COMPLETENESS OF SOME QUANTIFIED MODAL LOGICS

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0. Introduction

We give here simple completeness proofs for a wide range of quantified modal logics. The systems in question have the axioms and rules of some standard system of propositional modal logic, axioms for intensional identity, and quantificational principles drawn from free logic. The semantics we give is more general than usual, and perhaps that explains the simplicity of the completeness results.

Usually a semantics for quantified modal logic involves the introduction of a function D which assigns to each possible world d the set D(d) of individuals which «exist» in that world. We may think of D as the intension of a corresponding existence predicate 'E' in the object language.

In our semantics, we introduce a function Ψ, which assigns to each possible world d a set Ψ(d) of individual concepts, or world lines (i.e., functions from the set of possible worlds to the set of individuals). In this way we agree with Montague's [5] and Bressan's [1] more general treatment of the intension of a predicate in our description of the intension of 'E'. (1) As a result, the term position in 'E' becomes opaque, and not subject to substitution of identites. (2)

(1) This treatment is motivated by Montague [5] on p. 247 and following, and by Bressan [1] on p. 24 and following.

(2) It is possible to add other predicates which are given the more general semantical treatment of 'E' to the systems of this paper. The only changes required involve restriction of the substitution of identities to term -places of predicates which are not given the more general interpretation.

1. Formal Motivation

Why should we want to generalize the treatment of the intension of 'E' in this way ? The first set of reasons are formal. When D is used, there are several ways of introducing the truth definition for the quantifiers, and it is not easy to decide which is best. (3) At least one of these results in a definition of validity which is not axiomitizable (system Q2). On another definition (system Q3) the completeness results are highly sensitive to the strength of the modal operator. Strategies which work (say) to show the completeness of S4 strength quantified modal logics (for example those used to show completeness of Q3-S4) are not applicable to the weaker systems such as Q3-M or Q3-K. That difficulty does not arise with the more general interpretation of the quantifiers we give here. One simple strategy serves to provide results for any quantified modal logic whose modal fragment is determined by some non-empty class of Kripke frames, and for which a standard Henkin style completeness proof is available. In short, the methods of this paper yield simple completeness proofs for quantified versions of the vast majority of studied systems of modal logic.

2. Informal Motivation

Are there any reasons other than formal for accepting the approach of this paper ? I believe there are philosophical intuitions which recommend it, particularly when the quantified modal logics at issue are tense logics. It is generally felt that it is not part of the province of logic to determine which of the competing ontological theories is correct. It follows that semantics for quantified modal logics should not rule out any reasonable ontological theory.

(3) Thomason discusses these options in [6] pp. 134-140, and Garson [2] discusses similar matters for another intensional logic.

There is a venerable and pervasive ontology, which we might call ordinary onotology, or Aristotelian ontology, which seems to require the approach we have taken to the intension of the existence predicate. It is basic to this position that there are objects, and that objects change. So the formal counterpart of an object in a tense logic semantics must be a timè-worm or individual concept, and not a slice of a time worm, since it makes no sense to say of the latter that it changes. In a tense logic then, an object should be treated as a function from times into temporal slices of objects (or point-events), and hence objects are the intensions, not the extensions of the terms.

Now what does this have to do with whether the intension of 'E' should yield at time t a set of individuals or a set of individual concepts ? We know that there are many predicats P which are extensional, which means that the calculation of the truth-value of 'Pn' (say at t) depends soley on whether the time slice (individual) refered to by 'n' at t fails in the extension of 'P' (at t), and does not otherwise involve the intension of 'n'. Couldn't it turn out that 'E' is one of these extensional predicates ? I think there are stong intuitions which rule against this and which show that at least one rather basic treatment of existence in temporal situations requires the intensional approach.

To see this, imagine what we would say (at t) if a temporal slice (of finite, but small duration, if you like) of an object (say a slice of Gerald Ford) were to be present at t, in a context where the preceding and subsequent temporal slices of Gerald Ford were absent. Now it is true that we might be inclined to say that *something* existed at t, namely a temporal slice of Gerald Ford, or a fleeting «apparition» of Gerald Ford. It would be strange, however, to claim that Gerald Ford existed at t under these circumstances, for part of what makes us want to say that Gerald Ford exists at t, is the existence of previous and subsequent stages of Gerald Ford arranged in a certain coherent fashion.

This result should not be surprizing. The ordinary ontology takes objects as basic, so questions about existence and nonexistence of objects should be questions about objects and not about certain of their temporal parts. So it follows that if objects are to be treated as functions from times to times slices, then the referent of 'E' at a time t should be a set of these sorts of functions, and not a set of time slices.

3. The predicate 'E'

The predicate 'E' is included as a primitive of our systems. Its primitive inclusion is important to the generality of the results. In free logc, and in modal logics where the domain of quantification shifts from possible world to possible world, the usual axiom of universal instantiation is invalid, since it is possible for $\forall xA$ to be true at world d, and for An/x to be false when 'n' denotes something outside the domain of quantification for d. If we conditionalize the consequent of this axiom, a valid formula results, when the right condition is chosen. The condition we want is a formula which expresses that 'n' refers to some «object» in the domain of quantification for d. In free logic '$\exists xx=n$' serves that purpose. However, this formula will not work in an intensional logic which gives '=' the intensional or weak interpretation of identitiy, and which includes our approach to existence. That is because we now need a condition which says that the *function* refered to by 'n' falls into the domain $\Psi(d)$ of functions for the world d. But '$\exists xx=n$' says something weaker, namely that there is a function g such that g(d) is the extension of n at d. It does not improve matters to replace '$\exists xx=n$' with '$\exists x\Box x=n$', unless the underlying modal system is as strong as S4. (4) For '$\exists x\Box x=n$' says only that there is a function $g \in \Psi(d)$ which agrees with the intension of n at those arguments in the range of the Kripke relation R at d.

(4) When the modal logic is as strong as S4, $\forall x\Box Fx \supset (\exists x\Box x = n \supset \Box Fn)$ becomes valid on our semantics, in spite of the fact that $\exists x\Box x = n$ does not express that the function refered to by 'n' at d falls in $\Psi(d)$.

4. Rigidity of variables

One of the major simplifying assumptions made in Q3, Thomason's [6] approach to quantified modal logic, is to assume that the intensions of the variables are constant functions, that is to say, that their extensions do not change from possible world to possible world. On the other hand, he allows the terms to be non-rigid. This leads to formal and philosophical akwardness.

The formal akwardness is that the domain of quantification becomes the set of constant functions, or as Thomason believes he should call them, the substances. This invalidates the usual axiom for universal generalization. By adding an instantiation condition '$\exists x \Box x = n$', a valid axiom

A4' $\forall x A \supset (\exists x \Box x = n \supset Anx)$

results, but only when the underlying modal logic is S4 or stronger. In logics of weaker modality, A4' is invalid, and it is difficult to specify the instantiation condition that is needed for a sound and complete system. Hintikka [4] describes this matter in detail and gives the complicated axioms which must be used. Thomason's refusal to consider weaker modalities saves him from considerable complications, and even so, the rules for his system Q3 are elaborate.

Insisting on the rigidity of variables leads to formal complications, but I doubt that it is acceptable on philosophical grounds either. Thomason motivates his decision to «rigidify» the variables by claiming that this allows us to incorporate a concept of substance into modal logic. In choosing our variables to be rigid designators, we are supposed to get the effect of quantifying just over the substances. But why should we conclude that the substances are exactly the intensions of rigid designators. It is at the very least inconvenient to do so.

First, the insistence that a term for a substance refers to the same individual in each possible world immediately poses the problem of providing identity conditions for individuals across possible worlds. Is it sensible to assume that Plato (which I take to be a substance for the present discussion) in this world is exactly identical to the Plato of some other possible world ? Well, not unless Plato's identity does not depen don his accidental properties. But if trans-world identification is to depend only on essential properties, how can we be sure that there aren't possible worlds where two substances (say Aristotle and Plato) have exactly the some essence, and so are identical in that world, with the result that they must be ruled identical in all worlds ?

The inconvenience becomes more apparent when we give modal logics a tense interpretation. We would like to quantify over Gerald Ford as a member of our ontology, yet the extension of 'Gerald Ford' is not constant across times, since each time slice of his has its own peculiar properties. Insistence on the myth that 'Gerald Ford' *is* after all a rigid designator provokes a fruitless search for «something» which remains constant in all these time slices.

The difference between substances and other «things» is undoubtedly too subtle to be captured via the simple notion of the rigidity of designators. In fact, in order to be fair to any system of ontology, the best thing to do would be to make no assumptions at all about the semantical properties of the terms that happen to refer to substances.

We now find another reason for choosing the general approach of this paper. Our existence predicate E has a set of individual concepts as its extension, and this set can serve admirably as the boundary between substances and non-substances for those ontologies which bother to make the distinction. Since there are no conditions on this set in our definition of a model, any sort of ontology of substances may be accomodated. Since the substances so identified are not necessarily constant functions, the problem of trans-world identification does not need to bother us unless the ontology in question requires such a concept. In this way we leave the problem to the ontologist, not the logician.

Special logics designed to capture assumptions about substance or any other interesting sorts for quantification can be constructed by putting appropriate conditions on the semantical behavior of E in our definition of a model. So Thomason's approach can be reconstructed within ours. If we want a logic which does not close ontological issues, however, we should choose the generality of the approach of this paper.

4. Axiomatic Features of MQ

We will demonstrate the completeness of a system MQ, written in a morphology L for quantified modal logic with identity, which includes the primative predicate constant 'E' for 'exists'. MQ contains the principles of some propositonal modal logic M for which a standard Henkin-style completeness proof is available. (By standard, we mean to include that the definition of the Kripke relation R in the canonical model is given in the standard way: Rdc iff $\{A:\Box A\varepsilon d\}\subseteq c$.) It includes as well axioms for intensional identity:

(A=) $\vdash t=t$
(AS) $\vdash (s=t \supset (F(s) \supset F(t))$, where F(s) is atomic and F(t) is the result of replacing t for some occurrence of s in F(s).

A formula is atomic just in case it contains no logical constants. We consider 'E' to be a logical constant, so substitution guaranteed by (AS) does not apply to 'E'. MQ also contains the following quantificational principles drawn from free logic:

($\forall$G) $$\frac{\vdash (A \supset (Ex\supset B))}{\vdash (A\supset \forall xB)}$$

($\forall$I) $\vdash (\forall xA\supset(Et\supset At/x))$.

At/x is the result of replacing any term t properly for x wherever x appears in A.

When Γ is a set of formulas, '$\Gamma\vdash A$' means that either A is provable in MQ, or there is some conjunction G of members of Γ such that $\vdash_{MQ}(G\supset A)$. We use '$\bot$' for some preselected contradiction of propositional logic.

Let T be any set of terms of L, and let F(T) be the set of formulas of the morphology like L save that it has its only terms the members of T. Let T_Γ be the set of terms that appear in the set of formulas Γ.

A set of formulas Δ is a *MQ-set* iff for every formula A of $F(T_\Delta)$ i. $\Delta \nvdash \bot$, ii. if $A \notin \Delta$, then $\Delta \cup \{A\} \vdash \bot$, and iii. if $\sim \forall xA \in \Delta$, then there is a term t in T_Δ such that $(St \& \sim At/x) \in \Delta$. A MQ-set is just a saturated set with respect to the set $F(T_\Delta)$ of formulas of the morphology which contains only terms found in Δ, where saturation is defined as is appropriate for free logic.

MQ-set Lemma: If Γ is a set of formulas such that $\Gamma \nvdash \bot$, and there is an infinite set of terms T of L, none of which is in T_Γ, then there is a MQ-set Δ such that $\Gamma \subseteq \Delta$, and $\Delta \subseteq F(T_\Gamma \cup T)$.

Proof: Similar to the proof of the Lindenbaum Lemma.

MQ-sets obey the usual properties of saturated sets when attention is restricted to the formulas of their corresponding morphologies. In particular

P. When Δ is a MQ-set, the following properties hold for all $A, B \in F(T_\Delta)$:

1. if $\Delta \vdash A$, then $A \in \Delta$.
2. $\sim A \in \Delta$ iff $A \notin \Delta$.
3. $(A \supset B) \in \Delta$ iff $A \notin \Delta$ or $B \in \Delta$.
4. $\forall xA \in \Delta$ iff for all terms $t \in T_\Delta$, $(Et \supset Atx) \in \Delta$.

5. Semantics for MQ

An M-model U for L is a quintuple $\langle D, R, I, \Psi, u \rangle$, where $\langle D, R \rangle$ is a member of the set of Kripke frames which deter-

([5]) Following Hansson and Gardenfors [3] a Kripke frame is a pair $\langle D, R \rangle$ consisting of a non-empty domain D (of possible worlds) and a binary relation R on D. A propositional modal logic M is determined by mine M, [5] I is a non-empty set (of individuals), ψ is a function which assigns to each $d \in D$ a subset of the set of functions from D into I, and u is an interpretation function which assigns to each term t a function u(t) from D into I, and which assigns to each j-ary predicate letter P^j (other than E) a function from D into the power set of I^j. (We let $I^0 = \{\varnothing\}$, where $\varnothing$ is the null set, so that in the case of a propositional variable P^0, $u(P^0)(d)$ is $\{\varnothing\} = 1$ or $\varnothing$ (true of false).

We define '$U \vDash_d A$' (read 'A is true on U at d') as follows:

0) $U \vDash_d Et$ iff $u(t) \in \Phi(d)$
1) $U \vDash_d P^j t_1 \ldots t_j$ iff $< u(t_1)(d), \ldots, u(t_j)(d) > \in u(P^j)(d)$
2) $U \vDash_d s = t$ iff $u(s)(d)$ is $u(t)(d)$
3) $U \vDash_d \sim A$ iff not $U \vDash_d A$
4) $U \vDash_d (A \supset B)$ iff not $U \vDash_d A$ or $U \vDash_d B$
5) $U \vDash_d \forall xA$ iffUf/x or $f/x \vDash_d A$ for all $f \in U(d)$
   (Uf/x the model identical to U save that its interpretation function assigns f to x.)
6) $U \vDash_d \Box A$ iff for all $e \in D$, if Rde, then $U \vDash_e A$.

A set of formulas Γ of morphology L is MQ-satisfiable in L just in case there is an M-model for L en which each of the members of Γ is true. Validity is defined from satisfaction in the usual way.

It is a simple matter to show by induction that

S. $U^{u(t)}/_x \vDash_d A$ iff $U \vDash_d Atx$.

6. The completeness of MQ

Let us define a canonical model U_{Can} (or U for short) as the quintuple $< D, R, I, \Psi, u >$, where i. $d \in D$ iff d is a MQ-set, and there is an infinite set of terms of L not in T_d, and ii. Rdc iff $\{A : \Box A \in d\} \subseteq c$, and iii. $u(t)(d)$ is $\{s : s = t \in d\}$, and iv. $u(P^j)(d)$

a class K of Kripke frames just in case for every member $< D, R >$ of K, every model for $< D, R >$ satisfies all and only the theorems of M.

is $\{<u(t_1)(d), \ldots, u(t_j)(d)>: P^jt_1 \ldots t_j \in d\}$, and v. $i \in I$ iff $u(t)(d)$ is i for some term t of L and $d \in D$, and vi. $f \in \Psi(d)$ iff u(t) is f and $Et \in d$ for some term t of L.

If follows from this definition that

E. $Et \in d$ iff $u(t) \in U(d)$

Proof: The proof from left to right is trivial. Now suppose that $u(t) \in U(d)$. Then for some t', u(t) is u(t') and $Et' \in d$. But u(t) is u(t') only when t is t', for when t is not t' there is always a MQ-set $e \varepsilon D$ where $t \in T_e$ and $t' \notin T_e$, with the result that u(t)(e) is not $\varnothing$, while u(t')(e) is. It follows then that $Et \in d$.

We are not ready to prove

The Model Lemma For every formula A and each $d \in D$, if $A \in F(T_d)$, then $U_{Can} \models_d A$ iff $A \in d$.

The proof is by induction on the form of A. Cases for formulas having forms other than $\Box B$ are easy to provide given P., S., and E.. So we turn to the case for formulas having the form $\Box B$. Suppose $d \in D$ and $\Box B \in F(T_d)$. Now assume $U \models_d \Box B$. Then

(1) for all $c \in D$, if Rdc, then $U \models_c B$.

Now let $d_\Box$ be $\{A : \Box A \in d\}$, and suppose for *reductio* that $d_\Box \nvdash B$. Then $d_\Box \cup \{\sim B\} \nvdash \bot$. Since $d \in D$, there is an infinite set T of terms foreign to c, and since the members of $d_\Box \cup \{\sim B\}$ are all in $F(T_d)$, no member of T appears in $d_\Box \cup \{\sim B\}$. Now form two infinite disjoint sets T_1, T_2, such that T is $T_1 \cup T_2$. [6] By the MQ-set Lemma, there is an MQ-set e such that $d_\Box \cup \{\sim B\} \subseteq e$ and $e \subseteq F(T_d \cup T_1)$. There is an infinite set

[6] To do this, order T; then let T_1 consist of odd, and T_2 of even members of the ordering.

of terms (namely T_2) not in T_e, so $e \in D$. Furthermore Rde and $\sim B \in e$. Since $\Box B \in F(T_d)$, $B \in F(T_d)$, so by P. it follows that $B \notin e$. By the hypothesis of the induction and $B \in F(T_d)$ we conclude that not $U \models_e B$. So there is a member e of D such that Rde and not $U \models_e B$, which contradicts (1). Our *reductio* is complete, and so we know that $d_{\Box} \vdash B$. It follows that either $\vdash B$ or there are formulas $\Box A_1, \ldots, \Box A_n \in d$ such that $\vdash (A_1 \& \ldots \& A_n) \supset B$. In the first case $\vdash \Box B$, hence $\Box B \in d$ by P., and in the second case it follows by the principles of M (which is as strong as K) that $\vdash (\Box A_1 \& \ldots \& \Box A_n) \supset \Box B$. So $d \vdash \Box B$, and by P., $\Box B \in d$.

Now let e be any member of D and suppose that $\Box B \in d$ and Rde. Then $B \in e$, and so $B \in F(T_e)$. We employ the hypothesis of the induction and this last result to obtain $U \models_e B$. It follows that $U \models_d \Box B$.

Theorem. MQ is strongly complete.

Proof. We show that any consistent set of formulas is MQ-satisfiable. Let Γ be any consistent set of formulas in morphology L. Let L' be the morphology that results from adding to the terms of L an infinite set T of terms foreign to L. Now divide T into two disjoint infinite sets T_1, T_2. (6) By the MQ-set Lemma, there is an extension Δ of Γ such that $\Delta \subseteq F(T_{\Gamma} \cup T_1)$.

Now consider U_{Can}. We may use the principles of M to show that its Kripke frame $\langle D, R \rangle$ is a member of the class of Kripke frames which determines M, by reviewing the appropriate portion of the completeness proof for M, since the definition given for R in U_{Can} is standard. So U_{Can} is an M-model. Now there is an infinite set (namely T_2) of variables of L' which are not in T_Δ, and so it follows that $\Delta \in D$. By the Model Lemma, $A \in \Delta$ iff $U \models_\Delta A$. By forming the model like U_{Can} save that its interpretation function is restricted to the terms of L, we produce an M-model for L which satisfies the set of formulas of Δ which are written in L. Since Γ is a subset of this set, Γ ust be MQ-satisfiable in L.

We have been somewhat detailed in our exposition of this proof. That is because we have run a considerable risk of failure in introducing saturated sets which do not contain all the terms of our language. It was therefore important to show that the proof works in spite of the use of such improvershed sets, and so we paid particular attention to how mention of the language plays a role in this proof.

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