FORAYS INTO THE META-THEORY OF FUZZY SET THEORY

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1. *Introduction*

In a recent paper [2] I suggested a generalisation of the basic definitions and concepts of fuzzy set theory in which the values of fuzzy membership are mapped onto intervals within [0,1] rather than onto numbers between 0 and 1. The advantages obtained were partly conceptual and partly technical: the necessity of specifying an *exact* number-value to fuzzy membership, surely contrary to the purpose of the theory, is mitigated by assigning an interval value instead, which is less specific (though of course not fuzzy); and there is a structure-similarity between fuzzy sets and their interval values, absent from number-valued membership, in that both are susceptible to the same kinds of set-theoretic combination. These include not only the usual means of combination but also the arithmetical combinations of sets used in interval analysis [7]. I also further generalised the approach to make use of Young's theory of many-valued quantities [8], where the value of fuzzy membership is a sub-set of [0,1].

The modified theory is much richer in forms of combination than is its number-value-based predecessor, even though I deliberately restricted in [2] the number of order-relations that may apply between intervals. The range of such relations can be gauged from the interesting papers of Jahn (see especially [3] and [4]). His studies are independent of mine and are mainly confined to interval analysis itself, although he has noticed its possible bearing on fuzzy set theory. The purpose of this paper is not to develop further fuzzy set theory, but to explore the structure of its metatheory.

After recalling in section 2 the outline of the system defined in [2], I formulate in section 3 the ideas of derivation, validity and consistency that apply in this theory. Some special problems arise in connection with the definition of negation, and section 4 is devoted to them. Section 5 contains perhaps the most interesting and individual results, in dealing with the fuzziness of a certain part of the meta-theory. The meta-theory as a whole is elaborate, and so I have confined myself to principal definitions and properties, and have dealt only with interval-valued membership.

In the terminology of current studies in fuzzy logic, my system is a 'basic logic'. Some work, notably that of Bellman and Zadeh in [9], extends the notion of fuzziness to define such concepts as 'rather true'. I am not convinced of the validity of such concepts, and I have not introduced their analogues here.

2. *Resumé of the system*

I refer to [2] for the details of the definitions and proofs which I developed there, but I must recapitulate here the principal features.

2.1. *Brackets.* Peano dots are used to reduce the need for round brackets. Other types of brackets are used as follows:

[x, y]	closed interval with end-points x and y.
{x, y, ...}	unordered set of x, y, ...
⟨x, y, ...⟩	ordered set of x, y, ...
[ ]	indication of derivations: the references placed within refer to the definitions and/or theorems used to obtain the preceding definition or theorem.

2.2. *Logic and set theory.* Standard notations are used, save that '$\mathfrak{A}$' denotes a propositional function, 'A', 'B', ... denote fuzzy sets and '$\{A_u\}$' a class of them, 'R' and 'S' fuzzy relations, and 'I', 'J', ... '$\{I_u\}$' closed intervals within [0, 1].

2.3. *Properties on real numbers.*

Glb: $\bigwedge_u x_u = \text{df glb}\{x_u\}$ (2.3.1)

Lub: $\bigvee_u x_u = \text{df lub}\{x_u\}$ (2.3.2)

Subtraction: $x \dot{-} y = \text{df } x - y . \vee . 0 .$ (2.3.2)

2.4. *Interval arithmetic.* The interval I is expressed in terms of its end-points by

$$I = [i_l, i_r]; \quad (2.4.1)$$

'i_*' refers to either end-point. We also define

Width: $\text{wid}(A) = i_r - i_l.$ (2.4.2)

The interval-valued membership of x to A is symbolized by '$w_A(x)$'. 'Δ' is an operator symbol ranging over the means of combining intervals (or other kinds of set). '$\square$' similarly ranges over order-relations between intervals.

The definitions of interval combination follow as far as possible the schema

$$I \Delta J = \text{df } \{i \Delta j \mid i \varepsilon I,\ j \varepsilon J\},\ I \Delta J \subseteq [0,1]: \quad (2.4.3)$$

Addition: $I + J = \text{df } [i_l + j_l . \wedge . 1,\ i_r + j_r . \wedge . 1],$ (2.4.4)

Subtraction: $I \dot{-} J = \text{df } [i_l \dot{-} j_r,\ i_r \dot{-} j_l],$ (2.4.5)

Multiplication: $I \times J = \text{df } [i_l \times j_l,\ i_r \times j_r],$ (2.4.6)

Division: $I \div J = \text{df } [i_l \div j_r . \wedge . 1,\ i_r \div j_l . \wedge . 1],$ (2.4.7)

Complementation: $I - J = \text{df } \{x \mid x \varepsilon I . \wedge . x \notin J\},$ (2.4.8)

Minimisation: $I \wedge J = \text{df } [i_l \wedge j_l,\ i_r \wedge j_r],$ (2.4.9)

Maximization: $I \vee J = \text{df } [i_l \vee j_l,\ i_r \vee j_r].$ (2.4.10)

The ordering relations on intervals are based on these three:

Inferiority: $I \leqslant J = \text{df } i_l \leqslant j_l . \wedge . i_r \leqslant j_r,$ (2.4.11)

Improper inclusion: $I \subseteq J = \text{df } i_l \geqslant j_l . \wedge . i_r \leqslant j_r,$ (2.4.12)

Equality: $I = J = \text{df } i_l = j_l . \wedge . i_r = j_r;$ (2.4.13)

It is obvious how to define further relations, such as '$\geqslant$' and

'$\subset$'. Jahn's work has included defining such relations: for example, in [4], 116 he distinguishes for '$\leqslant$' between inferiority with partial overlap and with disjointness — that is, for $I \leqslant J$ between the cases $i_r \geqslant j_l$ and $i_r < j_l$ [(2.4.1)].

2.5. *Interval-valued fuzzy membership.* As with (2.4.3), I follow as far as possible the definition schema

$$w_{A\Delta B}(x) = \text{df } w_A(x) \,\Delta\, w_B(x). \qquad (2.5.1)$$

When Δ is $+$, $\dot{-}$, $\times$, $\div$, $\cup$ and $\cap$, (2.5.1) applies as it stands. In addition, I define

$$\text{Conjunction: } w_{A\wedge B}(x) = \text{df } w_A(x) \wedge w_B(x), \qquad (2.5.2)$$

$$\text{Disjunction: } w_{A\vee B}(x) = \text{df } w_A(x) \vee w_B(x), \qquad (2.5.3)$$

$$\text{Complementation: } w_A'(x) = \text{df } J \dot{-} w_A(x). \ [(2.4.5)] \qquad (2.5.4)$$

I have argued for (2.5.2) and (2.5.3), with their associations of conjunction with minimisation, and of disjunction with maximisation, in [2], 153. (2.5.4) differs from the definition given there, where [0,1] rather than J was used; the reasons will emerge in section 4.

2.6. *First-order mathematical logic.* Since the underlying logic is non-classical, I indicate each fuzzy connective by attaching an asterisk to the corresponding classical symbol. The valuation given in my [2], 157-158 is as follows (with a modification of negation to J, this time from [1, 1]), where Φ and Ψ are general well-formed formulae:

$$\text{Negation: } V(\neg^* \Phi) = \text{df } J \dot{-} V(\Phi), \qquad (2.6.1)$$

$$\text{Disjunction: } V(\Phi \vee^* \Psi) = \text{df } V(\Phi) \vee V(\Psi), \qquad (2.6.2)$$

$$\text{Conjunction: } V(\Phi \wedge^* \Psi) = \text{df } V(\Phi) \wedge V(\Psi) \qquad (2.6.3)$$

$$\text{Implication: } V(\Phi \rightarrow^* \Psi) = \text{df } V(\neg^* \Phi) + V(\Psi), \ [(2.4.4)] \qquad (2.6.4)$$

Equivalence: $V(\Phi \leftrightarrow^* \Psi) = \text{df } V(\Phi \rightarrow^* \Psi . \wedge^* . \Psi \rightarrow^* \Phi)$, (2.6.5)

Universal quantification:
$$V((\forall x)(\mathfrak{A}x)) = \text{df } \bigwedge_x V(\mathfrak{A}x) = \bigwedge_x w_A(x), \qquad (2.6.6)$$

Existential quantification:
$$V((\exists x)(\bar{A}x)) = \text{df } \bigvee_x V(\mathfrak{A}x) = \bigvee_x w_A(x). \qquad (2.6.7)$$

3. *Derivation, validity and consistency*

3.1. *Forms of definition.* Lee and Chang [6] have produced some straightforward definitions of validity and inconsistency in the ordinary form of fuzzy set theory, and these can be generalised as follows. *Derivation* is perhaps the most important idea, and it can be defined in various forms relative to a given valuation. Here are some:

Ψ *is I-derivable* from Φ *under the valuation* V, symbolically '$\Phi \overset{*}{\underset{I,V}{\vdash}} \Psi$', if

$$I \leqslant V(\Phi) \leqslant V(\Psi). \qquad [(2.4.11)] \qquad (3.1.1)$$

Ψ is *derivable from* Φ *under* V, $\Phi \overset{*}{\underset{V}{\vdash}} \Psi$, if there is an interval I for which it is I-derivable, so that

$$V(\Phi) \leqslant V(\Psi). \qquad (3.1.2)$$

Ψ is *classically derivable from* Φ *under* V,$\Phi \underset{V}{\vdash} \Psi$, if it is I-derivable from Φ under V for all I.

Ψ is *categorically I-derivable from* Φ, $\Phi \overset{*}{\underset{I}{\vdash}} \Psi$, if it is I-derivable for all V.

Ψ is *categorically derivable from* Φ, $\Phi \overset{*}{\vdash} \Psi$, if it is categorically derivable from Φ for all I.

To be more strict, categorical derivability should be defined only relative to some mutually consistent set of valuations; for otherwise it is unsatisfiable by any formula. Relative to this point, a number of theorems easily follow, such as:

If $I_1 \leqslant I_2$ and $\Phi \overset{*}{\underset{I_2,V}{\vdash}} \Psi$ then $\Phi \overset{*}{\underset{I_1,V}{\vdash}} \Psi$.

$\Phi \vdash \Psi$ if and only if $\Phi \overset{*}{\underset{[1,1],V}{\vdash}} \Psi$.

We can define similar properties for the *validity* of Φ. I give the formal definition of one:

Φ *is I-valid under* V, $\overset{*}{\underset{I,V}{\vdash}} \Phi$, if

$$I \leqslant V(\Phi). \tag{3.1.3}$$

Similarly we can have the properties:

Valid under V, $\overset{*}{\underset{V}{\vdash}} \Phi$;

Classically valid under V, $\underset{V}{\vdash} \Phi$;

Categorically I-valid, $\overset{*}{\underset{I}{\vdash}} \Phi$;

Categorically valid, $\overset{*}{\vdash} \Phi$.

The *inconsistency* of Φ can be treated in the same way, starting off from:

Φ is *I-inconsistent under* V, $\overset{*}{\underset{I,V}{\dashv}} \Phi$, if

$$I \geqslant V(\Phi). \tag{3.1.4}$$

Theorems similar to those for derivability can be proved; in particular, classical validity of Φ corresponds to its classical theoremhood. We can also make use of this theorem in fuzzy set theory ([2], 154):

Theorem 3.1.1. If $F\langle A_u\rangle$ is a combination of fuzzy sets under the operations $+$, $\times$, $\wedge$ and $\vee$, and $A_u \leqslant B_u$ for all u, then $F\langle A_u\rangle \leqslant F\langle B_u\rangle$.
The valuation (2.6.1)-(2.6.5) shows that we have the following structurally similar meta-theorem:

Theorem 3.1.2. If $\Phi_u \overset{*}{\underset{I,V}{\vdash}} \Psi_u$ for all u, and $F\langle\Phi_u\rangle$ is a logical combination of the $\{\Phi_u\}$ which involves no negations, then

$$F\langle\Phi_u\rangle \overset{*}{\underset{I,V}{\vdash}} F\langle\Psi_u\rangle. \qquad (3.1.5)$$

3.2. *The deduction theorem and axiomatisation.* The deduction theorem takes the following form in this system:

Theorem 3.2.1. If $\Phi \overset{*}{\underset{I,V}{\vdash}} \Psi$, then $\overset{*}{\underset{I,V}{\vdash}} \Phi \rightarrow^* \Psi$.

Proof. The premises show that

$$i* \leqslant \varphi* \leqslant \psi*. \qquad [(3.1.1)] \qquad (3.2.1)$$

Now

$$\begin{aligned} V(\Phi \rightarrow^* \Psi) &= [(j_l \dot{-} \varphi_r) + \psi_l \,.\, \wedge \,.\, 1,\ (j_r \dot{-} \varphi_l) + \psi_r \,.\, \wedge \,.\, 1] && [(2.6.4)] \\ &\geqslant [\psi_l, \psi_r] && [(3.2.1)] \\ &\geqslant I. && [(3.2.1)] \end{aligned}$$

Q.E.D.

This theorem is usually considered in connection with axiomatic systems, so that it is appropriate to mention here the possibility of axiomatising fuzzy set theory. Lake has proposed [5] that it can be done by means of a function-theoretic form of von Neumann's set theory; and while this should also be possible in my version, it may be laborious to derive useful theorems from the axioms, so that the freer approach adopted here may be more convenient.

4. *J-negation and I-validity*

In this section I examine some consequences of the definition (2.6.1) of negation, which uses J, in the context of I-validity discussed in the last section.

4.1. *Forms of the modus ponens.* In this system derivation is defined by (3.1.1) and its variants rather than by *modus ponens* and such rules; so it is interesting to explore the status of that rule. There has been discussion of the philosophical difference between its disjunctive and conditional forms (see [1], for example); and it is borne out here, for the disjunctive form contains more stringent conditions than does the conditional form, and so requires less additional assistance for its demonstration. The underlying reason is:

Theorem 4.1.1. If $\overset{*}{\underset{I,V}{\vdash}} (\neg^* \Phi \vee^* \Psi)$, then $\overset{*}{\underset{I,V}{\vdash}} (\Phi \rightarrow^* \Psi)$.

Proof. $$V(\neg^* \Phi \vee \Psi) = [j_l \dot{-} \varphi_r . \vee . \psi_l,\ j_r \dot{-} \varphi_l . \vee . \psi_r], \qquad [(2.6.1),\ (2.6.2)]$$
$$V(\Phi \rightarrow^* \Psi) = [(j_l \dot{-} \varphi_r) + \psi_l . \wedge . 1,\ (j_r \dot{-} \varphi_l) + \psi_r . \wedge . 1].$$
$$\therefore V(\neg^* \Phi \vee^* \Psi) \leqslant V(\Phi \rightarrow^* \Psi). \qquad [(2.6.4)]$$
The theorem now follows from (3.1.1) and (3.1.3). Q.E.D.

Here are a couple of theorems on the validity of the forms of the rule; the conditions have been chosen to show easily that those on the disjunctive form are weaker:

Theorem 4.1.2. If
$$j_r \leqslant \varphi_l + i_l, \qquad (4.1.1)$$
then if $\overset{*}{\underset{I,V}{\vdash}} \Phi$ and $\overset{*}{\underset{I,V}{\vdash}} (\neg^* \Phi \vee^* \Psi)$, then $\overset{*}{\underset{I,V}{\vdash}} \Psi$. (4.1.2)

Proof. The premises of (4.1.2) yield
$$\varphi_* \geqslant i_*, \qquad (4.1.3)$$
$$[j_l \dot{-} \varphi_r . \vee . \psi_l,\ j_r \dot{-} \varphi_l . \vee . \psi_r] \geqslant [i_l,\ i_r], \qquad (4.1.4)$$

Considering first the left hand end-point,

$$j_l \dot{-} \varphi_r \geqslant i_l \quad \text{or} \quad \psi_l \geqslant i_l. \tag{4.1.5}$$

But $j_l \dot{-} \varphi_r \leqslant j_r \dot{-} \psi_l \leqslant i_l$ by (4.1.1), so that $(4.1.5)_1$ is ruled out; while $(4.1.5)_2$ is the required result. Similarly, the right hand end-point gives

$$j_r \dot{-} \varphi_l \geqslant i_r \text{ or } \psi_r \geqslant i_r; \tag{4.1.6}$$

and the first case is impossible by (4.1.2) while the second is the condition sought. Q.E.D.

Theorem 4.1.3. If

$$j_r \leqslant \varphi_l, \tag{4.1.7}$$

$$\text{then if } \overset{*}{\underset{I,V}{\vdash}} \Phi \text{ and } \overset{*}{\underset{I,V}{\vdash}} \Phi \rightarrow^* \Psi, \text{ then } \overset{*}{\underset{I,V}{\vdash}} \Psi. \tag{4.1.8}$$

Proof. The premises of (4.1.8) yield

$$\varphi* \geqslant i*, \tag{4.1.9}$$

$$[(j_l \dot{-} \varphi_r) + \psi_l \,.\, \wedge \,.\, 1,\; (j_r \dot{-} \varphi_l) + \psi_r \,.\, \wedge \,.\, 1] \geqslant [i_l, i_r]. \tag{4.1.10}$$

Considering first the left hand end-point, (4.1.7) shows *a fortiori* that $j_l \leqslant \varphi_r$, so that $\psi_l \geqslant i_l$, as needed. For the right hand end-point, (4.1.10) reduces via (4.1.7) to $\psi_r \geqslant i_r$. Q.E.D.

The conditions (4.1.1) and (4.1.7) on these theorems are rather strong, or both pass by the premise $\overset{*}{\underset{I,V}{\vdash}} \Phi$ of the *modus ponens* rule. Consideration of alternative conditions involves the relation between I an J. For example, if I = J, then theorem 4.1.3 can be proved without the use of (4.1.1), for then (4.1.4) becomes

$$[i_l \dot{-} \varphi_r . \vee . \psi_l,\ i_r \dot{-} \varphi_l . \vee . \psi_r] \geqslant [i_l\ i_r], \tag{4.1.11}$$

which leads to

$$\psi_l \geqslant i_l; \text{ and } i_r \dot{-} \varphi_l \geqslant i_r \text{ or } \psi_r \geqslant i_r. \tag{4.1.12}$$

But $(4.1.12)_2$ is impossible, and the other two parts yield the required $\overset{*}{\underset{I,V}{\vdash}} \Psi$.

4.2. *Comparison with various components of classical systems.* In order to aid the comparison with classical logic I list a few representative theorems which correspond in a classical system either to an axiom, or an important result, or the definition of a connective. I omit the proofs, which are straightforward.

$$\Phi \overset{*}{\underset{V}{\vdash}} \Phi \vee^* \Psi.$$

$$\Phi \wedge^* \Psi \overset{*}{\underset{V}{\vdash}} \Phi.$$

$$\neg^* \Phi \vee^* \Psi \overset{*}{\underset{V}{\vdash}} \Phi \rightarrow^* \Psi \qquad \text{[Theorem 4.1.1]}$$

If $I \geqslant J$, then $\overset{*}{\underset{I,V}{\dashv}} \neg^* \Phi$. [(3.1.4)]

If $\varphi_l \geqslant j_r$, then $V(\Phi \rightarrow^* \Phi) = V(\Phi)$.

If $\underset{J,V}{\vdash} \Phi$, then $\overset{*}{\underset{J,V}{\vdash}} (\Phi \rightarrow^* \Phi)$.

If $I \geqslant J$ and $\overset{*}{\underset{I,V}{\vdash}} \Phi$, then

$$V(\Phi \vee^* \neg^* \Phi) = V(\Phi) \text{ (so that } \overset{*}{\underset{I,V}{\vdash}} (\Phi \vee^* \neg^* \Phi)).$$

If $I \leqslant J$ and $\overset{*}{\underset{I,V}{\dashv}} \Phi$, then $\overset{*}{\underset{I,V}{\dashv}} (\Phi \vee^* \neg^* \Phi)$.

If $I \geqslant J$ and $\overset{*}{\underset{I,V}{\vdash}} \Phi$, then $\overset{*}{\underset{I,V}{\vdash}} (\Phi \wedge^* \neg^* \Phi)$.

If $I \leqslant J$ and $\overset{*}{\underset{I,V}{\dashv}} \Phi$, then $(\Phi \wedge^* \neg^* \Phi) \overset{*}{\underset{V}{\vdash}} \Phi$

(so that $\overset{*}{\underset{I,V}{\dashv}} (\Phi \wedge^* \neg^* \Phi)$.

If $\Phi \overset{*}{\underset{I,V}{\vdash}} \Psi$, then $\neg^* \Psi \overset{*}{\underset{I,V}{\vdash}} \neg^* \Phi$.

$\mathfrak{A}x \overset{*}{\underset{I,V}{\vdash}} (\exists y)\mathfrak{A}y$. [(2.6.6)]

$(\forall x)\, \mathfrak{A}x \overset{*}{\underset{I,V}{\vdash}} \mathfrak{A}y$. [(2.6.7)]

There are many such theorems, many involving special conditions for their validity. A detailed inventory is not needed here, since the method of analysing the relevant inequalities is clear. But I mention that the conversion into conjunctive and disjunctive normal forms is very appropriate, since under the valuation used here the valuations have respectively a maximin and minimax form, and these are well-known in interval analysis.

5. *The possession of a membership grade: fuzzy meta-theory*

5.1. *Basic definitions.* In [2], 160 I mentioned that the definition of complementation ((2.5.4) here, although in [2], 153 I used [0,1] rather than J) gave grounds for thinking that a fuzzy set is defined not only by its membership function but also by its range of significance; for the edges of this range are not easy to specify. (In other words, the universe(s) of discourse are themselves fuzzy.) Developing the example given in [2], 160, while it is clear that granite does not have a membership grade to the property of laziness and that Olympic sprinters do (a very low one in that capacity !), there can be objects for which it is not clear whether membership is or is not possessed (daffodils or snails, say). Thus '*the range of significance of a fuzzy set is itself a fuzzy set*' ([2], 160), so that the property of having a membership grade to a fuzzy set is itself a fuzzy meta-property.

We may symbolise and define this meta-property as follows: x *has a membership grade to* A, symbolically '$\varepsilon\langle x, A\rangle$', if

$$(\exists I) . (I \subseteq [0,1] . \wedge . w_A(x) = I). \qquad (5.1.1)$$

Just as we gave interval values to fuzzy set membership, so this fuzzy meta-property has interval-valued membership also. Since fuzzy set membership of x to A was symbolised by '$w_A(x)$', then fuzzy-membership possession of x to A, a fuzzy meta-relation, may be symbolised by '$\varepsilon\langle x,A\rangle$'. (The theory of fuzzy relations is structurally similar to that of fuzzy sets, and is sketched out in [2], 154-155.) Its interval valution may be written:

$$V(\varepsilon\langle x, A\rangle) = w_\varepsilon\langle x, A\rangle. \qquad (5.1.2)$$

Corresponding to the fuzzy set A we have the fuzzy meta-set A_ε.

In this way the structure-similarity which appplies between fuzzy sets A and their interval values $w_A(x)$ continues further from fuzzy set theory to its fuzzy meta-theory, with the membership relation $\varepsilon\langle x, A\rangle$ and its interval value $w_\varepsilon\langle x, A\rangle$. Hence our definition schema (2.5.1) for fuzzy membership to combinations of sets applies here also:

$$w_\varepsilon\langle x, A \Delta B\rangle = \mathrm{df}\ w_\varepsilon\langle x, A\rangle \Delta w_\varepsilon\langle x, B\rangle.$$

This schema is straightforward to execute when Δ is $\wedge$, $\vee$, $\cap$, $\cup$, $+$, $\dot{-}$, $\times$ and $\div$; but again complementation causes difficulties. Suppose that we are convinced that $\varepsilon\langle x, A\rangle$ (so that $w_A(x) = [1, 1]$); then we shall be equally certain that $\varepsilon\langle x, A'\rangle$ (though of course the *value* of the membership grade, given by (2.5.4), is quite different). Similarly, if we are unsure whether or not $\varepsilon\langle x, A\rangle$, then our doubts will remain in comparable degree over $\varepsilon\langle x, A'\rangle$. Thus an appropriate definition for complementation is

$$w_{\varepsilon}\langle x, A'\rangle = \text{df } w_{\varepsilon}\langle x, A\rangle. \qquad (5.1.4)$$

We may similarly define non-membership of x to A:

$$w_{\neg\varepsilon}\langle x, A\rangle = \text{df } w_{\varepsilon}\langle x, A'\rangle, \qquad (5.1.5)$$

so that

$$w_{\varepsilon}\langle x, A\rangle = w_{\neg\varepsilon}\langle x, A\rangle; \qquad (5.1.6)$$

that is, we are equally unsure whether $\varepsilon\langle x, A\rangle$ or not. But it does not follow that each has a membership grade of $[1/2, 1/2]$, for this part of the meta-theory is fuzzy, so that the law of excluded middle does not apply.

5.2. *Ordering meta-relations.* Just as the $\{w_A(x)\}$ are subject to ordering relations in fuzzy set theory, so are the meta-relations' values $\{w_{\varepsilon}\langle x, A\rangle\}$; we may talk about $w_{\varepsilon}\langle x, A\rangle$ being less $w_{\varepsilon}\langle x, B\rangle$, just as we might discuss whether or not $w_A(x) \leqslant w_B(x)$. For convenience I shall use the same notations for the various ordering relations as I did in fuzzy set theory, and so I denote the meta-relation just described by '$\leqslant_{\varepsilon}\langle x; A, B\rangle$', and define it, and notate its interval valuation, as follows:

$$\leqslant_{\varepsilon}\langle x; A, B\rangle = \text{df } w_{\varepsilon}\langle x, A\rangle \leqslant w_{\varepsilon}\langle x, B\rangle, \qquad (5.2.1)$$

$$V(\leqslant_{\varepsilon}\langle x; A, B\rangle) = w_{\leqslant}\langle w_{\varepsilon}\langle x; A\rangle, w_{\varepsilon}\langle x, B\rangle\rangle. \qquad (5.2.2)$$

We may define similarly $\leqslant_{\varepsilon}\langle x, y; A\rangle$, $\leqslant_{\varepsilon}\langle x, y; R, S\rangle$ (where we understand that both arguments satisfy each fuzzy relation); and we could also discuss $w_{\varepsilon}\langle x, A\rangle \leqslant w_{\varepsilon}\langle y, B\rangle$ if desired, and so on. We can also define inferiority for the fuzzy meta-sets A_ε and B_ε:

$$\leqslant_\varepsilon \langle A_\varepsilon, B_\varepsilon \rangle = \text{df } (\forall x)(w_\varepsilon\langle x, A\rangle \leqslant w_\varepsilon\langle x, B\rangle), \quad (5.2.3)$$

$$V(\leqslant_\varepsilon \langle A_\varepsilon, B_\varepsilon \rangle) = w_\leqslant\langle A_\varepsilon, B_\varepsilon\rangle. \quad (5.2.4)$$

The structure-similarity between theory and meta-theory provides useful theorems. For example, theorem 3.1.1 above suggests another meta-theoretic analogue:

Theorem 5.2.2. If $\leqslant_\varepsilon \langle x; A_u, B_u\rangle$ for all u, and $F\langle A_u\rangle$ is as defined in theorem 3.1.1, then $\leqslant_\varepsilon \langle x; F\langle A_u\rangle, F\langle B_u\rangle\rangle$.

Such meta-theory may be developed for *any* ordering relation $\square$ between the $\{\varepsilon\langle x, A\rangle\}$. (5.2.1) and (5.2.2) become

$$\square_\varepsilon \langle x; A, B\rangle = \text{df } w\varepsilon\langle x, A\rangle \ \square \ w_\varepsilon\langle x, B\rangle, \quad (5.2.5)$$

$$V(\square_\varepsilon \langle x; A, B\rangle) = w_\square\langle w_\varepsilon\langle x, A\rangle, w_\varepsilon\langle x, B\rangle\rangle. \quad (5.2.6)$$

The meta-theorem corresponding to theorem 5.2.1 is:

Theorem 5.2.2. If $\square_\varepsilon \langle x; A_u, B_u\rangle$ for all u, and $F_\square\langle A_u\rangle$ is a combination of fuzzy sets $\{A_u\}$ under some collection $\Delta_\square$ of the means of set combination defined in sub-section 2.4, then

$$\square_\varepsilon \langle x; F_\square\langle A_u\rangle, F_\square\langle B_u\rangle\rangle.$$

For example, when $\square$ is $\subseteq$ or $=$, Δ takes all values defined in subsection 2.4; the corresponding theorem in fuzzy set theory, given in [2], 154, excludes complementation for Δ when $\square$ is $\subseteq$, but the change in definition of complementation provided by (5.1.3) extends this meta-theorem.

The structure-similarity from theory to meta-theory allows us also to extend theorems on meta-ordering to combinations of fuzzy meta-sets. For example, corresponding to this theorem in fuzzy set theory:

$$\text{If } A \leqslant B \wedge C, \text{ then } A \leqslant B \,.\wedge.\, A \leqslant C, \tag{5.2.7}$$

we have the meta-theorem:

$$\text{If } \leqslant_\varepsilon \langle A_\varepsilon, B_\varepsilon \wedge C_\varepsilon \rangle, \text{ then}$$
$$\leqslant_\varepsilon \langle A_\varepsilon, B_\varepsilon \rangle \,.\wedge.\, \leqslant_\varepsilon \langle A_\varepsilon, C_\varepsilon \rangle. \quad [5.2.3)] \tag{5.2.8}$$

(For convenience I use the same symbol for 'and'.) There are very many such results, and are easy to generate when Δ is $\vee$, $\wedge$, $\cup$ and/or $\cap$, and $\square$ is $\leqslant$ and/or $\geqslant$. However, there is no clear underlying structure to the results when Δ is $+$, $\dot{-}$, $\times$ and/or $\div$, and $\square$ is $\subseteq$ and/or $\supseteq$.

5.3. *Special sets for meta-theory*. The applications of fuzzy set theory often revolve around the values of fuzzy set membership and the range of such values taken by some sub-set of the arguments. The use of interval membership can enhance the study of such questions, for they occur in interval analysis anyway (see [7], ch. 5 and elsewhere). We may define the following numbers and sub-sets of [0, 1] to express various properties of fuzzy sets:

Vagueness: $\text{vag}(A) = \text{df} \sup_x \{\text{wid } w_A(x) \mid \varepsilon\langle x, A\rangle\}$, [(2.4.2)] (5.3.1)

Specificity: $\text{spec}(A) = \text{df} \inf \{\text{wid } w_A(x) \mid \varepsilon\langle x, A\rangle\}$, (5.3.2)

Spread: $\text{spr}(A) = \text{df} \bigcup_x \{w_A(x) \mid \varepsilon\langle x, A\rangle\}$, (5.3.3)

Kernel: $\text{kern}(A) = \text{df} \bigcap_x \{w_A(x) \mid \varepsilon\langle x, A\rangle\}$. (5.3.4)

spr(A) may be a union of intervals while kern(A) may be empty. Ordering and other relations between these quantities can be examined in the usual way. Similar definitions can be given for fuzzy meta-sets, for example:

$$\text{vag}(A_\varepsilon) = \text{df} \sup_x \{\text{wid } w_\varepsilon \langle x, A\rangle\}. \tag{5.3.5}$$

There is a difference between (5.3.1) and (5.3.5) in that the former incorporates the condition $\varepsilon\langle x, A\rangle$. It could be omitted, since only defined interval values can be considered in the definition of vag(A) anyway; but alternatively we may seek conditions to be placed on the definition of vag(A_ε). This question touches on the meta-meta-theory of fuzzy set theory, on which I shall pass a few general remarks to conclude this section.

5.4. *Meta-meta-theory for fuzzy set theory.* The structure-similarity from theory to meta-theory can obviously be iterated into higher types of meta-theory. We can in principle ask how possible it is that $\varepsilon\langle x, A\rangle$, and so formulate the meta-meta-property $\varepsilon\langle\varepsilon; x, A\rangle$, with valuation

$$V(\varepsilon\langle\varepsilon; x, A\rangle) = w_\varepsilon\langle\varepsilon; x, A\rangle, \qquad (5.4.1)$$

fuzzy meta-meta-set $A_{\varepsilon_\varepsilon}$, and so on. (Strictly speaking, we should use a symbol different from 'ε' to denote the new kind of membership.) The various ordering relations may be defined, for example:

$$\subseteq\langle\leqslant; x; A, B\rangle =\text{df } V(\leqslant_\varepsilon\langle x, A\rangle) \subseteq V(\leqslant_\varepsilon\langle x, B\rangle). \quad [5.2.2)] \qquad (5.4.2)$$

I only wish to make the point that such forays into meta-meta-... theory are *possible* (and are fairly easy to execute), in contrast to such theory for most other (classical or non-classical) logics. The question of the merit or importance of such properties is not at issue here.

6. *Some conclusions*

Fuzzy set theory is properly of prime interest for its applications: my own proposal of interval membership in [2] was made chiefly to aid certain structural and computing features of the subject. However, the meta-theory is worth exploring, for it is rather different from the kinds of meta-theory to which we have accustomed. Some of the meta-properties defined — membership-possession, for example — are unusual, while the structural individuality of interval analysis leads to results of a novel kind concerning derivability and validity. Further, traditional questions for meta-theory, such as completeness and satisfaction, seem not to be prominent. Most striking of all is the fact that those parts of the meta-theory which are concerned with membership-possession are structurally similar to fuzzy set theory in being themselves fuzzy. This situation is reminiscent of Gödel's 'arithmeticisation' of the meta-theory of first-order arithmetic into a system which is structurally isomorphic with the arithmetic itself; but in this case the similarity can be carried over into higher types of meta-theory. A very interesting consequence is that at *no level of meta-meta-...* theory will a classical system, based on the law of excluded middle, be obtained.

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