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Complete (Rigorous) Induction & Fermat's Great Theorem.

0. In this paper, I am going to describe a non-traditional understanding of complete (rigorous) induction. I will follow the logical conception developed in my previous works, particulary [1 → 5]. But the ideas of this paper can be understood independently.

Afterwards I am going to use this understanding of complete induction for the logical analysis of the problem associated with Fermat's Great Theorem.

Here is a list of the symbols used, in the order of their mutual precedence, when appearing on one line:

(a) $\in$ — for inclusion of individuals into classes (sets).

(b) ⇻ — for inclusion of terms according to value;

(c) $\top$ — for the (unary) predicate of logical truth (provability);

(d) $\sim$, & , V , → (or ⟶) — for the operators of negation, conjunction, disjunction and implication in order of precedence;

(e) ∃...: , ∀...: — for the existential and universal quantifiers;

(f) $\vdash$ — for the (binary) predicate of logical inference.

The expression " *a* ⇻ *b* " is to be read as " Any object designated by *a* is designated by *b*. " ([1]). The expression " $x \vdash y$ " reads as " *y* follows (can be inferred from) *x* ". The expression " $\top\, x$ " reads as " *x* is logically true " or " *x* is provable ".

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[1]) Example: *dog* ⇻ *mammal*.

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1. I consider the Theory of Complete Induction as a part of Proof Theory. The latter represents a branch of logic in which properties of the predicate *provable* in combinations with other expressions are being established. From my point of view, there is only one source for the provability of sentences: definitions of linguistic expressions and their consequences. Special cases of those are implicit definitions of the form $\vdash$.

Below I shall formulate some additions to my Deduction Theory as a basic fragment of my Proof Theory.

1.1. *Additions to the definition of sentential formulas:*

(1) if x is a sentential formula, then so is $\mathrm{T}\, x$.

(2) if x and y are sentential formulas, then so is $x \vdash y$.

1.2. *Additional axioms (or axiom schemata):*

A(1) $\mathrm{T}\, x \vdash x$

A(2) $\mathrm{T}\, x \vdash \mathrm{T}\, \mathrm{T}\, x$

A(3) $\sim \mathrm{T}\, x \vdash \mathrm{T} \sim \mathrm{T}\, x$

A(4) $(x \vdash y) \vdash \mathrm{T}\, (x \vdash y)$

A(5) $\sim (x \vdash y) \vdash \mathrm{T} \sim (x \vdash y)$

A(6) $(x \vdash y) \vdash (\mathrm{T}\, x \vdash \mathrm{T}\, y)$

B(1) $\mathrm{T}\, (x \vee \sim x)$

B(2) $\mathrm{T}\, x \,\&\, \mathrm{T}\, y \vdash \mathrm{T}\, (x \,\&\, y)$

B(3) $\mathrm{T}\, x \vee \mathrm{T}\, y \vdash \mathrm{T}\, (x \vee y)$

C(1) $(x \vdash y) \vdash \mathrm{T}\, (x \rightarrow y)$

C(2) $(x \,\&\, z \vdash y) \,\&\, \sim (z \vdash y) \,\&\, \mathrm{T}\, z \vdash \mathrm{T}\, (x \rightarrow y)$

C(3) $\mathrm{T} \sim (x \rightarrow y) \vdash \sim (x \vdash y)$

$C(4)$ $\sim \text{T}\ (x \rightarrow y) \quad \vdash \quad \text{T} \sim (x \rightarrow y)$

$C(5)$ $\text{T} \sim (x \rightarrow y) \quad \vdash \quad \sim \text{T}\ (x \rightarrow y)$

$$D(1) \quad \exists\ Q: \left[\begin{array}{l} \text{T}\ x \rightarrow Q(\delta x)\ \& \\ \exists\ A:\ Q(\delta A)\ \&\ Q(\delta{\sim}B) \quad \vdash \quad \text{T} \sim (A \rightarrow B) \end{array}\right.$$

$$D(2) \quad \exists\ Q: \left[\begin{array}{l} \text{T}\ x \rightarrow Q(\delta x)\ \& \\ \exists\ A:\ Q(\delta A)\ \&\ \sim Q(\delta B) \quad \vdash \quad \text{T} \sim (A \rightarrow B) \end{array}\right.$$

where expressions of the form " (δx) " are to be read as " The sentence x ".

$E(1)$ $(W\ \&\ \text{T}\ x \ \vdash\ \text{T}\ y\ \&$

$W\ \&\ \text{T}\ y\ \vdash\ \text{T}\ x)$

$\vdash\ \&\ \text{T}\ z\ \vdash\ \text{T}\ v$

where v is obtained from z by replacing one or more occurrences x in z by y.

1.3. From my point of view the Deduction Theory must contain negative rules of the form $\sim (x \vdash y)$ and the Proof Theory must contain negative rules of the form $\sim \text{T}\ x$. I suggest the following basic negative rules:

(1) if a variable a occurs in y and does not occur in x, then: $\sim (x \vdash y)$

(2) if a variable a occurs freely in x and the statement " $\exists\ a: \sim x$ " is true then: $\sim \text{T}\ x$;
if variables $a^1,\dots,a^n$ $(n \geq 2)$ occur freely in x and " $\exists\ a^1 \dots a^n: \sim x$ " is true, then: $\sim \text{T}\ x$.

(3) if $\sim \text{T}\ x$, then: $(\sim \text{T}\ x\ \&\ y)$

(4) if $\sim (x \vdash y)$, then: $\sim (x \vdash y\ \&\ z)$

(5) if $\sim \text{T}\ (A \rightarrow B)$ and at least one variable occuring freely in A and in B does not occur in C, then $\sim \text{T}\ (A\ \&\ C \rightarrow B)$.

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2. In the Theory of Complete Induction, the properties of the universal quantifier in association with the provability predicate for ordered sets of terms must be established. I make the following preliminary assumptions :

(1) A term variable is a term.

(2) If a is a term from the value-range of variable b, then: $a \rightleftharpoons b$

(3) A group of terms is a special kind of term: if $a^1,\ldots,a^n$ $(n \geq 2)$ are terms, then $< a^1\ldots a^n >$ is a group of terms. The rules for groups of terms are:

(3_1)	$< a^1\ldots a^n > \rightleftharpoons < b^1\ldots b^n >$	$\vdash a^1 \rightleftharpoons b^1 \ \& \ \ldots \ \& \ a^n \rightleftharpoons b^n$
(3_2)	$a^1 \rightleftharpoons b^1 \ \& \ \ldots \ \& \ a^n \rightleftharpoons b^n$	$\vdash < a^1\ldots a^n > \rightleftharpoons < b^1\ldots b^n >$
(3_3)	$\forall < a^1\ldots a^n >: x$	$\vdash \forall a^1\ldots\forall a^n: x$
(3_4)	$\forall a^1\ldots\forall a^n: x$	$\vdash \forall < a^1\ldots a^n >: x$
(3_5)	$\exists < a^1\ldots a^n >: x$	$\vdash \exists a^1\ldots\exists a^n: x$
(3_6)	$\exists a^1\ldots\exists a^n: x$	$\vdash \exists < a^1\ldots a^n >: x$

The group of terms $< a^1\ldots a^n >$ is said to occur freely in x, iff all of $a^1,\ldots,a^n$ occur freely in x.
The substitution of $< a^1\ldots a^n >$ in place of $< b^1\ldots b^n >$ in x is the substitution of a^i (for $i = 1,\ldots,n$) in place of b^i, wherever b^i occurs freely in x.

2.1. *Degenerate Complete Induction:*

Let V be the sentence " The term a is an individual ". I accept the following rules for Degenerate Complete Induction:

A. $V \ \& \ \mathrm{T}\, x \ \vdash \ \mathrm{T}\, (\forall a: x)$

2.2. *Finite Complete Induction:*

Let M be a finite set. Let it be partitioned into a finite number of nonempty and pairwise disjoint subsets $M^1, \ldots, M^n$ $(n \geq 2)$.

I shall use the following notations:

(a) a is a variable for the terms belonging to M.

(b) ${}^{i}a$ is a variable for the terms belonging to M^i.

(c) ${}^{i}x$ is a sentence obtained from x by substituting ${}^{i}a$ for a wherever a occurs freely in x, and none of the ${}^{i}a$ occurs in x.

(d) W is the sentence " ${}^{i}a \neq a$ & ... & ${}^{n}a \neq a$, and $b \neq a \rightarrow (b \neq {}^{i}a \vee \ldots \vee b \neq {}^{n}a)$ is true for any term b ". I adopt the following rules for Finite Complete Induction:

B(1) $W \,\&\, \text{T}\,\forall\, {}^{i}a{:}\ {}^{i}x \,\&\, \ldots \,\&\, \text{T}\,\forall\, {}^{n}a{:}\ {}^{n}x \vdash \text{T}\,\forall\, a{:}\ x$

B(2) $W \,\&\, \text{T}\,\forall\, a{:}\ x \vdash \text{T}\,\forall\, {}^{1}a{:}\ {}^{1}x \,\&\, \ldots \,\&\, \text{T}\,\forall\, {}^{n}a{:}\ {}^{n}x$

2.3. *Infinite Non-conditional Complete Induction:*

Let M be a set of terms, partitioned into the non-empty, finite, pairwise disjoint subsets M^1, M^2, M^3,...

Let these subsets be ordered in the following way:

(1) M^1 is the first in the ordering.

(2) M^{i+1} directly follows M^i (i.e. M^{i+1} surpasses M^i in the ordering, and none of M^1, M^2, M^3,... lies between them).

Let there be rules by which we can construct all elements of M^{i+1} if only all the elements of M^i are given.

Assume that for all sets M^i $(i = 1,2,3\ldots)$ all elements of M^{i+1} can be constructed. This means that the sets M^1, M^2, M^3,... form an ordered series. Both the series M^1, M^2, M^3,... and M itself are infinite.

I use the following notations :

(a) a is a variable for the terms of M.

(b) x is a sentence in which a occurs freely.

(c) ${}^{i}a$ is a variable for the terms of M^{i} such that ${}^{i}a \neq a$, and ${}^{i}a$ does not occur in x.

(d) ${}^{i}x$ is a sentence obtained from x by substituting ${}^{i}a$ for all free occurrences of a.

(e) W is a sentence about the relations between ${}^{i}a$ and a (i = $1, 2, 3, \ldots$)

I adopt the following rules:

C(1)
$$W \,\&$$
$$\mathrm{T}\ (\forall\ a:\ x)\ \&$$
$$\mathrm{T}\ [\ (\forall\ {}^{i}a:\ {}^{i}x) \rightarrow (\forall\ {}^{i+1}a:\ {}^{i+1}x)\ ]$$
$$\vdash \mathrm{T}\ (\forall\ a:\ x)$$

C(2)
$$W \,\&$$
$$\mathrm{T}\ (\forall\ a:\ x)$$
$$\vdash \mathrm{T}\ (\forall\ a^{1}:\ {}^{1}x)\ \&$$
$$\mathrm{T}\ [\ (\forall\ {}^{i}a:\ {}^{i}x) \rightarrow (\forall {}^{i+1}a:\ {}^{i+1}x)\ ]$$

2.4. *Infinite Conditonal Complete Induction:*

Let p be a term variable such that the contents of M and its partition into M^{1}, M^{2}, M^{3}...depend on the value p. Assume that the value-range of p is the following:

(1) the first element in the ordering is given.

(2) for any value of p we can construct another value which follows it in the ordering.

Let us construct for any value of p all elements of M, and assume that we can construct all elements of M^{i+1} if all elements of M^{i} are given. The relation between M^{1}, M^{2}, M^{3},... is similar to the previous one. Let W^{*} be the sentence about the relations between p, a and ${}^{i}a$.

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I take the following rules of Infinite Conditional Complete Induction:

D(1) W^* &

$\mathrm{T}\ (\forall\ p,\ {}^1a:\ {}^1x)$

$\mathrm{T}\ [\ (\forall\ p,\ {}^ia:\ {}^ix) \rightarrow (\forall\ p,\ {}^{i+1}a:\ {}^{i+1}x)\ ]$

$\vdash \mathrm{T}\ (\forall\ p,\ a:\ x)$

D(2) W^* &

$\mathrm{T}\ (\forall\ p,\ a:\ x)$

$\vdash \mathrm{T}\ (\forall\ p,\ {}^1a:\ {}^1x)$ &

$\mathrm{T}\ [\ (\forall\ p,\ {}^ia:\ {}^ix) \rightarrow (\forall\ p\ {}^{i+1}a:\ {}^{i+1}x)\ ]$

Consequences of D:

D(3) W^* &

$[\ \sim \mathrm{T}\ (\forall\ p,\ {}^1a:\ {}^1x) \vee$

$\sim \mathrm{T}\ (\forall\ p,\ {}^ia:\ {}^ix) \rightarrow (\forall\ p,\ {}^{i+1}a:\ {}^{i+1}x]$

$\vdash\ \sim \mathrm{T}\ (\forall\ p,\ a:\ x)$

D(4) W^* &

$\mathrm{T}\ (\forall\ p,\ a:\ x)$

$\vdash \sim \mathrm{T}\ (\forall\ p,\ {}^1a:\ {}^1x) \vee$

$\sim \mathrm{T}\ [\ (\forall\ p,\ {}^ia:\ {}^ix) \rightarrow (\forall\ p,\ {}^{i+1}a:\ {}^{i+1}x)\ ]$

2.5. *Infinite Complete Induction with Limitation:*

Let D be a sentence such that:

$(D \vdash z \in M)\ \&\ (z \in M \vdash D)$

where z is a subject variable.

Let $b^1,\ldots,b^m$ be all terms included in M^{i+1}, and $B^1,\ldots,B^m$ sentences containing $b^1,\ldots,b^m$ respectively, such that the following statement [R] is true:

$\mathrm{T}\ [\ B^1\ \&\ \ldots\ \&\ B^m \rightarrow (\forall\ {}^{i+1}a:\ {}^{i+1}x)$ &

$$(\forall\ {}^{i+1}a:\ {}^{i+1}x) \rightarrow B^1 \ \&\ \dots\ \&\ B^m\ ] \qquad [R]$$

Let S be the statement:

$$\mathrm{T}[\ (\forall\ {}^{i}a:\ {}^{i}x) \rightarrow B^1\ \&\ \dots\ \&\ B^k\ ] \qquad [S]$$

where $1 \leq k \leq m$. Let $D^{k+1},\dots,D^m$ be all sentences obtained from D by replacing all free occurrences of z in D by $b^{k+1},\dots,b^m$ respectively. Let T be the statement:

$$\mathrm{T}\ (\forall\ {}^{i}a:\ {}^{i}x) \rightarrow B^{k+1} \lor \mathrm{T}\ (D^{k+1} \rightarrow B^{k+1})$$

$$\&\ \dots\ \&$$

$$\mathrm{T}\ (\forall\ {}^{i}a:\ {}^{i}x \rightarrow B^m) \lor \mathrm{T}\ (D^m \rightarrow B^m) \qquad [T]$$

I adopt the following rules:

$E(1)$ $\quad W\ \&\ R\ \&\ \mathrm{T}\ (\forall\ {}^{1}a:\ {}^{1}x)\ \&\ S\ \&\ T \vdash \forall\ a:\ x$

$E(2)$ $\quad W\ \&\ R\ \&\ \mathrm{T}\ (\forall\ {}^{1}a:\ {}^{1}x)\ \&\ S\ \&\ \mathrm{T}\ (\forall\ a:\ x) \vdash T$

$F(1)$ $\quad W^*\ \&\ R\ \&\ \mathrm{T}\ (\forall\ p,\ {}^{1}a:\ {}^{1}x)\ \&\ S\ \&\ T \vdash \forall\ p,\ a:\ x$

$F(2)$ $\quad W^*\ \&\ R\ \&\ \mathrm{T}\ (\forall\ p,\ {}^{1}a:\ {}^{1}x)\ \&\ S\ \&\ \mathrm{T}\ (\forall\ p,\ a:\ x) \vdash T$

Consequences of E:

$F(3)$ $\quad W\ \&\ R\ \&\ \mathrm{T}\ (\forall\ {}^{1}a:\ {}^{1}x)\ \&\ S\ \&\ \sim T \vdash \sim (\forall\ a:\ x)$

$F(4)$ $\quad W\ \&\ R\ \&\ \mathrm{T}\ (\forall\ {}^{1}a:\ {}^{1}x)\ \&\ S\ \&\ \sim (\forall\ a:\ x) \vdash \sim T$

3. I now consider:

> number as terms;

> signs of arithmetic operations as term-generating operators;

> the signs $<$, $>$, $=$ as the predicates indicating respectively the relations of preceding, following or being equal to, in the given ordering. Properties of these predicates are established in logical relations. It would be sufficient here to accept axioms $(1) \div (6)$ below (where the letters stand for subject variables). We use the expression " $a < b$ " as an equivalent of " $\sim b > a$ " and the expression " $a = b$ " as an abbreviation of " $\sim a > b\ \&\ b > a$ ".

(1) $\top \sim a > a$

(2) $a > b \quad \vdash \sim b > a$

(3) $\sim a > b \quad \vdash b > a \vee a = b$

(4) $a > b \;\&\; b > c \quad \vdash a > c$

(5) $a = b \;\&\; b > c \quad \vdash a > c$

(6) $a > b \;\&\; b = c \quad \vdash a > c$

I consider the following fragment of formal arithmetic, which is sufficient for examining the problem of Fermat's Great Theorem. I shall call it MFA *(Minimal Formal Arithmetic)*.

3.1. *Alphabet of* MFA:

(1) 1 : the constant number *one;*

(2) $+$, $\times$, $\uparrow$: the operators of addition, multiplication, exponentiation, in reverse order of precedence. I may write $a.b$ for $a \times b$, and a^b for $a \uparrow b$.

3.2. *Definition of a number in* MFA:

(1) 1 is a number in MFA;

(2) if $x_1, \ldots, x_n$ $(n \geq 2)$ are numbers in MFA, so are $x_1 + \ldots + x_n$ and $x_1 \times \ldots \times x_n$.

(3) if x and y are numbers in MFA, then so is $x \uparrow y$.

(4) something can be a number in MFA only by virtue of (1) → (3).

The formulas in MFA contain only numbers in MFA; the arithmetic predicates $<$, $>$, $=$; the logical operators $\sim$, $\&$, $\vee$, $\rightarrow$ and the logical predicates $\top$, $\vdash$.

3.3. *Axiom schemata of* MFA:

(1) $\top\; a+1 > a$

(2) $\top\; (a > b \rightarrow a > b+1 \vee a = b+1)$

(3) ⊤ $a_1+a_2+\ldots+a_n = b$

where b differs from $a_1+a_2+\ldots+a_n$ by any changes of operator precedences (rearrangement of brackets).

(4) ⊤ $a\times 1 = a$

(5) ⊤ $a \times (b+c) = (a\times b) + (a\times c)$

(6) ⊤ $a_1\times a_2\times\ldots\times a_n = b$

where b differs from $a_1\times\ldots\times a_n$ by altered operator precedences.

(7) ⊤ $a\uparrow 1 = a$

(8) ⊤ $a\uparrow(b+c) = a\uparrow b \times a\uparrow c$

It is obvious that for any number x in MFA, one can find a number y in MFA such that:

(a) ⊤ $x = y$ is a theorem in MFA;

(b) the operators × and ↑ do not occur in y.

In this sense, MFA is sufficient for our purpose.

3.4. I use the substraction operator only for the sake of simplicity in the presentation.

The expressions *2, 3, 4,...* are abbreviations respectively for *1+1, 1+1+1, 1+1+1+1*, etc...

The language of MFA contains no variable. Statements concerning the numbers of MFA are metastatements, but the formulas of MFA are not. The former contain letters that play the role of implicit variables.

3.5. I make the following additions to the Proof Theory for such expressions:

A. *Definition of numeric expression (num.exp.):*

(1) a number in MFA is a *num.exp.*;

(2) a variable for numbers in MFA is a *num.exp.*;

(3) if x_1, x_2,..., x_n are *num.exp.*, then $x_1+...+x_n$ is a *num.exp.*.

B. *Additional axiom schemata:*

(1) $a > b \;\&\; c > d \rightarrow a+c > b+d$

(2) $a > b \rightarrow a+c > b$

(3) $a > b \;\&\; c = d \rightarrow a+c > b+d$

C. *Additional negative rules:*

C₁. If e is a variable for numbers in MFA (or a pair, triplet, etc...) of such variables occuring freely in $c < d$, and if $\exists\, e: c < d$, then:

(1) $\sim \mathrm{T}\ (a > b \rightarrow a+c > b+d)$

(2) $\sim \mathrm{T}\ (a > b \rightarrow \sim a+c > b+d)$

(3) $\sim \mathrm{T}\ (a > b \rightarrow a+c < b+d)$

(4) $\sim \mathrm{T}\ (a > b \rightarrow \sim a+c < b+d)$

(5) $\sim \mathrm{T}\ (a > b \rightarrow a+c = b+d)$

(6) $\sim \mathrm{T}\ (a > b \rightarrow \sim a+c = b+d)$

C_2. If a variable for the numbers in MFA occurs in c, then:

(1) $\sim \mathrm{T}\ (a > b \rightarrow a > b+c)$

(2) $\sim \mathrm{T}\ (a > b \rightarrow \sim a > b+c)$

(3) $\sim \mathrm{T}\ (a > b \rightarrow a = b+c)$

(4) $\sim \mathrm{T}\ (a > b \rightarrow \sim a = b+c)$

C_3. If p is a variable for the numbers in MFA (or pair, triplet, etc... of such variables) occuring freely in $a < r \;\&\; c > t$, and if $\exists\, p: q < r \;\&\; s < t$, then:

(1) $\sim \mathrm{T}\ (a > b \;\&\; c > d \rightarrow a+q > b+r \;\&\; c+s > d+t)$

(2) $\sim \mathrm{T}\ (a > b \;\&\; c > d \rightarrow a+q < b+r \;\&\; c+s < d+t)$

(3) $\sim \mathrm{T}\ (a > b \;\&\; c > d \rightarrow a+q = b+r \;\&\; c+s = d+t)$

Axiom schemata of MFA are axiom schemata in the extended systems (MFA^*) given above.

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4. I shall now use the above Theory of Complete Induction for the analysis of the problem associated with Fermat's Great Theorem (*FGT*).

FGT is formulated in the following way:

If a, b, c, n are integers $\neq 0$, and $n \geq 3$, then:

$$\sim c^n = a^n + b^n \qquad [FGT]$$

It suffices to consider the case then the statement D is true:

$$c > a \ \& \ c > b \ \& \ a > n \ \& \ b > n \ \& \ c \text{ is odd } \& \ a \text{ is even } \& \ b \text{ is odd } \& \ n \geq 3 \qquad [D]$$

It is obvious that *FGT* is a metastatement relative to the numbers in MFA.

Let a, b, c, n be numeric variables satisfying restriction [D];

Let c_1, c_2, c_3,... be values of c;

a_1, a_2, a_3,... be values of a;

b_1, b_2, b_3,... be values of b;

Let $c_1 = n+3$ & $a_1 = n+2$ & $b_1 = n+1$ if n is even;

$c_1 = n+4$ & $a_1 = n+1$ & $b_1 = n+2$ if n is odd.

Other values of c, a, b are the following:

$$c_{i+1} = c_{i+2} \ \& \ a_{i+1} = a_{i+2} \ \& \ b_{i+1} = b_i+2$$

where $i = 1, 2, 3, \ldots$

We shall consider the set M of triplets of numbers $< c, a, b >$.

Let us divide M into subsets M^1, M^2, M^3,...in the following way:

(1₁) If n is odd, then M^1 contains just two triplets of numbers $< c_1, a_1, b_1 >$, i.e. $< n+3, n+2, n+1 >$.

(1_2) If n is odd, then M^1 contains just two triplets of numbers $< c_1, a_1, b_1 >$ and $< c_1, a_2, b_1 >$, i.e. $< n+4, n+1, n+2 >$ and $< n+4, n+3, n+2 >$.

(i) The subset M^i contains all such triplets of numbers $< c_i, a_r, b_s >$ that satisfy the condition:

(i_1) if n is even, then $a_1 \le a_2 \le a_i$;

(i_2) if n is odd, then $b_1 \le b_s \le b_i$.

If $n \ge 3$, then the formulas:

$$(n+3)^n > (n+2)^n + (n+1)^n$$

$$(n+4)^n > (n+1)^n + (n+2)^n$$

$$(n+4)^n > (n+3)^n + (n+2)^n$$

are provable, i.e. *FGT* is <u>true</u> for all elements of M^1.

Let *FGT* be true for all elements of M^i if $n \ge 3$. The case when $c^n < a^n + b^n$ for all elements of M^i is excluded. Two cases remain:

(1) $c^n > a^n + b^n$ for all elements of M^i;

(2) $c^n > a^n + b^n$ for some elements of M^i and $c^n < a^n + b^n$ for the remaining elements of M^i.

Thus we can formulate our assumption in the form:

* *FGT* is provable (possibility *A*) or:
* the following statements are provable (possibility *B*):

$$c_i^n > a_i^n + b_i^n$$

...

$$c_i^n > a_k^n + b_i^n$$

where a_k is a number from the value-range of a, such that:

$$a_1 \le a_k \le c_i - 3;$$

$$c_i^n > (a_k+2)^n + b_{l_1}^n \ \& \ c_i^n < (a_k+2)^n + (b_{l_1}+2)^n$$

...

$$c_i^n > (a_k+2.m)^n + b_{l_m}^n \;\&\; c_i^n < (a_k+2.m)^n + (b_{l_m}+2)^n$$

where:

> $m \geq 1$;

> $a_{k+2}, \ldots, a_k+2.m$ are numbers from the value-range of a, such that $(a_k+2.m) < c_i-1$;

> $b_{l_1}, \ldots, b_{l_m}$ are numbers from the value-range of b such that $b_r \leq b_i$ (possibility B).

Let us consider the sequences following from the assumption $A \vee B$ for the set M^{i+1}. Let the possibility B be true. From A follow the statements:

$$(c_i+2)^n > (c_i+1)^n + (c_i-2)^n$$
$$(c_i+2)^n > (c_i-1)^n + c_i^n$$

The only element of M^{i+1}, which is not included in the above cases is:

$$< c_i+2,\ c_1+1,\ c_i >$$

Consider possibility B. From B follow the statements:

$$(c_i+2)^n > a_1^n + (b_i+2)^n$$
$$\ldots$$
$$(c_i+2)^n > a_k^n + (b_i+2)^n$$

$$(c_i+2)^n > (a_k+2)^n + (b_{l_1}+2)^n \;\&\; (c_i+2)^n < (a_k+4)^n + (b_{l_1}+4)^n$$
$$\ldots$$
$$(c_i+2)^n > (a_k+2m)^n+(b_{l_n}+2)^n \;\&\; (c_i+2)^n < (a_k+2m+2)^n+(b_{l_m}+4)^n$$
$$(c_i+2)^n > (c_i-1)^n + (b_r+2)^n \;\&\; (c_i+2)^n < (c_i+1)^n + (b_r+4)^n$$

$$(c_i+2)^n > (a_k+4)^n + b_{l_1}^n$$

...

$$(c_i+2)^n > (a_k+2.m+2)^n + b_{l_m}^n$$

$$(c_i+2)^n > (c_i+1)^n + b_r^n$$

The following elements of M^{i+1}

$$< c_i+2,\ a_k+2,\ b_{l_1}+4 >,\ < c_i+2,\ a_k+4,\ b_{l_1}+2 >$$

...

$$< c_i+2,\ a_k+2.m,\ b_{l_m}+4 >,\ < c_i+2,\ a_k+2.m+2,\ b_{l_m}+2 >$$

$$< c_i+2,\ c_{i-1},\ b_r+4 >,\ < c_i+2,\ c_{i+1},\ b_r+2 >$$

have no similar consequences. I designate them by E.

In case B it is obvious that

$$c_i^n < (a_k+2)^n + (b_{l_1}+2)^n$$

...

$$c_i^n < (a_k+2)^n + b_i^n$$

The special case is $b_{l_1}+2 = b_i$.

It is also obvious that we have not considered the triplets of numbers:

$$< c_i+2,\ a_k+4,\ b_{l_1}+2 >$$

...

$$< c_i+2,\ a_k+4,\ b_i >$$

I introduce the notation:

(1) V : for the statement " $\sim (c_i+2)^n = (c_i+1)^n + c_i^n$ ";

(2) D^A : for the statement " c_i *is odd* & $n \geq 3$ & $c_i > n$ ";

(3) W : for statements containing the triplets of numbers E;

(4) D^B : for the statement " c_i is odd & a_k is even & b_{l_1} is

odd & ... & b_{l_m} is odd & b_r is odd & $a_k+2m < c_i+2$ & $a_k+2 > n$ & $b_{1_1} > n$ & ... & $b_{1_m} > n$ & $b_r > n$ & $n \geq 3$ ".

The task is now reduced to the following problem:

(1) to find out if V follows from A or not, and if V follows from D^A or not;

(2) to find out if all of W follow from B or not and if all of D^B follow from D^B or not.

By virtue of rules of MFA^*, expressions of the form $x^n \lesseqgtr y^n + z^n$ (where $\lesseqgtr$ is $>$, $<$ or $=$) reduce to such expressions as $A \lesseqgtr B$, where A and B contain only number variables, 1 and $+$.

The relations between assumptions A and B (on one hand) and their consequences for M^{i+1} (on the other hand) are reduced to the following:

if an assumption is $x > y$ (or $x < y$) then its consequences are:

$z > u$ (resp. $z < u$)

where $z = x + \alpha$, $u = y + \beta$ and $\alpha > \beta$ (resp. $\alpha < \beta$).

In short, all consequences of A and B are due to the rules:

$\mathrm{T}\ (a > b \ \&\ \alpha > \beta \ \rightarrow\ a + \alpha > b + \beta)$

$\mathrm{T}\ (a < b \ \&\ \alpha < \beta \ \rightarrow\ a + \alpha < b + \beta)$ etc...

And any other consequence of A and B is not valued.

But the statements V and W are exceptions. In case A we have:

$$\exists\, c_i,\ n:\ (c_i+2)^n - c_i^n < (c_i+1)^n + c_i^n - (c_i-1)^n - (c_i-2)^n$$

$$\sim \mathrm{T}\ [\ c_i^n > (c_i-1)^n + (c_i-2)^n \ \rightarrow\ \sim (c_i+2)^n = (c_i+1)^n + c_i^n\ ]$$

In case B, the statement is the following. Let us consider the statement:

$$\sim (c_i+2)^n = (c_i+1)^n + (b_r+2)^n$$

The only line in B containing the letter b_r is:

$$c_i^n > (c_i-1)^n + b_r^n \ \& \ c_i^n < (c_i-1)^n + (b_r+2)^n$$

We have:

$$\exists\ c_i,\ b,\ r_n:\ (c_i+2)^n - c_i^n < (c_i+1)^n + (b_r+2)^n - (c_i-1)^n - b_r^n$$

$$\sim \top\ [\ c_i^n > (c_i-1)^n + b_r^n \ \rightarrow\ \sim (c_i+2)^n = (c_i+1)^n + (b_r+2)^n\ ]$$

$$\forall\ c_i,\ b_r,\ n:\ (c_i+2)^n - c_i^n < (c_i+1)^n + (b_r+2)^n - (c_i-1)^n - (b_r+2)^n$$

$$\sim \top\ [\ c_i^n < (c_i-1)^n + (b_r+2)^n \rightarrow \sim (c_i+2)^n = (c_i+1)^n + (b_r+2)^n\ ]$$

$$\exists\ c_i,\ b_r,\ n:\ (c_i+2)^n - c_i^n < (c_i+1)^n + (b_r+2)^n - (c_i-1)^n - b_r^n \ \&$$

$$(c_i+2)^n - c_i^n > (c_i+1)^n + (b_r+2)^n - (c_i-1)^n - (b_r+2)^n$$

$$\sim \top\ [\ c_i^n > (c_i-1)^n + b_r^n \ \&\ c_i^n < (c_i-1)^n + (b_r+2)^n \rightarrow \sim (c_i+2)^n = (c_i+1)^n + (b_r+2)^n\ ]$$

We shall consider now the relations between D^A and V, and that between D^B and W. I adopt the following rules of interpretation:

(1) if y is a letter, then x^y is translated as $x+y$;

(2) if a is a letter, and a occurs in $(b_1+b_2+...+b_n)$ more than once, then all its occurrences except one are striked out;

(3) after that the values $1, 2, 3, ...$ are ascribed to the letters;

(4) if x and y are letters, if the value α is ascribed to x and the value β to y, then the value $max(\alpha,\beta)$ is ascribed to $x+y$.

Consider the expression:

$$(c_i+2)^n = (c_i+1)^n + c_i^n$$

According to condition *D*, we may write it as:

$$(2.k+3)^n = (2.k+2)^n + (2.k+1)^n$$

According to the accepted rules, we have:

$$2.k + 3 + n = 2.k + 2 + n + 2.k + 1 + n$$

$$2.k + 3 + n = 2.k + 3 + n$$

We can ascribe to it the value *T*. We can also ascribe the value *T* to D^A and to all theorems of logical systems independently of the accepted rules, because number variables do not occur in these theorems. So we can ascribe the value *T* to the expressions:

$$\sim \sim (c_i+2)^n = (c_i+1)^n + c_i^n$$

And according to rule *D* of the Proof Theory, we have:

$$\sim \mathrm{T}\ (D^A \rightarrow V)$$

Similarly, in case *B*, we can ascribe the value *T* to the expression:

$$(c_i+2)^n = (a_k+4)^n + b_i^n$$

where $c_i = b_i+2$. We have:

$$(2.k+5)^n = (a_k+4)^n + (2.k+1)^n$$

$$2.k + 5 + n = 2.k + 5 + n$$

So that we have:

$$\sim \mathrm{T}\ (D^B \rightarrow W)$$

According to rule *E(3)* of the Complete Induction, we obtain:

$$\sim \mathrm{T}\ \forall\ c,\ a,\ b: \sim c^n = a^n + b^n$$

And according to rule *A(3)* of the Proof Theory, we have:

$$\mathrm{T} \sim \mathrm{T}\ \forall\ c,\ a,\ b: \sim c^n = a^n + b^n$$

This means that *FGT* is unprovable! □

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