A$ is a consequence of $(t/x)A$ *and* $(\exists x)$ $(x=t)$, where t is not x (7).

5. The three systems of logic presented might have been constructed in reverse order, except for the familiarity of the first; then, removal of a restriction could be viewed as the addition of an assumption just as addition of VQE to Lambert and van Fraasen's system is. A case is not made here for regarding universally free logic as basic; but it should be noted that the interpretation of the quantifiers, which assumes neither that singular terms denote nor that the domain is non-empty, works well in explaining the quantifier rules for all three systems, and that for the rules of universal quantifier introduction and

(5) Reiterate $\sim(\exists x)(x=x)$ into the subproof; enter $t=t$ as an item by id int and apply e q int re II to get $(\exists x)(x=x)$; then apply neg elim to get A. By u q int, $(x)A$ is a direct consequence of the subproof; therefore, under the assumption that the domain is empty, any universally quantified sentence is derivable.

(6) Use reit, u q elim re II, id elim, and then e q elim.

(7) Use reit, id int, id elim (twice), e qu int re II, and then eq elim.

existential quantifier elimination, which belong to all three systems, it is all that is needed.

APPENDIX

A proof is a column of items boarded on the left by a vertical line extending the length of the column. Each item is either a sentence or another proof. A proof which is an item of another proof is a subordinate proof (subproof). A subproof is subordinate to any proof in which it is an item, or an item of an item, and so on, but is directly subordinate only to the proof in which it is itself an item. A subproof may be either a regular subproof or a general subproof. A general subproof has an individual constant or variable to the immediate left of the upper part of the vertical line associated with it, a regular subproof has none.

A sentence item of a proof may be either (i) an hypothesis, (ii) a direct consequence of preceding items of that proof, by one of the rules of direct consequence, or (iii) in case the proof is a subproof, a reiterate by the rule of reiteration of a sentence (a) that precedes the subproof as an item in the proof to which the subproof is directly subordinate and (b) in which t does not occur free in case the subproof is general with respect to t. The hypotheses, if any, are the first items of the column and are separated from the others by a short horizontal line extending out to the right of the vertical line. A proof with hypotheses is a hypothetical proof; a proof without is categorical. A theorem is the last item of a categorical main proof. A main proof is a proof not subordinate to any other proof.

The rule of reiteration is simply the following:

Rule of reiteration («reit»). Each sentence is a reiterate of itself.

Among the rules of direct consequence, one introduction and one elimination rule is given for each sentential connective in the language. The usual two valued interpretation of these connectives is assumed. All other sentential connectives of two valued logic are definable in terms of negation and implication and the rules for them are derivable from the rules for negation and implication.

Negation introduction («neg int»). $\sim A$ is a direct consequence (hereafter abbreviated as «d.c.») of a regular subproof that has A as its only hypothesis and has both B and $\sim B$ among its items.

Negation elimination («neg elim»). A is a d.c. of any pair of sentences B and $\sim B$.

Implication introduction («imp int»). $A \supset B$ is a d.c. of a regular subproof with A as its only hypothesis and B among its items.

Implication elimination («imp elim»). B is a d.c. of A and $A \supset B$.

The rules of negation elimination and implication elimination should be obvious; in the other two rules, however, a sentence is a direct consequence not just of a pair of sentences but of a regular subproof. As suggested earlier, a subproof functions as an auxiliary to the proof to which it is directly subordinate as a sort of argument for the sake of the argument. By the rule of reiteration, a regular subproof may have as its items all of the sentences preceding it in the proof to which it is directly subordinate. If it is hypothetical, it has, of course, its own hypotheses as items. From this, it follows that any item of a regular subproof could be derived in the proof to which the subproof is directly subordinate if the hypotheses of the subproof were available. Thus, the rule of negation introduction, in effect, says: $\sim A$ may be entered as an item of a proof if it is shown previously that a sentence and its negate are derivable from the sentence items of that proof plus A. And the rule of implication introduction says: $A \supset B$ may be entered as an item of a proof if it is shown previously that B is derivable from the sentence items of that proof plus A.

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