A NATURAL DEDUCTION SYSTEM FOR 'IF THEN'

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Some years ago I published in this journal [1] some doubts I had entertained concerning the compatibility of maintaining the transitivity of «if ... then» sentences and the thesis that an «if ... then» sentence expresses some connection of relevance between its antecedent and consequent. The form of transitivity there considered was

if p then q
if q then r

∴ *if p then r*

and as one counterexample of this form I gave

if I knock this typewriter off the desk then it will fall
if it falls then it is heavier than air

∴ *if I knock this typewriter off the desk then it is heavier than air*

The counterexample has true premises and, on any relevancy account, a false conclusion.

Once truth-functionality of «if ... then» is given up (i.e. the identity of «if ... then» with «$\supset$» since this is the only truth-function that could sensibly be argued to be so identical) the problem for those logicians holding non-truth-functional accounts is how to determine which sentences involving «if ... then» are logically true.

[1] A. J. Dale, 'The Transitivity of «if ... then»', *Logique et Analyse* 1972, pp. 439-441.

In a later article ([2]) I noticed that a certain form of transitivity seemed unassaiable because it relied only on what I called the *sufficiency condition* for the *falsity* of «if ... then» sentences, namely, that an «if ... then» sentence is false when its antecedent is true and its consequent false Thus using only this sufficiency condition it is possible to show that

if p then q
if q then r
p
———————
$\therefore$ *r*

is a valid argument, since if the premises *if p then q* and *p* are true then *q* must be true and if the premise *if q then r* is true and *r* is false then *q* is false. Since it is impossible for the premises to be true and the conclusion false the schema *if, if p then q . if q then r . p, then r* is logically true. I shall, without prejudice, from now on use «$\rightarrow$» for «if, ... then».

The sufficiency condition I argued establishes the logical truth of the following schemata

1. $(A . B) \rightarrow A$
2. $(A . B) \rightarrow B$
3. $\{(A \rightarrow B) . (B \rightarrow C) . A\} \rightarrow C$
4. $\{(A \rightarrow B) . (A \rightarrow C) . A\} \rightarrow (B . C)$
5. $\{(A \rightarrow B) . A\} \rightarrow B$
6. $A \rightarrow (A \vee B)$
7. $B \rightarrow (A \vee B)$
8. $\{(A \rightarrow B) . (C \rightarrow B) . (A \vee C)\} \rightarrow B$
9. $\sim\sim A \rightarrow A$
10. $A \rightarrow \sim\sim A$
11. $\{(A \rightarrow B) . \sim B\} \rightarrow \sim A$

In fact the sufficiency condition is only needed for those sche-

([2]) A. J. Dale, 'A Defence of Material Implication', *Analysis* 1974, pp. 91-95.

mata involving nested «if ... then» sentences, i.e. 3, 4, 5, 8, 11; the remainder can be seen to be logically true by considering the appropriate truth-functions. If these schemata are treated as axiom schemata and to them are added the rule of modus ponens and the further schema of exportation $\{(A . B) \rightarrow C\} \rightarrow \{A \rightarrow (B \rightarrow C)\}$ (schema 12) then the resulting system is identical to the traditional propositional calculus. Thus it follows that the defence of material implication as the meaning of «if ... then» reduces to a defence of this schema. In that article I defended this schema for some uses of «if ... then» but this defence does not concern me here.

For the remainder of this paper I shall be concerned with an attack on this defence of material implication which claims that I have not shown that the sufficiency condition plus the schema of exportation is enough to establish that some uses of «if ... then» (those satisfying the exportation schema) reduce to the material conditional.

In a recent paper [3] P. Gibbins has argued that my defence is unsuccessful since the sufficiency condition can only show the validity of arguments but not the logical truth of corresponding sentences. Briefly, although allowing that the sufficiency condition does guarantee the validity of

if p then q
if q then r
p
———
∴ *r*

it does not guarantee the logical truth of *if, if p then q . if q then r . p, then r* since the sufficiency condition is compatible with «if ... then» always having the truth-value false. Now although it is certainly true that the sufficiency condition is so compatible, it does not follow that the validity of the argument does not settle the truth of the corresponding schema. To

(3) P. Gibbins, 'Material Implication, the Sufficiency Condition, and Conditional Proof', *Analysis* 1979 pp. 21-24.

show that an argument is valid *is* to show that if the premises are true then the conclusion is true. To make this explicit, there is also a sufficiency condition for the truth of an «if ... then» sentence, namely if it is impossible for its antecedent to be true and its consequent false ([4]). This uncontroversial principle would have to be used even when my original sufficiency condition is irrelevant as in establishing the logical truth of schemata 1, 2, 6, 7, 9 and 11. It was not my intention to discard this principle even if it was not there stated explicitly. After all it is an odd objection that the truth-table for «$\vee$» does not show that the schema «if A then A $\vee$ B» is logically true on the grounds that the falsity of all «if ... then» sentences is compatible with the truth-table for «$\vee$».

However this may be, I would still challenge anyone to give good reasons for rejecting any of axiom schemata 1-11; they would all certainly appear to be beyond suspicion. So, once again, if the schema of exportation in some contexts is accepted then in those contexts «if ... then» behaves as a material conditional.

Gibbins then purports to detect the inadequacy of my defence by showing that it does not work in a corresponding natural deduction system. Since my original sufficiency condition justifies the eleven arguments corresponding to my eleven uncontroversial schemata, eleven corresponding natural deduction rules are valid. Gibbins then claims that even allowing a twelfth rule corresponding to my twelfth schema (exportation) I do not have a system sufficient for deriving all the tautologies of the propositional calculus. Furthermore, if the rule of conditional proof is added then the system is a complete system for propositional calculus but conditional proof is itself not justifiable by my sufficiency condition. Hence my defense can be seen to fail.

([4]) Naturally, for those worried about impossible antecedents and necessary consequents this condition may need modifying but all my schemata satisfy additional conditions of a more stringent type imposed by Geach etc., See A. J. Dale «Geach on Entailment», *Philosophical Review* 1973, pp. 215-219.

Suppose then the transformation of schemata into rules is made, so that we have

1. $\frac{A.B}{A}$ 2. $\frac{A.B}{B}$ 3. $\frac{(A \rightarrow B),\ (B \rightarrow C),\ A}{C}$

4. $\frac{(A \rightarrow B),\ (A \rightarrow C),\ A}{B.C}$ 5. $\frac{(A \rightarrow B),\ A\ (^5)}{B}$

6. $\frac{A}{A \vee B}$ 7. $\frac{B}{A \vee B}$

8. $\frac{(A \rightarrow B),\ (C \rightarrow B),\ (A \vee C)}{B}$ 9. $\frac{\sim\sim A}{A}$

10. $\frac{A}{\sim\sim A}$ 11. $\frac{(A \rightarrow B),\ \sim B}{\sim A}$ 12. $\frac{(A.B) \rightarrow C}{A \rightarrow (B \rightarrow C)}$

as the twelve rules.

Now it is certainly true that these rules do not yield the full propositional calculus. But Gibbins is incorrect in maintaining that the rule of conditional proof is needed for their completion. The rule of conditional proof

$$\frac{\Gamma\ \ , A \vdash B}{\Gamma \vdash A \rightarrow B}$$

where Γ is a sequence (possibly empty) of well-formed formulae, would certainly complete the system but, as Gibbins points out, this rule is as questionable as accepting material implication as the correct interpretation of «if ... then». Fortunately, however, it is not necessary to add this rule of con-

(5) This rule takes the place both of schema 5 and the rule of modus ponens of the original axiom system.

ditional proof to the twelve rules: instead the rules can be supplemented by a restricted conditional rule

$$13.\ \frac{A \vdash B}{A \rightarrow B}$$

which corresponds to the sufficiency condition for the truth of an «if, ... then» sentence which I referred to above. This restricted rule, which allows only one hypothesis to operate at any one time, is as uncontroversial as that sufficiency condition [6]. To Prove that this natural deduction system does produce all the tautologies of the propositional calculus it is necessary only to prove that the twelve axiom schemata are derivable since the rule of modus ponens is rule 5 of the system. It is obvious how this should be done but I will illustrate by proving schema 4.

1. $(A \rightarrow B) . (A \rightarrow C) . A$	hypothesis
2. $(A \rightarrow B) . (A \rightarrow C)$	1. rule 1
3. $A \rightarrow B$	2. rule 1
4. $A \rightarrow C$	2. rule 2
5. A	1. rule 2
6. $B . C$	3, 4, 5 rule 4
7. $(A \rightarrow B) . (A \rightarrow C) . A \rightarrow (B . C)$	1, 6 rule 13.

Of course, this system is inelegant and somewhat degenerate as a natural deduction system, the power of which is a result of having many hypotheses in operation at once. But it was not my intention to produce an aesthetically satisfying natural deduction system since I was interested only in showing that if the schema of exportation is accepted so too must the material conditional account of «if ... then». Now with the above natural deduction system the defence of material implication again reduces to that of exportation, this time in its

(6) Even such iconoclasts as Belnap and Anderson accept this rule, at least if they allow the truth of «A entails B» to entail the truth of «if A then B» See ANDERSON and BELNAP, *Entailment* London 1975, p. 7.

rule guise rather than as a schematic sentence. If Gibbins or anyone else doubts this he must show which of the twelve rules (other than exportation) he would give up for «if ... then» and explain why.

One more point, Gibbins also claims that my defence must cover counterfactuals. But this is not so, for presumably any theory of the counterfactual must reject the exportation schema for the counterfactual use of «if ... then». I have nowhere claimed that all uses of «if ... then» obey the schema of exportation but only that there is certainly a very common use which does.

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