A TENSE SYSTEM WITH SPLIT TRUTH TABLE*

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Abstract

On his thesis [3], J.A.W. Kamp introduced a tense system *US* for real numbers time with the two *binary* connectives *Since* and *Until*. Kamp showed that these two connectives are strong enough to express any other n-place tense connective for any n. (See remark 10 below for the precise result). This paper introduces a tense system τ for real numbers time with the two *unary* operators *F* (*will*) and *P* (*was*) that is equivalent in strength to Kamp's system. In fact, *Since* and *Until* are definable in τ using *F* and *P*.

We furthermore show that the operators *F* and *P* of τ are intuitive in the sense that they are used in English.

1. *Background and Statement of Results.*

Let $(\lambda, <)$ be the linearly ordered real numbers flow of time, with $<$ ir-reflexive and transitive earlier-later relation. Let L_0 be a propositional tense logic with the usual classical connectives $\sim$, &, $\vee$, $\rightarrow$ and the additional two binary tense connectives $U(A, B)$ and $S(A, B)$ (reading «B until A» and «B since A» respectively). L_0 is interpreted semantically as follows (see [3]). If g is a function assigning for each atomic p a subset $g(p) \subseteq \lambda$, then in $(\lambda, <, g)$ the truth value of a wff A at t denoted by $||A||^g_t$ is defined by induction as follows (1 is truth, O is falsity). (We omit g when possible.)

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(1) $||p||_t = 1$ iff $t \in g(p)$, p atomic
$||\sim A||_t = 1$ iff $||A||_t = 0$.
$||A \vee B||_t = 1$ iff $||A||_t = 1$ or $||B||_t = 1$
$||A \& B||_t = 1$ iff $||A||_t = ||B||_t = 1$.
$||A \rightarrow B||_t = 1$ iff $||A||_t = 0$ or $||B||_t = 1$.

(2) $||S(A, B)||_t = 1$ iff for some $s < t$, $||A||_s = 1$ and for all t', if $s < t' < t$ then $||B||_{t'} = 1$.

(3) $||U(A, B||_t = 1$ iff for some $s > t$, $||A||_s = 1$ and for all t', if $t < t' < s$ then $||B||_{t'} = 1$.

The usual tense operators P^+A and F^+A are definable by

(4) (a) $P^+A = S(A, A \rightarrow A)$.
(b) $F^+A = U(A, A \rightarrow A)$.

Let *US* be the set of all wff valid in every t of $(\lambda, <)$ and every assignment g. That is for all A,
$A \in US$ iff for all $t \in \lambda$ and all g, $||A||^g_t = 1$.

An axiomatization $\alpha(US)$ of *US* was given by Kamp and is quoted in Hoepelman [4].
We thus have, for the finite axiomatization $\alpha(US)$:

$$\alpha(US) \vdash A \quad \text{iff } A \in US.$$

The reader should note that *S* and *U* are not definable via P^+ and F^+ alone.
Systems with *S* and *U* were applied by Hoepelman [4] to the study of the tenses of natural language. Other systems also use operators of at least similar strength [5].
Consider now the following truth table, for a system τ in a language L_1 with unary *P* and *F* only. The notation ${}^g[A]^t_s$ reads the truth value of A at the point s, with respect to the reference point t, and assignment g. (We omit g when possible.):

(5) $[p]^t_s = 1$ iff $s \in g(p)$, for p atomic.

(6) The conditions for &, $\vee$, $\sim$, $\rightarrow$ are the usual.

(7) $[FA]^t_s = 1$ iff

(a) $t = s$ and for some $s' > s$, $[A]^t_{s'} = 1$.

(b) $t < s$ and $[A]^s_s = 1$.

(c) $t > s$ and for some $s < u < t$ $[A]^u_u = 1$.

(8) $[PA]^t_s = 1$ iff

(a) $t = s$ and for some $s' < s$ $[A]^t_{s'} = 1$.

(b) $t < s$ and for some $t < u < s$ $[A]^u_u = 1$.

(c) $t > s$ and $[A]^s_s = 1$.

Let τ be the set of all wff A such that $[A]^t_t = 1$ for all t in $(\lambda, <)$ and all assignments g.

Thus we have, for any wff A:
$A \in \tau$ iff for all g, t, ${}^g[A]^t_t = 1$.

The main purpose of part 1 of this note is to show that *US* and τ are inter-translatable into each other. This means that:

Theorem 9

(a) For each wff A of the language of *US* there exists a wff A^* of the language of τ such that:
$A \in US$ iff $A^* \in \tau$
or equivalently, since $\alpha(US)$ is an axiomatization of *US*,
$\alpha(US) \vdash A$ iff $A^* \in \tau$

(b) For each B of the language of τ there exists a $B^{\#}$ of the language of *US* such that:

$$B \in \tau \text{ iff } B^{\#} \in US \text{ (or } \alpha(US) \vdash B^{\#}).$$

Remark 9

The previous theorem yields an axiomatization $\alpha(\tau)$ of τ, obtained from the axiomatization $\alpha(US)$ of US.
We shall see that later. The two functions $* : A \to A^*$ and $\# : B \to B^{\#}$ are defined effectively and inductively on the complexity of the wffs.
τ is of particular interest since its connectives are *unary* (P and F) and they have the same expressive power as the *binary* connectives U and S. Furthermore, Kamp [3] has shown that US is a very strong system in the following sense:

Remark 10

Consider a language for $(\lambda, <)$ with a symbol «$<$» for the order, variables $t, s, t_1, s_1, \ldots$ ranging over elements of λ and variables $Q_1, Q_2, \ldots$ ranging over subsets of λ. Add the usual classical connectives together with quantifiers on the elements of λ (i.e. quantifiers of the form $\forall t, \exists t$).
This is really the first order theory of $(\lambda, <, Q_1, Q_2, \ldots)$. Let $\psi(t, Q_1, \ldots, Q_n)$ be any wff of this language with variable t free and containing exactly the set variables $Q_1, \ldots, Q_n$. In [1], page 140, ψ is called an n-place 1-dimensional truth table (for the flow of time $(\lambda, <)$). Kamp shows that for any such $\psi(t, Q_1, \ldots, Q_n)$ there exists a wff of US of the form $A(q_1, \ldots, q_n)$, with exactly the n atoms $q_1, \ldots, q_n$ such that for any assignment g:

$$||A(q_1, \ldots, q_n)||^g_t = 1 \quad \text{iff} \quad (\lambda, <) \models \psi(t, g(q_1), \ldots, g(q_n)).$$

Kamps theorem holds for $(\lambda, <)$ or any complete flow of time. It does not hold for e.g. the case of rational time $(\eta, <)$. Kamp gave the following example of a table for a connective $K(p)$, not definable in $(\eta, <)$ using Since and Until.

$$\psi(t_o, P) = (\exists t_1 < t_o)\,(P(t_1) \wedge \forall t_2(t_1 < t_2 < t_o \rightarrow (P(t_2) \wedge \exists t_3(t_2 < t_3 < t_o \wedge \forall t_4(t_2 < t_4 < t_3 \rightarrow P(t_4)))) \vee (\sim P(t_2) \wedge (\exists t_3 < t_2)\,\forall t_4(t_3 < t_4 < t_2 \rightarrow \sim P(t_4))))).$$

2. *The equivalence of US and τ.*

We begin with the proof of (9a).

Definition 11. Let * be the following translation from *US* into τ.

(a) $(p)^* = p$, p atomic
(b) $(A \,\&\, B)^* = A^* \,\&\, B^*$
$(\sim A)^* = \sim A^*$
$(A \vee B)^* = A^* \vee B^*$
$(A \rightarrow B)^* = A^* \rightarrow B^*$
(c) $S(A, B)^* = P(PA^* \,\&\, \sim F \sim B^*)$
(d) $U(A, B)^* = F(FA^* \,\&\, \sim P \sim B^*)$.

Lemma 12. Under the translation *, $A \in US$ iff $A^* \in \tau$.

Proof: Obvious from the semantical conditions. In fact, for any assignment g, and any $t \in \lambda$,

$$||A||^g_t = {}^g[A^*]^t_t.$$

Remark. Note in particular, that F^+ of *US*, is translated into
$(F^+A)^* = U(A, A \rightarrow A)^* = F(FA^* \,\&\, \sim P \sim (A \rightarrow A)^*)$
which is equivalent to FFA^*.
Similarly for P^+. This shows, what is semantically evident, that F^+ can be taken as FF and P^+ as PP in τ.
We now proceed to prove (9b), and define the translation # from τ into *US*. For this purpose we need some auxiliary results.
Remark 13. Below are some axioms and rules valid in τ.

(a) All wffs of the form below and their mirror images (obtained by interchanging P and F).

Let $Hx = \sim P \sim x$ and $Gx = \sim F \sim x$.

(1) $A \& HA \& GA$
For any A which is a substitution instance of a truth functional tautology.

(2) Let $U^+(x, y) \overset{\text{def}}{=} F(Fx \& Hy)$

$S^+(x, y) \overset{\text{def}}{=} P(Px \& Gy)$.
Then for any A of *US* such that $A \in US$ let A^* be the wff obtained from A by replacing each *U* by U^+ and *S* by S^+, then $A^* \in \tau$.
(Compare with 11).

(3) $G(A \& B) \leftrightarrow GA \& GB$
$G(A \rightarrow A)$
$GH(A \rightarrow A)$
$GH\ A \leftrightarrow G^+A$
$Fx \leftrightarrow FFx$, x atomic
$G(FA \& FB \leftrightarrow F(A \& B))$
$G(F \sim A \leftrightarrow \sim FA)$
$G(HA \& HB \leftrightarrow H(A \& B))$.

(4) $A \& H^+A \rightarrow G(HA \rightarrow FH^+A)$

$$G(FA \rightarrow \bigvee_{i=1}^{m} HB_i) \rightarrow G^+(\sim B_1 \rightarrow G(FA \rightarrow \bigvee_{i=2}^{m} HB_i)$$

(5) For any finite set Δ, let $y^{o*}_{+}(A, \Delta, C)$ be defined as

follows, by induction on $\bar{\bar{\Delta}}$:

$$\bar{\bar{\Delta}} = O : y^{o*}_{+}(A, \varnothing, C) \overset{\text{def}}{=} U^+(A, C).$$

$$\bar{\bar{\Delta}} = n : y^{o*}_{+}(A, \Delta, C) =$$

$$\bigvee_{\substack{\Delta' \subseteq \Delta \\ \Delta' \neq \varnothing}} U^+(C \& \bigwedge \Delta' \& y^{o*}_{+}(A, \Delta - \Delta', C), C).$$

The following is valid in τ.

$$F(FA \,\&\, HC \,\&\, \bigwedge_{i=1}^{n} PB_i) \leftrightarrow y^{o*}_{+}(A, \{B_1, \ldots, B_n\}, C).$$

(6) $G^+ (FHA \rightarrow P^+FH\,A) \rightarrow G^+ (FF^+HA \rightarrow FH\,A)$.

(b) The following rules are valid in τ.

(1) $$\frac{\vdash A, \vdash A \rightarrow B}{B}$$

(2) $$\frac{\vdash P^+B \,\&\, F^+A \rightarrow C}{\vdash A \rightarrow H(PB \rightarrow GC)}$$

(3) $$\frac{\vdash A \leftrightarrow B}{\vdash \varrho(A) \leftrightarrow \varrho(B)}$$

where $\varrho(x)$ is any wff built up using U^+ and S^+ only (i.e. for some ψ of US, $\varrho = \psi^*$), and x is atomic.

Lemma 14. (a) In τ, any wff of the form FD is a equivalent to a disjunction $\bigvee_{i=1}^{n} Q_i$, where each Q_i has the form $y_+(A, \{B_j\}, C)$, where

$$y_+(A, \{B_j\}, C) = F(FA \,\&\, HC \,\&\, \bigwedge_j PB_j).$$

(b) Similarly for the mirror image case of PD, we can obtain a $y_-(A, \{B_j\}, C)$.

Proof: Write D in disjunctive normal form, regarding anything of the form FX or PX as atoms. Thus D is equivalent to a disjunction $\bigvee D_i$, where each D_i has the form $\bigwedge_j X_j$, where X_j is either atomic or of the form FA_j or of the form PB_j or a negation of these forms.
FD is equivalent to $\bigvee FD_i$.

We know (see 13) that $F(p \& A)$ is equivalent to $F(Fp \& A)$, whenever p is an atomic wff. Therefore each FD_i is equivalent to $F \wedge X_j$, where each X_j is either of the form FA or $\sim FA$ or PB or $\sim PB$. Furthermore, since

$G(\sim FA \leftrightarrow F \sim A) \in \tau$

we can assume that the form $\sim FA$ does not occur. Furthermore since

$G(FA \& FB \leftrightarrow F(A \& B)) \in \tau$

we can assume that there exists at most one X_j of the form FA. Now further since $\sim PA$ is the same as $H \sim A$ and

$G(HA \& HB \leftrightarrow H(A \& B)) \in \tau$

we can assume that there exists at most one wff X_j of the form HC. Thus we are left with at most one FA, at most one HC and possibly several PB_i. We can assume that at least one FA and HC appear because $G(A \rightarrow A) \in \tau$

$$GH(A \rightarrow A) \in \tau.$$

We can conclude that each FD_i is equivalent to a wff of the form $y_+(A, \{B_j\}, C)$.

Lemma 14a. For any wff E, there exist A_i, B_{ij}, C_j such that

$$G(E \leftrightarrow \bigvee_i (FA_i \& HC_i \& \bigwedge_j PB_{i,j})).$$

Proof: Follows from the proof of the previous lemma. Similarly for the mirror image case.

Definition 15. For any wff of the form $y_+(A, \{B_i\}, C)$, $i = 1, \ldots, n$ we define, by induction on n, a wff y°_{+} of US.

We write $y_+(\{B_1, \ldots, B_n\})$ for brevity: For $n = O$ let

$$y_+(A, \varnothing, C) \stackrel{\text{def}}{=} F(FA \& HC).$$

(a) $n = O \quad y^{\circ}_{+} = U(A, C).$

$n = 1 \quad y^{\circ}_{+} = U(B_1 \& C \& U(A, C), C).$

(b) Case $n = 2$ $y^{\circ}_{+} = U(B_1 \& B_2 \& C \& U(A, C), C) . \quad \vee$

$$U(C \& B_1 \& y_+(\{B_2\})^{\circ}, C) \vee$$
$$U(C \& B_2 \& y_+(\{B_1\})^{\circ}, C).$$

(c) *Case n + 1* $y_+(\{B_1, \ldots, B_{n+1}\})^o =$

$$\bigvee_{\substack{\Delta \subseteq \{B_1 \ldots B_n\} \\ \Delta \neq \emptyset}} U(C \,\&\, \wedge\, \Delta \,\&\, y^o_+(A, \Theta, C), C)$$

where $\Theta = \{B_1 \ldots B_n\} - \Delta$.

Lemma 16. For any assignment g and $t \in \lambda$ and any atomic A, B_i, C

$$^g[y_+(A, \{B_i\}, C)]^t_t = ||y^o_+(A, \{B_i\} C)||^g_t.$$

Proof: By induction on the number n of B_i.

Case n = O: $[y_+(A, \emptyset, C)]^t_t = 1$ iff

$[F(FA \,\&\, HC)]^t_t = 1$ iff (by the semantic truth condition and the fact that A and C are atomic) $||U(A, C)||_t = 1$.

Case n + 1: Let $\Delta \neq \emptyset$ be a subset of $\{B_1 \ldots B_{n+1}\}$ and let Φ be $\{B_1 \ldots B_{n+1}\} - \Delta$. Then
$||y^o_+(A, \{B_1, \ldots, B_{n+1}\}, C)||_t = 1$ iff

$$||\bigvee_{\substack{\Delta \subseteq \{B_1, \ldots, B_{n+1}\} \\ \Delta \neq \emptyset}} U(C \,\&\, \wedge\, \Delta \,\&\, y^o_+(A, \Phi, C), C)||_t = 1$$

iff for some t', $t < t'$ and some $\Delta \subseteq \{B_1, \ldots, B_{n+1}\}$, $\Delta \neq \emptyset$ we have that the following holds:

(a) For all $t < x \leqslant t'$, $||C||_x = 1$

(b) $||\wedge\, \Delta||_{t'} = 1$; $||y^o_+(A, \Phi, C)||_{t'} = 1$.

Since A, C, B_i are all atomic, and by the induction hypothesis for $\bar{\bar{\Theta}} < n + 1$ we get that (a) is equivalent to (a') and (b) to (b') where:

(a') For all $t < x < t'$, $[C]^x_x = 1$

(b') $[\wedge \Delta]^{t'}_{t'} = 1$; $[y_+(A, \Theta, C)]^{t'}_{t'} = 1$.

Thus we have established that:
$||y^o_+(A, \{B_1 \dots B_{n+1}\}, C||_t = 1$ iff for some $t' > t$ and Δ as above, conditions (a') and (b') hold.

Now regard the following:

$[y_+(A, \{B_1 \dots B_{n+1}\}, C)]^t_t = 1$ iff by definition,

$$[F(FA \mathbin{\&} HC \mathbin{\&} \bigwedge_i PB_i)]^t_t = 1$$

iff for some s and u_i, $i = 1, \dots n+1$, $t < s$ and $t < u_i < s$ and $[A]^s_s = 1$ and $[B_i]^{u_i}_{u_i} = 1$ and for all y, if $t < y < s$ then $[C]^y_y = 1$.

The u_i need not necessarily be different. Let $t' \in \{u_i\}$ be the smallest (i.e. nearer to t) from among the u_i and let $\Delta = \{B_i | u_i = t'\}$. Then at t' the following holds:

(a') For all y, $t < y \leqslant t'$ implies $[C]^y_y = 1$.

(b') $[C \mathbin{\&} \wedge \Delta \mathbin{\&} F(FA \mathbin{\&} HC \mathbin{\&} \wedge (\{B_1 \dots B_{n+1}\} - \Delta))]^{t'}_{t'} = 1$

i.e. $[C \mathbin{\&} \wedge \Delta \mathbin{\&} y_+(A, \{B_i\} - \Delta, C)]^{t'}_{t'} = 1$.

Conversely, if (a') (b') hold at some t' for some Δ then certainly $[y_+(A, \{B_1, \dots, B_{n+1}\}, C)]^t_t = 1$.

We can thus conclude that $[y_+(A, \{B_1, \dots, B_{n+1}\}, C]^t_t =$ $= ||y^o_+(A, \{B_1 \dots B_{n+1}\}, C||_t$ since both sides of the equation are equal 1 iff (a') and (b') both hold.

Thus the induction step is completed and lemma 16 follows.

Remark 16a. In lemma 16 we assumed that A, B_i, C are atomic.

This assumption was used only to show that ${}^{g}[X]^{t}_{t} = {}^{g}||X||_{t}$, for $X \in \{A, B_i, C\}$. Therefore the following version of 16 is true.

Let A, $A^{\#}$, B_i, $B_i^{\#}$, C, $C^{\#}$ be such that for any $X \in \{A, B_i, C\}$ any g and any$^{t}_{g}$ ${}^{g}[X]^{t}_{t} = ||X^{\#}||^{g}_{t}$

Then for any g and t

${}^{g}[y_+ (A, \{B_i\}, C]^{t}_{t} = ||y^{o}_{+}(A^{\#}, \{B_i^{\#}\}, C^{\#})||^{g}_{t}.$

(17)

We can now define the translation # of τ into *US*.

(a) $A^{\#} = A$, for A atomic

(b) $(A \,\&\, B)^{\#} = A^{\#} \,\&\, B^{\#}$

$(\sim A)^{\#} = \sim A^{\#}$

$(A \vee B)^{\#} = A^{\#} \vee B^{\#}$

$(A \rightarrow B)^{\#} = A^{\#} \rightarrow B^{\#}$

(c) For the case of *FD*, we have only to define the translation for wffs of the form $y_+(A, \{B_j\}, C)$, in view of (14). Hence let

$$y_+(A, \{B_j\}, C)^{\#} = y_+(A^{\#}, \{B_j^{\#}\}, C^{\#})^{o}.$$

(d) Similar to (c) for the case of *PD*.

Lemma 17. For any A of τ and any assignment g and a moment t,

$^g[A]^t_t = ||A^{\#}||^g_t$

Proof: By induction, in view of 16a.

Lemma 17 yields theorem 9 b. Now that theorem 9 is established, we can use it to find an axiomatization $\alpha(\tau)$ of τ. Let $\alpha^*(US)$ be the *-translations of all the axioms and rules of $\alpha(US)$. That is, the set of axioms in $\alpha^*(US)$ is the set of all A^*, for A an axiom of $\alpha(US)$ and the set of rules of $\alpha^*(US)$ is the set of all

rules of the form $\dfrac{\vdash A^*_i}{\vdash B^*}$, $i = 1, \ldots n.$, where $\dfrac{\vdash A_i}{\vdash B}$, $i = 1 \ldots n$

is a rule of $\alpha(US)$.
Now let $\alpha(\tau)$ be the axiom system $\alpha^*(US)$ together with the set of all axioms of the form $A \leftrightarrow A^{\#*}$ for A in the language of τ, and the additional rule of modus ponens. We now prove:

(18) $\alpha(\tau) \vdash A$ iff $A \in \tau$.

Proof: That all axioms and rules are τ valid, follows from 9a and 9b.

Now assume $^g[B]^t_t = 1$, for all t and g. Then certainly $||B^{\#}||^g_t = 1$, for all g and t and hence

$\alpha(US) \vdash B^{\#}$.

Hence $\alpha^*(US) \vdash B^{\#*}$

and hence $\alpha(\tau) \vdash B$, since $\alpha(\tau) \vdash B^{\#*} \to B$.
Thus 18 is proved and $\alpha(\tau)$ is an axiomatization of τ.
It is desirable to find a simple axiomatization for τ.
Any set of axioms and rules that is sound and can prove all the axioms and rules of $\alpha(\tau)$ is of course complete.

One final remark. The truth table given for P and F of is not arbitrary. P and F do occur in English.

(a) Clause (7a) captures the behaviour of «will» in «He will come».

(b) Clause (7b) captures the behaviour of the second will in «I will come and bring the parcel».
«He will say he is coming».

(c) (7c) captures the meaning of «would» in
«He said he would come».
Or better, its mirror image, (8c), captures the future perfect.

To make our connectives P and F more plausible to the reader, here is an example of how to express «until» using «will» and «will have» «When the postman delivers the parcel he will ask what's in it and add that he has been asking what's in it all the time».
This has the form
$F(F\text{p} \ \& \ H\text{p})$
or U(deliver, ask).
(i.e. asks until delivers).

§ 3 *Axiomatization of* τ.

This section proposes an axiomatization $\beta(\tau)$ of τ and proves a direct completeness theorem for $\beta(\tau)$. The proof will also suggest a simple axiomatization $\beta(US)$ of US to be compared with the axiomatization α(US) quoted in Hoepelman [4].
Let $\beta(\tau)$ be the axiom system with the following axioms and rules and their mirror images:
(Recall that $F^{+} = FF$, $P^{+} = PP$, $G = \sim F \sim$ and $H = \sim P \sim$.)

(19) *The system* $\beta(\tau)$:

(a) All substitution instances of truth functional tautologies with the rule of modus ponens.

(b) The usual axioms for rational time with P^+ and F^+ i.e.

1) $G^+(A \to B) \to (G^+ A \to G^+B)$
2) $G^+A \to G^+G^+A$
3) $\sim A \to G^+P^+A$
4) $G^+(G^+A \to B) \vee G^+(B \,\&\, G^+B \to A)$
5) $G^+G^+A \to G^+A$
6) $\dfrac{\vdash A}{\vdash G^+A}$
7) $F^+(A \to A)$

(c) 1) $G(p \leftrightarrow Fp)$, p atomic
2) $G(\sim FA \leftrightarrow F \sim A)$
3) $G(F(A \,\&\, B) \leftrightarrow FA \,\&\, FB)$
4) $G(HA \,\&\, HB \leftrightarrow H(A \,\&\, B))$
5) GA, for all A that are substitutions of truth functional tautologies.

(e) $x \,\&\, H^+x \,\&\, G^+ \sim x \to$
$\to [F(FC \,\&\, \wedge PB_i \,\&\, HD) \leftrightarrow F^+(C \,\&\, H^+(x \vee D) \,\&$
$\wedge P^+(B_i \,\&\, \sim x))]$

(g) $$\frac{\vdash A \leftrightarrow A', \ \vdash B_i \leftrightarrow B_i', \ \vdash C \leftrightarrow C'}{\vdash F(FA \,\&\, HC \,\&\, \bigwedge_i PB_i) \leftrightarrow F(FA' \,\&\, HC' \,\&\, \bigwedge_i PB_i').}$$

(h) $G^+(FHA \to P^+F\,HA) \to G^+(F^+FHA \to FHA)$.

(k) $$\frac{\vdash A_1 \to \Box_1(A_2 \to \Box_2(A_3 \to \ldots \Box_{n-1}(A_n \to (G^+x \to x))\ldots)}{\vdash A_1 \to \Box_1(A_2 \to \Box_2(A_3 \to \ldots \Box_{n-1} \sim A_n) \ldots)}$$
Where x is an atom not appearing in $A_1, \ldots, A_n$ and $\Box_i$ is either G^+ or H^+.

Lemma 20 $\beta(\tau) \vdash A$ iff $A \in \tau$.

Proof: To show one direction, check every axiom and rule of $\beta(\tau)$ and verify they are all valid and are in τ. For the direction, namely, $A \in \tau \to \beta(\tau) \vdash A$, we need a series of constructions.

Definition 21

a) Let I be the set of all nonzero integers and let I^* denote the set of all finite sequences of elements of I. Let $R \subseteq I^* \times I^*$ denote the relation of «being an initial segment of», thus, for $x, y \in I^*$, $x R y$ iff $x = y$ or y is a longer sequence continuing x. Let $\varnothing$ denote the empty sequence. $\varnothing \in I^*$.
b) Let * denote concatenation of sequence and let t' be the sequence obtained from t by deleting the last element of t. Let $t^{(n)}$ denote $t'^{\cdots'}$ n-times. Thus $s = t^{(n)}$ iff for some $m_1, \ldots, m_n \in I$, $t = s*(m_1, \ldots, m_n)$.

Let $\bar{t}$ denote the last element of the sequence t (for $t \neq \varnothing$).

Thus $t = t'*(\bar{t})$, for $t \neq \varnothing$.

Definition 22

An ordered pair $\Theta = (S, \delta)$ is called a $\beta(\tau)$ finite table or just a table, iff the following holds.
(a) $S \subseteq I^*$, S is finite.
(b) $\varnothing \in S \;\&\; (t \in S \rightarrow t' \in S)$.
(c) For each $t \in S$, $\delta(t)$ is a wff of $\beta(\tau)$ which is $\beta(\tau)$ consistent and $\delta(t') \vdash \Diamond_{\bar{t}}\delta(t)$

where $\Diamond_x = \begin{cases} F^+ & \text{if } x > O \\ P^+ & \text{if } x < O \end{cases}$

Definition 23

Let $\Theta = (S, \delta)$ be a table and $t \in S$ and let x be a wff. Define the following structure $\Theta[x, t]$, which is a candidate for another table.

(a) Let $\delta(x, t) = \delta(t) \;\&\; x$
and by induction,
$\delta(x, t^{(i+1)}) = \delta(t^{(i+1)} \;\&\; \Diamond_{\overline{t^{(i)}}}\delta(x, t^{(i)})$.

Now let $\delta(x, s) = \delta(s)$, for all s such that $\sim sRt$.
Let $\Theta[x, t] = (S, \delta(x, s))$.

Lemma 24

(a) $\sim\delta(x, t^{(i)}) = \delta(t^{(i)}) \rightarrow \Box_{\overline{t^{(i-1)}}} (\delta(t^{(i-1)}) \ldots \Box_{\overline{t}} (\delta(t) \rightarrow x) \ldots).$

where $\Box = \sim \Diamond \sim$.

(b) $$\frac{\vdash \sim \delta(x, t^{(i)})}{\vdash \sim \delta(x, t^{(i+1)})}$$

(c) $\vdash \sim \delta(x, t^{(i)}) \;\&\; \sim \delta(\sim x, t^{(i)}) \rightarrow \sim \delta(t^{(i)})$.

Proof: a) is checked by direct computation or by induction.
b) follows from a) and the axioms of $\beta(\tau)$.
c) is proved by induction on i.

Case i = O: It is clear that

$$(\delta(t) \rightarrow x) \;\&\; (\delta(t) \rightarrow \sim x) \rightarrow \sim\delta(t).$$

Case i + 1: $\sim\delta(x, t^{(i+1)}) \;\&\; \sim\delta(\sim x, t^{(i+1)})$ imply, by definition

$$[(\delta(t^{(i+1)}) \rightarrow \Box_{\overline{t^{(i)}}} \sim\delta(x, t^{(i)})] \;\&\; [\delta(t^{(i+1)}) \rightarrow \Box_{\overline{t^{(i)}}} \sim\delta(\sim x, t^{(i)})]$$

which implies

$$\delta(t^{(i+1)}) \rightarrow \Box_{\overline{t^{(i)}}} \sim\delta(x, t^{(i)}) \;\&\; \sim\delta(\sim x, t^{(i)}))$$

by the induction hypothesis, this implies

$$\delta(t^{(i+1)}) \rightarrow \Box_{\overline{t^{(i)}}} \sim \delta(t^{(i)})$$

but since $\delta(t^{(i+1)}) \vdash \Diamond_{\overline{t^{(i)}}} \delta(t^{(i)})$

we conclude

$$\sim\delta(t^{(i+1)}).$$

This proves c).

Lemma 25.

Let $\Theta = (S, \delta)$ be a table and $t \in S$ and x a wff. Then either $\Theta\,[x, t]$ is a table or $\Theta\,[\sim x, t]$ is a table.

Proof: For $\Theta\,[x, t]$ to be a table, all we need is that each $\delta(x, t^{(m)})$ be $\beta(\tau)$ consistent, for all m. So if (25) is not true, we get for some m, n that

$\vdash \sim\delta\,(x, t^{(m)}),\ \vdash \sim\delta\,(\sim x, t^{(n)})$.

By lemma 24b we can assume $m = n$ and by lemma 24c $\vdash \sim\delta\,(t^{(n)})$, contradicting the consistency of $\delta\,(t^{(n)})$.
It is clear, that if $\delta\,(x, s)$ is consistent, it satisfies the other conditions of a table.

Lemma 26.

Let $\Theta = (S, \delta)$ be a table and $t \in S$. Assume $\delta\,(t) \vdash G^{+}x$, then the following (S, δ^{*}) is a table; where

$\delta^{*}(s) =$

$\delta\,(s)\,\&\,x$ if $s' = t$ and $\bar{s} > O$
$\delta\,(s)$ otherwise.

Proof: We must show that

(a) δ^{*} is consistent.
(b) $\delta^{*}\,(s') \vdash \underset{s}{\Diamond_{-}}\,\delta^{*}(s)$.

To show (a) assume otherwise that $\vdash x \rightarrow \sim\delta\,(s)$ hence

$\vdash G^{+}x \rightarrow G^{+}\sim\delta\,(s)$

and since $\delta\,(s') \vdash G^{+}x$ we get $\delta\,(s') \vdash G^{+}\sim\delta\,(s)$ but since $\bar{s} > O$ then by assumption $\delta\,(s') \vdash F^{+}\delta\,(s)$, which is a contradiction.

To show (b) observe that since $\vdash G^{+}x\,\&\,F^{+}y \rightarrow F^{+}(x\,\&\,y)$, (this can be proved from the axioms and rules of 19b), we get

$\delta^*(s') \vdash G^+x \,\&\, F^+\delta(s) \rightarrow F^+(x \,\&\, \delta(s))$ hence

$\delta^*(s') \vdash F^+\delta^*(s)$.

A similar lemma holds for H^+.

Definition 27.

Let N be a function associating with each wff A a positive Goedel Number $N(A)$. Leet Θ be a table (S, δ) and let $t \in S$ be such that $\delta(t) \vdash F^+x$. Let q be a propositional variable not appearing in any $\delta(s)$, $s \in S$. Then by axiom 19k,

$\delta(t) \vdash \sim F^+(x \,\&\, \sim q \,\&\, G^+q)$.

Define the following table

$$(S_{F^+x}, \delta^*).\ S_{F^+x} = S \cup \{t^*(N(A))\}$$

$$\delta^*(t^*(N(A))) = x \,\&\, \sim q \,\&\, G^+q$$

$$\delta^*(t) = \delta(t) \,\&\, F^+(x \,\&\, \sim q \,\&\, G^+q)$$

and by induction

$$\delta^*(t^{(i+1)}) = \delta(t^{(i+1)}) \,\&\, \Diamond_{\overline{t^{(i)}}}\, \delta^*(t^{(i)}).$$

$$\delta^*(s) = \delta(s) \text{ for any } s \in S, \text{ such that } \sim \exists i(s = t^{(i)}).$$

We can assume that N(A) is a number not appearing anywhere, i.e. that $t^*(N(A)) \notin S$.

We must show that we obtain a table. For this purpose show that $\delta^*(t^{(i)})$ is consistent.

Otherwise

$$\vdash \delta(t^{(i)}) \rightarrow \Box_{\overline{t^{(i-1)}}}\ (\delta(t^{(i-1)}) \ldots$$
$$\rightarrow G^+(x \rightarrow (G^+q \rightarrow q)) \ldots .$$

hence by 19)

$$\vdash \delta(t^{(i)}) \rightarrow \Box^*_{\overline{t^{(i-1)}}} \rightarrow \ldots G^+(\sim x) \ldots)$$

which is the same as

$$\vdash \delta(t^{(i)}) \rightarrow \sim\Diamond_{\overline{t^{(i-1)}}}\, \delta(t^{(i-1)})$$

and is impossible.

A similar result holds for the case of $P^{+}x$. In this case add $t^{*}(-N(A))$ and use $\sim q \,\&\, H^{+}q$.

Definition 28.

A sequence of tables (S_n, δ_n) is called regular iff it satisfies the the following conditions:

(a) $S_n \subseteq S_{n+1} \subseteq \ldots \subseteq S = \bigcup_n S_n$

(b) $\delta_{n+1}(t) \vdash \delta_n(t)$, for all $t \in S_n$.

(c) For any wff x and any $t \in S$ there exists an n such that $\delta_n(t) \vdash x$ or $\delta_n(t) \vdash \sim x$.

(d) For any $G^{+}x$ and $t, s \in S$ and m such $\delta_m(t) \vdash G^{+}x$ and $s' = t$ there exists an $n > m$ such that $\delta_n(s) \vdash x$. Similarly for H^{+}.

(e) For any $t \in S$ and m such that $\delta_m(t) \vdash F^{+}x$ there exists an $n > m$ and s such that $s' = t$ and $s > O$ and $\delta_n(s) \vdash x$. Similarly for P^{+}.

(f) For any $t \in S$ there exists a q and an n such that $\delta_n(t) \vdash \sim q \,\&\, G^{+}q$ or $\delta_n(t) \vdash \sim q \,\&\, H^{+}q$.

Lemma 29.

Let A be a consistent wff, then there exists a regular sequence of tables, (S_n, δ_n) such that $\delta_o(\emptyset) \vdash A$.

Proof: Since A is consistent so is $A \& \sim q \& G^+q$, for q atomic, q not appearing in A. Let $S_0 = \{\emptyset\}$, $\delta_0(\emptyset) = A \& \sim q \& G^+q$. To define (S_n, δ_n) for all $n \geq 1$, we need an auxiliary function f as follows. Let f be a function on the natural numbers such that $f(m) = (t_m, B_m)$, where $t_m \in I^*$ and B_m a wff. We can always arrange that each pair (t, B), $t \in I^*$ and B a wff is obtained in the range of f infinitely many times for odd m and infinitely many times for even m. We can now construct by induction the sequence (S_n, δ_n).
Assume (S_n, δ_n) has been defined. To define (S_{n+1}, δ_{n+1}) let $f(n) = (t, B)$.

We distinguish several cases:

(a) If $t \notin S_n$ let $S_{n+1} = S_n$, $\delta_{n+1} = \delta_n$.

(b) If $t \in S_n$ and n is even, then we know that either (S_n, δ_n) [B, t] is a table, or (S_n, δ_n) [$\sim$B, t] is a table.

So let (S_{n+1}, δ_{n+1}) be (S_n, δ_n) [B, t] if it is a table and otherwise let $(S_{n+1}, \delta_{n+1}) = (S_n, \delta_n)$ [$\sim$B, t].

(c) $t \in S_n$ and n is odd and B is of the form F^+C or P^+C, then let (S_{n+1}, δ_{n+1}) be the corresponding table constructed in in (27). If B is of the form G^+C or H^+C, let (S_{n+1}, δ_{n+1}) be the table constructed in (26).

(d) If $t \in S_n$, n is odd and B is not of the form of c), then proceed as in case b).

It is easy to verify that we obtain a regular sequence.

Definition 30.

Let (S_n, δ_n) be a regular sequence. Let $S = \bigcup_n S_n$. For each $t \in S$ let $\Delta_t = \{A|$ for some n, $\delta_n(t) \vdash A\}$. Let $T = \{\Delta_t| \; t \in S\}$. Define $<$ on T by $\Delta < \Theta$ iff for all G^+A, $G^+A \in \Delta \Rightarrow A \in \Theta$.

Lemma 31.

(a) Each Δ_t is a complete and consistant $\beta(\tau)$ theory.

(b) For each Δ_t there exists a q such that $\sim q \,\&\, G^+q \in \Delta_t$ and r such that $\sim r \,\&\, H^+r \in \Delta_t$.

(c) If $\Delta_1 < \Delta_2$ and $H^+A \in \Delta_2$ then $A \in \Delta_1$.

(d) $(T, <)$ is a transitive, dense linear order without endpoints and $<$ is irreflexive.

(e) If $F^+A \in \Delta \in T$ then for some Δ', $A \in \Delta'$, $\Delta' \in T$ and $\Delta < \Delta'$. Similarly the mirror condition for P^+.

Proof:

(a) Follows from conditions 28b) and 28c).

(b) Follows from the construction. Each t when introduced into S_n, its $\delta_0(t)$ contains a q with either $\sim q \,\&\, H^+q$ or $\sim q \,\&\, G^+q$. From the axiom it follows that a similar mirror r is also available.

(c) Follows from axiom $19b_3$).

(d) Transitivity follows from axiom $19b_2$). Ir-reflexivity follows from b) above.
To obtained linearity, first show connectedness: Using c) and transitivity show that for all $t \neq \varnothing$, $t \in \bigcup_n S_n$, $\Delta_{\varnothing} < \Delta_t$ or $\Delta_t < \Delta_{\varnothing}$. This would give connectivity.
It can be shown by induction on the length of t, using 28d-e). Now linearity follows from $19b_4$) and density from $19b_5$). The order is non ending since axiom $19b_7$) is available, together with e) below.

(e) If $F^+A \in \Delta_t$, then for some m, $\delta_m(t) \vdash F^+A$. By condition

28e), there exists an $n > m$ and s such that $\bar{s} > O$ and $t = s'$ with $\delta_n(s) \vdash A$. Thus $A \in \Delta_s$. We now show that $\Delta_t < \Delta_s$. Let $G^+x \in \Delta_t$, then for some m', $\delta_{m'}(t) \vdash G^+x$, and by 28e) we can assume $m' > n$. So by 28d) for some n', $\delta_{n'}(s) \vdash x$, i.e. $x \in \Delta_s$.

From now on we deal with $(T, <)$ obtained above. The properties that interest us in $(T, <)$ are summarised in (32) below.

(32)

(a) $(T, <)$ is isomorphic to the rationals in order, with $<$ irreflexive.

(b) All members of T are $\beta(\tau)$ complete and consistent theories.

(c) $<$ is defined by $\Delta < \Theta$ iff for all A, $G^+A \in \Delta$ implies $A \in \Theta$ or equivalently for all A, $H^+A \in \Theta$ implies $A \in \Delta$.

(d) $F^+A \in \Delta$ iff for some Θ, $\Delta < \Theta$ and $A \in \Theta$.
Similarly the mirror condition for P^+A.

Definition 33:

(a) Let Γ be a set of wffs. We say Γ is $\beta(\tau)G$ consistent iff for no $A_1, \ldots, A_n \in \Gamma$ do we have $\beta(\tau) \vdash G{\sim} \bigwedge_i A_i$.

(Notice we use G here and not G^+!).
Similarly define the notion of $\beta(\tau)H$ — consistency.

(b) Let Γ be a complete (i.e. for all x, $x \in \Gamma$ or ${\sim}x \in \Gamma$) $\beta(\tau)G$ consistent set.
Define $\Gamma^+ = \{A \mid FA \in \Gamma\}$.

Similarly define Γ^- for a complete $\beta(\tau)H$ consistent set.

Lemma 34.

Any $\beta(\tau)G$ consistent set can be extended to a complete $\beta(\tau)G$ consistent set.

Proof: It is sufficient to show that if Γ is $\beta(\tau)G$ consistent and A is any wff then $\Gamma \cup \{A\}$ or $\Gamma \cup \{\sim A\}$ is $\beta(\tau)G$ consistent. For otherwise for some $B_i, C_j \in \Gamma$

$$\vdash G(\wedge B_i \rightarrow \sim A)$$
$$\vdash G(\wedge C_i \rightarrow A).$$

From the axioms of 19c) it follows $\vdash G(\bigwedge_i (C_i \& B_i) \rightarrow A \& \sim A)$

hence $\vdash G\sim (\bigwedge_i C_i \& \bigwedge_i B_i)$, a contradiction.

Lemma 35.

Let $\Delta, \Theta \in T$ and $\Delta < \Theta$, then there exists a unique $\beta(\tau)G$ — consistent and complete $\Gamma = \underline{\underline{\alpha}}(\Delta, \Theta)$ such that:

(1) $GA \in \Delta \rightarrow A \in \Gamma$

(2) $C \in \Theta \rightarrow FC \in \Gamma$

(3) If D is such that for all $y \in T$, $\Delta < y < \Theta \rightarrow D \in y$ then $HD \in \Gamma$.

(4) If B is such that for some, $\Delta < y < \Theta$ and $B \in y$ then $PB \in \Gamma$.

Proof: To show uniqueness, assume that Γ_1, Γ_2 have the above properties. Show by induction on A,
$A \in \Gamma_1 \leftrightarrow A \in \Gamma_2$.
For atomic p, since $\vdash G(p \leftrightarrow Fp)$ we get $p \in \Gamma_i \leftrightarrow p \in \Theta$.
The case of *F*A is clear. So are the cases of *H*A and *P*A.

To show existence, let $\Gamma_0 = \{A \mid GA \in \Delta\} \cup \{FC \mid C \in \Theta\}$

$\cup \{PB | B \in y$, for some y such that $\Delta < y < \Theta\}$
$\cup \{HD |$ for all y, $\Delta < y < \Theta \rightarrow D \in y\}$.

We claim Γ_0 is $\beta(\tau)G$ consistent. Otherwise for some $A_j, C_k, B_i, D_m, \vdash G \sim (\wedge HD_m \& \wedge A_j \& \wedge FC_k \& \wedge PB_i)$. From the axioms of 19c) we get $\vdash G \sim (HD \& A \& FC \& \wedge PB_i)$

where $A = \wedge A_j$, $C = \wedge C_k$, $D = \wedge D_m$.

So $\vdash GA \rightarrow G \sim (HD \& FC \& \wedge PB_i)$

and since $GA \in \Delta$, we get

$\sim F(HD \& FC \& \wedge PB_i) \in \Delta$.

Since $\sim \Delta < \Delta$, we can find an x such that $x \& H^+x \& G^+ \sim x \in \Delta$ hence by (19e) $\sim F^+ (H^+ (D \vee x) \& C \& \bigwedge_i P^+ (B_i \& \sim x) \in \Delta$
but this is impossible since

$C \& H^+ (D \vee x) \& \wedge P^+ (B_i \& \sim x) \in \Theta$ and $\Delta < \Theta$.

Thus Γ_0 is $\beta(\tau)G$ consistent and can be extended to an $\underline{\underline{\alpha}} (\Delta, \Theta)$ as required.
A similar mirror lemma holds for the existence of $\underline{\underline{\alpha}} (\Theta, \Delta)$ for $\Delta < \Theta$.

Lemma 36.

For every wff A, there exists a wff B of the form
$B = \vee_j (Fx_j \& Hu_j \& \bigwedge_i Py_{ij})$ such that $\vdash \beta(\tau) G (A \leftrightarrow B)$.

Proof: Follow the steps of the proof of lemma 14a and note that axioms (19c) and (19g) allow you to prove the equivalence in $\beta(\tau)$.
A similar lemma holds for the case of *PA*.

Lemma 37.

For $\Delta \in T$,
$C = F(Fx \,\&\, Hu \,\&\, \bigwedge_i Py_i) \in \Delta$ if, and only if for some $\Theta \in T$,
$\Delta < \Theta$ and
$Fx \,\&\, Hu \,\&\, \bigwedge Py_i \in \underline{\underline{\alpha}}(\Delta, \Theta)$.

Proof: From axiom 19 it follows, since for some A,
$A \,\&\, H^{+}A \,\&\, G^{+} \sim A \in \Delta$, that
$C \leftrightarrow F^{+}(x \,\&\, H(u \vee A) \,\&\, \& P^{+}(y_i \,\&\, \sim A)) \in \Delta$
and hence $C \in \Theta$ iff for some Θ and some $\Delta < t_i < \Theta$, $y_i \in t_i$, and $x \in \Theta$ and for all $\Delta < t < \Theta$, $u \in t$. Thus by construction of $\underline{\underline{\alpha}}(\Delta, \Theta)$ iff

$Fx \,\&\, Hu \,\&\, \bigwedge Py_i \in \underline{\underline{\alpha}}(\Delta, \Theta)$.

A similar lemma holds for the mirror case.

Lemma 38.

Let $\underline{\underline{\alpha}}(\Delta, \Delta) = \Delta$ and let $g(x) = \{\Delta \in T \mid x \in \Delta\}$, for x atomic. Then for $(T, <, g)$ and any wff A,

$${}^{g}[A]^{\Delta}_{\Theta} = 1 \text{ iff } A \in \underline{\underline{\alpha}}(\Delta, \Theta).$$

Proof: By induction on A. For atomic A, this holds by definition of g and the fact that axiom $(19c_1)$ is available.
The cases of $\&$, $\vee$, $\sim$, $\rightarrow$ present no difficulty.

Case of F and $\Delta = \Theta$:

$FA \in \Delta$, iff by lemma 36, $\bigvee_j F(Fx_j \,\&\, Hu_j \,\&\, \bigwedge_i Py_{ij}) \in \Delta$ iff some disjunct j is in Δ, iff by 37, for some Θ $\Delta < \Theta$ and $Fx_j \,\&\, Hu_j \,\&\, \bigwedge_i Py_{ij} \in \underline{\underline{\alpha}}(\Delta, \Theta)$

iff by 36 for some Θ, $\Delta < \Theta$ and $A \in \underset{=}{\alpha}(\Delta, \Theta)$ iff by the induction hypothesis ${}^{g}[A]_{\Delta}^{\Theta} = 1$ for some Θ such that $\Delta < \Theta$ iff ${}^{g}[FA]_{\Delta}^{\Delta} = 1$.

Case F and $\Delta < \Theta$:

$FA \in \underset{=}{\alpha}(\Delta, \Theta)$ iff by construction $A \in \Theta$ iff by the induction hypothesis ${}^{g}[A]_{\Theta}^{\Theta} = 1$ iff ${}^{g}[FA]_{\Theta}^{\Delta} = 1$.

Case F and $\Theta < \Delta$:

$FA \in \underset{=}{\alpha}(\Theta, \Delta)$ iff by construction for some t, $\Theta < t < \Delta$ and $A \in t$ iff by the induction hypothesis ${}^{g}[A]_{t}^{t} = 1$ iff ${}^{g}[FA]_{\Theta}^{\Delta} = 1$.
The case of P is the mirror of the case of F.
Thus lemma 38 is proved.
What we have proved is the following:

Theorem 39:

Let A be a $\beta(\tau)$ consistent sentence, then there exists a structure of the form $(\eta, <, g)$, with $(\eta, <)$ the rationals such that
(a) $\beta(\tau) \vdash B$ implies ${}^{g}[B]_{t}^{t} = 1$, for all $t \in \eta$ and all B.

(b)ττFor some $t_0 \in \eta$, ${}^{g}[A]_{t_0}^{t_0} = 1$.

Theorem 40:

Let $\beta(\eta)$ be the axioms and rules of $\beta(\tau)$ without axiom 19h). Then $\beta(\eta)$ is complete for all structures $(\eta, <, g)$ with raitonal time.

Proof: In the completeness proof of theorem 39, we never used axiom (19h). Thus theorem 39 is valid for $\beta(\eta)$ as well and since all axioms of $\beta(\eta)$ are valid in rational time, we get completeness.

We still have to prove the completeness theorem for $\beta(\tau)$. To do this, let us start with a structure of the form $(\eta, <, g)$ in which all theorems of $\beta(\tau)$ are valid (i.e. 39a, 39b hold) and convert it to a structure of the form $(\lambda, <, g^*)$.

So let us approach our problem generally, we are going to construct, from a given structure $(\eta, <, g)$ in which (19h) is everywhere valid, another structure $(\lambda, <, g^*)$, in which $(\lambda, <)$ is the completion of $(\eta, <)$ and for any A and $t \in \eta$, $^*[A]^t_t =$ $[A]^t_t$.

So let $(\eta, <, g)$ be given and let S_1, S_2 be an open cut in η, i.e. $S_1 \cap S_2 = \varnothing$, $\eta = S_1 \cup S_2$, and $t \in S_1 \wedge s \in S_2 \rightarrow t < s$ and S_1 has no last element, and S_2 does not have a first element.

Lemma 41.

If for some $t \in S_1$ and wff A we have $[A]^s_s = 1$, for all $s \in S_1$, $s > t$ then for some $t' \in S_2$ we have $[A]^s_s = 1$ for all $s < t'$, $s \in S_2$.

Proof: Otherwise, take any $s \in S_2$. Then $[HH(PGA \rightarrow FFPGA)]^s_s$ $= 1$ but $[HH(PPPGA \rightarrow PGA)^s_s = O$.

Let $(\lambda, <)$ is *the completion* of $(\eta, <)$, so for each open cut in η, λ contains an element t defining the cut. We want to define a theory Θ^x_x for each $x \in \lambda$. For $x \in \eta$, let $\Theta^x_x = \{A \mid [A]^x_x = 1\}$. We now want to define Θ^x_x for $x \in \lambda - \eta$.

Lemma 42.

For $x \in \lambda - \eta$, the following set Δ^x_x is τ consistent.

$\Delta^x_x = \Delta^x_1 \cup \Delta^x_2 \cup \Delta^x_3 \cup \Delta^x_4 \cup \Delta^x_o$, where $\Delta^x_o = \{E|$ for some $t < x < s$, $[E]^y_y = 1$, for all $y \in \eta$, $t < y < s\}$.

$\Delta^x_1 = [F(HA \mathbin{\&} \bigwedge_i PC_i \mathbin{\&} FB) |$ for some $t \in \eta$, $t > x$, $[B]^t_t = 1$, and for some $s_i \in S$, $x < s_i < t$, $[C_i]^{s_i}_{s_i} = 1$ and for all $s \in \eta$, $x < s < t$, $[A]^s_s = 1\}$

$\Delta^x_2 = \{ \sim F(HA \mathbin{\&} \bigwedge_i PC_i \mathbin{\&} FB) |$ the previous condition of Δ^x_1 does not hold$\}$

$\Delta^x_3 = \{P(GA' \mathbin{\&} \bigwedge_i FC'_i \mathbin{\&} PB' |$ the mirror of the condition of Δ^x_1 holds$\}$

$\Delta^x_4 = \{ \sim P(GA' \mathbin{\&} \bigwedge_i FC'_i \mathbin{\&} PB' |$ the condition of Δ^x_3 does not hold$\}$

Proof: If Δ^x_x is not consistent then for some finite $\Delta^+_i \subseteq \Delta^x_i$, $\cup \Delta^+_i$ is not consistent. We will show that for some $t \in \eta$ near enough to x, $[A]^{t'}_{t'} = 1$ for all $A \in \cup \Delta^+_i$ and all $x < t' < t$, $t' \in \eta$. Certainly for any t if t is near enough x, any of the members of Δ^+_o hold at (t, t). As for the members of Δ^+_2, if we look on t's such that $x < t$, then for any near enough t, all members of Δ^+_1 are valid. Now take any membe2r of Δ^+_2, say $\sim F(HA \mathbin{\&} \bigwedge_i PC_i \mathbin{\&} FB)$ and assume that for no $t > x$, no matter how near, can we make this sentence true at all (t', t'), $t' \in S$, $x < t' < t$. This implies that for any near enough t,

$[F(HA \,\&\, \bigwedge_i PC \,\&\, FB]^{t_o}_{t_o} = 1$, for some $t_o \in \eta$, $x < t_o < t$. Take such a $t_o \in \eta$ and note that further for any $x < t < t_o$, $t \in S$ $[A]^t_t = 1$. But this contradicts the fact that the above statement is in Δ^+_2.

Turning to Δ^+_3, let $D = P(GA' \,\&\, \bigwedge_i FC'_i \,\&\, PB') \in \Delta^+_3$. By the condition of admittance to Δ^x_3, there exists a $t_o < x$, $t_o \in \eta$ such that $[B']^{t_o}_{t_o} = 1$, and for some $t_o < s_i < x$, $s_i \in \eta$, $[C'_i]^{s_i}_{s_i} = 1$ and for all $t_o < s < x$, $[A']^s_s = 1$. By lemma 41 there exists a $t_* \in \eta$, $x < t_*$ such that for all $s \in \eta$, $x < s < t_*$ $[A']^s_s = 1$. But this means now that for any $t \in \eta$, $x < t < t_*$ $[D]^t_t = 1$. Thus for near enough $t > x$, all members (finite in number) of Δ^+_3 can be made true at an (t', t'), $x < t' < t$, $t' \in \eta$.

Turning now to Δ^+_4, let $D = P(GA' \,\&\, \bigwedge_i FC'_i \,\&\, PB')$ and $\sim D \in \Delta^+_4$.

Suppose for any $t \in \eta$ $x < t$ there exists a $t' \in \eta$, $x < t' < t$, with $[D]^{t'}_{t'} = 1$. This assumption implies that

(a) For some $t_* > x$, $t_* \in \eta$ we have that $[A']^s_s = 1$, for all $x < s < t_*$, $s \in \eta$. By lemma 41, for some $s_* < x$, $s_* \in \eta$, $[A']^s_s = 1$ for all $s_* < s$ x, $s \in \eta$.

(b) For each C'_i, (in view of lemma 41) and for each $s_i < x < t_i$, there exists $s_i < s'_i < x < t'_i < t_i$ such that

$[C'_i]^{s_i}_{s_i} = [C'_i]^{t_i}_{t_i} = 1$. $s_i, s'_i \in \eta$.

(c) Similarly for B.

But then (a), (b), (c) contradict the condition of admission of $\sim D$ into Δ_4^+.

We have thus shown that for each $A \in \cup \Delta_i^+$, there exists an $s \in \eta$, $x < s$ such that $[A]_{s'}^{s'} = 1$, for all $s' \in \eta$, $x < s' < s$. Thus lemma 42 is proved.

Since Δ_x^x is a τ consistent it can be extended to a complete and consistent τ theory Θ_x^x.

(43) Notice that by construction for any $t < x < s$ there exist $t', s' \in \eta$ $t < t' < x < s' < s$ such that $[A]_{t'}^{t'} = [A]_{s'}^{s'} = 1$.

Lemma 44.

Let $\Delta_{s'}^t$, $t < s$, $t, s \in \lambda$ be

$$\Delta_s^t = \{A \mid GA \in \Theta_t^t\} \cup \{FC \mid C \in \Theta_s^s\} \cup \{HD \mid D \in \Theta_u^u \text{ for all } t < u < s\}$$

$$\cup \{PB \mid B \in \Theta_u^u \text{ for some } t < u < s\}.$$

Then:

(a) Δ_s^t is $\beta(\tau)G$ consistent.

(b) For $t, S \in \eta$ we have
$A \in \Delta_s^t \rightarrow [A]_s^t = 1.$

Proof: To show consistency assume otherwise and reach a conradiction. Assume that some A_i, D, C, B_j from the respective theories we have:

$E = \wedge A_i \rightarrow \sim(HD \,\&\, \wedge PB_i \,\&\, FC))$ and $\vdash GE$.

We can take only one C and one D since we have axioms (19c).

And so

$E_1 = \sim F(HD \,\&\, FC \,\&\, \wedge PB_i) \in \Theta^t_t$.

Case 1. $t \in \eta$.

Then $[E_1]^t_t = 1$. From what is given and (43) there exists a $u \in \eta\; t < u < s$ such that for some $u_i \in \eta$, $t < u_i < u$, $[B_i]^{u_i}_{u_i} = 1$. This means in view of the condition on HD, that for any $s' \in \eta$, $u < s' < s$, $[\wedge PB_i \,\&\, HD]^t_{s'} = 1$ and hence since $[E_1]^t_t = 1$, $[\sim C]^{s'}_{s'} = 1$ for all $u < s' < s$. Now if $s \in \eta$, then $[\wedge PB_i \,\&\, HD]^t_s = 1$, and hence by the same reasoning $\sim C \in \Theta^s_s$, a contradiction. If $s \in \lambda - \eta$, then by construction $\sim C \in \Theta^s_s$ and again a contradiction.

Case 2. $t \in \lambda - \eta$.

By the condition of admission into Θ^t_t, for no point $t < s'$, do we have that the condition in the definition of Δ^x_1 of (42) is satisfied. Thus by following the same reasoning as in case 1, we get a contradiction.

(b) follows from (43).

We now define Θ^t_s, for $t < s$. If $t, s \in \eta$, let $\Theta^t_s = \{A \mid [A]^t_s = 1$. In this case $\Delta^t_s \subseteq \Theta^t_s$ in view of (44b). Otherwise let Θ^t_s be some fixed complete $\beta(\tau)G$ consistent extension of Δ^t_s.

We can similarly define Δ^t_s, $t > s$ and Θ^t_s, $t > s$ using mirror-argument.

We can no wdefine g^*. Let $g^*(p) = \{t \in \lambda \mid p \in \Theta^t_t\}$.

Lemma 45.

For any A, in the structures $(\lambda, <, g^*)$:
$*[A]^t_s = 1$ iff $A \in \Theta^t_s$.

Proof: By induction.

The cases of A atomic and the cases of the classical connectives are obvious.
$*[FA]^t_s = 1$, for $t < s$, iff $*[A]^s_s = 1$ iff by induction $A \in \Theta^s_s$

iff by construction $A \in \Theta^t_s$.

$*[FA]^t_s = 1$ $t < s$, iff for all $s < u < t$
$*[A]^u_u = 1$, iff (by the induction hypothesis) $A \in \Theta^u_u$ for all $s < u < t$ iff (by definition of Θ^t_s) $FA \in \Theta^t_s$.

$*[FA]^t_t = 1$ implies that for some $s > t$, $*[A]^t_s = 1$ and hence $A \in \Theta^t_s$ and hence $FA \in \Theta^t_t$ by construction of Θ^t_s, $t < s$.
Now assume $FA \in \Theta^t_t$, then if $t \in \eta$, then $[FA]^t_t = 1$ and hence for some $s > t$, $s \in \eta$, $[A]^t_s = 1$ and hence $A \in \Theta^t_s$ and hence $*[A]^t_s = 1$ and hence $*[FA]^t_t = 1$.
If $t \in \lambda - \eta$, then by lemma 14,

$\vdash FA \leftrightarrow \bigvee_j F(FC^j \,\&\, \bigwedge_i PB^j_i \,\&\, HD^j)$.

Now since the disjunction is in Θ^t_t, one of the disjuncts $\in \Theta^t_t$, $t \notin \eta$, say the j-th. By construction, for some s_i, $s \in \eta$, $t < s_i < s$ $[C^j]^s_s = 1$, $[B^j_i]^{s_i}_{s_i} = 1$ and for all $u \in \eta$, $t < u < s$, $[D^j]^u_u = 1$.
By construction of Θ^x_x we have for s, s_i, as above and any

$t < u < s$, $u \in \lambda$, $D \in \Theta^u_u$, $B_i \in \Theta^{s_i}_{s_i}$, $C \in \Theta^s_s$. By the induction hypothesis

$$*[FC^j \;\&\; \bigwedge_i PB_i^j \;\&\; HD^j]^t_s = 1$$

i.e. $*[FA]^t_t = 1$.

Thus the proof of lemma 45 is complete.

We can now conclude

Theorem 46.

Every $\beta(\tau)$ consistent sentence A is valid in some $(\lambda, <, g^*)$. Thus $\beta(\tau)$ is complete for validity in real numbers time.

§ 4 *Further Remarks.*

Let *KC* be Prior's logical system of all wffs of the language with F^+ and P^+ valid in real time $(\lambda, <)$. *KC* can be axiomatised using axiom group (19a), (19b) and the following axiom for real time (and its mirror image):

(47) $G^+(G^+A \rightarrow P^+G^+A) \rightarrow G^+(F^+G^+A \rightarrow G^+A)$.

We now define a translation$^+$ from *US* into *KC* such that
US $\vdash$ A iff *KC* $\vdash$ A^+.

Definition 48.

Let $A(x_1, \ldots, x_n)$ be a wff of *US* with the atoms $x_1, \ldots, x_n$. We define by induction on A a wff $A^+(x_1, \ldots, x_n, y_1, \ldots, y_m)$ of *KC* with atoms $x_1, \ldots, x_n, y_1, \ldots, y_m$, where m is the number of occurences of *U* and *S* in A.

(a) $x^+ = x$; x atomic.

(b) $(\sim A)^+ = \sim A^+$

(c) Let $A(x_1, \ldots, x_m)$, $B(u_1, \ldots, u_k)$ be given and let $A^+(x_1, \ldots, x_m, y_1, \ldots, y_n)$ $B^+(u_1, \ldots, u_k, v_1, \ldots, v_r)$ be their translations. We can assume that all $y_1, \ldots, y_n, v_1, \ldots, v_r$ are different atoms.
Let $(A \& B)^+ = A^+ \& B^+$
$(A \vee B)^+ = A^+ \vee B^+$
$(A \to B)^+ = A^+ \to B^+$.

(d) Under the assumption on A, B in (c) let
$U(A, B)^+ = z \& H^+z \& G^+ \sim z \& F^+(A^+ \& H^+(B^+ \vee z))$
$S(A, B)^+ = z \& G^+z \& H^+ \sim z \& P^+(A^+ \& G^+(B^+ \vee z))$
where Z is a new atom not appearing in A^+ or B^+.

Lemma 49.

Let g, g' be two assignments. Write $g =_{x_i} g'$ iff $g(x_i) = g'(x_i)$, $i = 1, \ldots, n$. Then
$||A(x_1, \ldots, x_n)||^g_t = 1$ iff $\exists g' =_{x_i} g\ ||A^+(x_i, y_i)||^{g'}_t = 1$.

Proof: By induction on A. For A atomic this is clear.

$$||U(A, B)||^g_t = 1 \text{ iff } \exists s > t\ \forall r\ (t < r < s \to$$
$$||A||^g_s = 1\ \&\ ||B||^g_r = 1) \text{ iff for some } g', g'' =_{x_i} g$$
$$\exists s > t\ \forall r\ (t < r < s \to ||A^+||^{g'}_s = 1\ \&\ ||B^+||^{g''}_r = 1).$$

Since he extra parameters of A^+ and B^+ can be taken to be all different, we can assume that $g' = g''$. Let g^* agree with g', g'' on all the parameters involved and let $g^*(z) = \{r \mid r \leqslant t\}$, hence

$$||U(A, B)||^g_t = 1 \text{ iff } \exists g^* =_{x_i} g\ ||U(A, B^+||^{g^*}_t = 1.$$

Corollary 50: $US \vdash A$ iff $KC \vdash A^+$.

Proof: From the previous lemma.

The above corollary suggests the following axiomatization of *US*. Let $F^+A = U(A, A \to A)$, $P^+A = S(A, A \to A)$. Let $\gamma(US)$ contain all the axioms and rules of *KC* for F^+ and P^+ together with the following axiom and its mirror image:

$$Z \,\&\, H^+Z \,\&\, G^+ \sim Z \to [U\,(A, B) \leftrightarrow F^+(A \,\&\, H^+(B \vee Z))].$$

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REFERENCES

[1] D. M. Gabbay, *Investigation in modal and tense logic with application to problems in philosophy and linguistics,* D. Reidel 1976.

[2] D. M. Gabbay and J. Moravcsik, *Tense Logics and the Tenses of English II,* in preparation.

[3] J. A. W. Kamp, *Tense Logics and the theory of linear order,* Thesis, Univ. of California, 1968.

[4] J. Hoepelman, *Tense logics and the semantics of the Russian aspects,* Theoretical Linguistics, Vol. 1, 1974, pp. 158-182.

[5] L. Åqvist and F. Guenther, *Fundamentals of a theory of verb aspects and events within the setting of an improved tense logic,* in: F. Guenther & C. Rohrer (eds.) *Studies in Formal Semantics,* N. Holland (in press).

[6] A. N. Prior, *Past, Present, Future,* Oxford 1969.