UNSPECIFIED CONSTANTS IN PREDICATE CALCULUS AND FIRST-ORDER THEORIES.

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In this paper, I will present arguments to show that unspecified constants, that is, constants which have no axiomatic characterisation placed on them by the formal system, should not be included in Predicate Calculus or in a first-order theory, both with a semantics, with the exception of unspecified individual constants in first-order theories which can only have one-element domains in their semantics. I will also indicate an improvement on Tarski's semantics of Predicate Calculus. I will deal firstly with unspecified individual constants in predicate calculus and then deal with other unspecified constants and with 1st-order theories by using similar arguments. The practice of including unspecified individual constants in Predicate Calculus and 1st-order theories is quite widespread (e.g. in [2], [5], [7] and [8]).

§1. *Proof-Theoretic Considerations.*

In order to point out the problems of including unspecified individual constants, I will use the following axiomatisation of Predicate Calculus, similar to that used by Leblanc in [6], pp. 2-3. I will use the symbols p, q, r, ... for sentential variables; x, y, z, ... for free individual variables; x', y', z', ... for bound individual variables; a, b, c, ... for individual constants; f, g, h, ... for predicate variables. The formation rules for the primitives $\sim$, $\vee$ and $\forall$ are as follows:

1. A sentential variable is a wff.
2. If f is an n-place predicate variable and $t_1, \ldots, t_n$ are terms, then $ft_1 \ldots t_n$ is a wff. (A term is an individual constant or a free individual variable.)

3. If A and B are wffs then $\sim A$ and $A \vee B$ are wffs.
4. If A is a wff then $(\forall x')\, A\,(x'/x)$ is a wff, where $A(x'/x)$ consists of the wff A with all occurrences of the free individual variable x replaced by the bound individual variable x'.

   The definitions of &, $\supset$ and $\equiv$ are as usual.

   $(\exists x')\, A\,(x'/x) =\text{df} \sim (\forall x') \sim A(x'/x)$.

Any axiomatisation of sentential calculus will do, provided the axioms are in schematic form. The axioms and rule dealing with the quantificational part are as follows:

Axioms.

1. $(\forall x')\, A\,(x'/x) \supset A(t/x')$, where the wff $A(t/x')$ is $A(x'/x)$ with the term t substituted for all occurrences of x' which are not in the scope of a quantifier, $(\forall x')$ or $(\exists x')$.
2. $(\forall x')\, (A \supset B(x'/x)) \supset . A \supset (\forall x')\, B\,(x'/x)$.

Rule.

$\vdash A \Rightarrow \vdash (\forall x')\, A\,(x'/x)$, where each free individual variable x in the wff A does not appear in the scope of a quantifier, $(\forall x')$ or $(\exists x')$.

Consider a proof in this formal system of the wff A(c), where c is an individual constant occurring in A. There are two ways in which this constant c could have been introduced into the proof. Firstly, c could have been introduced using the first axiom in the form, $(\forall x)\, B\,(x/x) \supset B(c/x')$, where at least one substitution of c has occurred. Secondly, c could have been introduced as part of a wff which has been substituted for a schematic letter. Let z be a free individual variable which does not occur at all in the proof of A(c). At each occurrence when the constant c was introduced into the proof of A(c), z could equally well have been introduced instead of c. Since z does not occur in the proof, z cannot be generalised upon using the rule and hence cannot be used in a way in which c could not be used, if z is substituted for c at each introduction of c in the proof. Hence A(z) is also provable. We have shown that if $\vdash$ A(c) then $\vdash$ A(z), where z does not occur in the proof of A(c). This is a rule that is used by Church on p. 242, [1], and by Mendelson on p. 75, [8], without proof or reference.

To sharpen this rule, we can generalise A(z), using the transformation rule, to prove $(\forall z')\, A(z'/z)$, where each z in A(z) does not appear in the scope of a quantifier, $(\forall z')$ or $(\exists z')$. Using the first axiom, we obtain $(\forall z')\, A(z'/z) \supset A(y/z')$, where y is substituted in $A(z'/z)$ for all occurrences of z' which are not in the scope of a quantifier, $(\forall z')$ or $(\exists z')$. Hence, we can prove A(y), where y is substituted for all occurrences of c in A(c). Henceforth, let us assume this condition when writing 'A(y)', after 'A(c)'. We have now shown that if $\vdash$ A(c) then $\vdash$ A(y). If we add the condition, that y does not occur in A(c), to guarantee that the occurrences of y in A(y) are at exactly the same places as the occurrences of c in A(c), then, if $\vdash$ A(y) then $\vdash$ A(c), and hence: (*) $\vdash$ A(c) iff $\vdash$ A(y), where y does not occur in A(c).

In the terminology of Hiż, on p. 194 of [4], and of Corcoran, on p. 432 of [3], (*) is an admissible rule, as it is thesis-preserving. Admissible rules, whether derived rules or not, can be added to the transformation rules of a system without affecting the set of theses of the system. Admissible rules are obtained by using meta-theoretic methods and, unlike derived rules, may not hold when the system they apply to is extended by the addition of extra axioms or extra primitives. However, as will be shown, (*) will continue to hold when the Predicate Calculus is extended, provided the individual constants a, b, c, ... are not axiomatically characterised.

Because Predicate Calculus is complete, (*) will yield the validity-preserving rule:

(+) A(c) is logically valid iff A(y) is logically valid, where y does not occur in A(c).

Hence, one set of symbols, either those for individual constants or those for free individual variables, is redundant, since the two sets of symbols are interchangeable in theses and logically valid wffs. Within the logic, there is no way of distinguishing the individual constants from the free individual variables, because they both have the same logical properties. The only difference that appears in the axiomatisation is that the transformation rule generalises on free individual variables rather than individual constants, but, by applying (*), it can be shown that this generalisation can also apply to individual constants.

In the next section, we will decide which it is better to eliminate from the formal system, individual constants or free individual variables.

§2. Semantic Considerations.

In order to make this decision, we will prove (+) using semantical methods. We will use the Tarski semantics of Predicate Calculus, as set out in Hunter, [5], pp. 141-149.

Consider the wff A(c) and the wff A(y), with the above convention applying. Let A(c) be logically valid. Then A(c) is true for every interpretation I and hence, for all interpretations I, every denumerable sequence s_D of members of the domain D of I satisfies A(c).

Putting this in symbols, using restricted quantification, ($\forall$ domains D) ($\forall$ interpretations I with domain D) ($\forall s_D$) (s_D satisfies A(c)). Hence,

($\forall$ domains D) ($\forall d \in D$) ($\forall$ interpretations I with domain D and with c assigned as d) ($\forall s_D$) (s_D satisfies A(c)).

Let y be the k-th variable in the enumeration of free individual variables. By specializing the sequences s_D, we obtain,

($\forall$ domains D) ($\forall d \in D$) ($\forall$ interpretations I with domain D and with c assigned as d) ($\forall s_D$ with d as its k-th member) (s_D satisfies A(c)). Such sequences s_D with d as their k-th member will also satisfy A(y), and hence,

($\forall$ domains D) ($\forall d \in D$) ($\forall$ interpretations I with domain D and with c assigned as d) ($\forall s_D$ with d as its k-th member) (s_D satisfies A(y)).

Since all the occurrences of c in A(c) are now replaced by y, this result would apply for all interpretations I with domain D, independently of the assignment given to c. Hence, ($\forall$ domains D) ($\forall$ interpretations I with domain D) ($\forall d \in D$) ($\forall s_D$ with d as its k-th member) (s_D satisfies A(y)). Hence, ($\forall$ domains D) ($\forall$ interpretations I with domain D) ($\forall s_D$) (s_D satisfies A(y)). Hence A(y) is true for every interpretation I and A(y) is logically valid.

We have shown that if A(c) is logically valid then A(y) is logically valid. Since, if A(y) is logically valid then A(c) is logically valid, where y does not occur in A(c), we have: (+) A(c) is logically valid iff A(y) is logically valid, where y does not occur in A(c).

Since a logically valid wff is one which is true for every interpretation, one has to consider all interpretations of A(c) to determine its logical validity. By keeping the domain D fixed, one can vary the interpretations of A(c) by varying the assignment of the individual constant c in the domain D. To obtain the rest of the interpretations of A(c), one would have to vary the domain D, as in the above proof.

Intuitively, an individual constant c should be assigned a unique member of a given domain D, whereas a free individual variable should be allowed to range over all members of the domain D. *By varying the interpretations of A(c)* in the above way, it can be seen that the individual constant c is allowed to range over all members of the domain, despite the fact that Tarski's semantics deals with individual constants and free individual variables differently. The way the free individual variable y is allowed to range over the domain D is by allowing the k-th member of sequences to be any member of D.

As also indicated in the proof, this type of assignment for individual constants accounts for the inter-substitutivity of individual constants and free individual variables, since both of these are allowed to range over a given domain D. However, such an interpretation for an individual constant is counter-intuitive and so it is better to eliminate the individual constants rather than the free individual variables from the formal system, given that Tarski's semantics is used.

However, if one changes the semantics in such a way as to give the intuitive interpretation to individual constants, that is, to assign to each individual constant a unique member of each domain, then one needs to be able to specify the appropriate member of each domain. One will have to assign to each individual constant a, b, c, ... of the formal system, a unique member of D, for each domain D. That is, for each individual constant, say a, there must be a function $\varnothing_a$, associated with it, so that a is assigned $\varnothing_a(D)$, for each domain D. In order for this to be so, there must be some characterising properties which are sufficient to pick out each member $\varnothing_a(D)$ of D, for each D. In order to formalise these properties, one would need a first-order theory with predicate constants and with non-logical axioms, each of which would express these properties of the individual constants. Hence, this semantics would then be more than just Predicate Calculus semantics as it deals with predicate constants in addition to the primitives of Predicate Calculus. Note that, in Predicate Calculus, unlike first-order theories, one cannot add axioms characterising individual constants without adding some predicate constants as well.

In the case of a domainless semantics for Predicate Calculus, the only way to provide an interpretation for an individual constant would be to give a set of characterising properties. However, this would yield a semantics for a first-order theory, as above.

Therefore, in any semantics for Predicate Calculus, individual constants cannot be given their intuitive interpretation and so should not be included in the formalisation of Predicate Calculus, with a semantics.

However, individual constants may appear in a formalisation of Predicate Calculus, provided they are not given their intuitive interpretation. They may be interpreted as variables or not given an interpretation at all. They may be included for reasons of technical expediency. They may also be included to show how specified individual constants would behave in the context of Predicate Calculus, if such individual constants were introduced.

§ 3. Other Arguments.

(i) One may want to say that the unspecified individual constants are arbitrary constants, thus allowing such an individual constant to be assigned any member of a domain. But then there would be no difference between the interpretations of 'free individual variable' and 'arbitrary individual constant' and the introduction of a concept of arbitrary individual constant would be superfluous.

Note that in Tarski's semantics the above two concepts would be distinguished using his notion of an interpretation, but he is artificially distinguishing two concepts which turn out to be the same. Moreover, in § 5, it will be shown that Tarski's notion of an interpretation can be replaced by an alternative notion which would not distinguish between the two concepts, if they were introduced.

(ii) One may argue that unspecified individual constants should be included in Predicate Calculus so as to form a predicate logic that is common to those first-order theories which contain individual constants. However, the individual constants, unspecified in Predicate Calculus, would be interpreted in many different ways in the first-order theories in which they are specified and could not be given any specific interpretation in any semantics of Predicate Calculus. Hence, as above, they would have to be left uninterpreted.

(iii) Some may argue that individual constants are essential for substituting into free individual variables. If this type of substitution is legitimate in the formal system, it is not essential for any such substitution to be made to show validity or to indicate the range of the variables. Moreover, the assignments to free individual variables are allowed to range over a domain in the semantics, and the substitution into free individual variables, that is needed, is semantic rather then syntactic. Hence the individual constants do not have to be introduced into the syntax for this purpose.

§ 4. Other Conclusions.

In this section, we will extend our conclusion about individual constants in Predicate Calculus to other constants and to first-order theories and Natural Deduction systems.

(i) The arguments that have been used for showing that unspecified individual constants should not be included in Predicate Calculus, with a semantics, can be modified in such a way as to apply to unspecified predicate and sentential constants in Predicate Calculus.

In formalisations of Predicate Calculus, authors have used either predicate variables (as in § 1), predicate «letters» or predicate «symbols», but not both predicate constants and predicate variables. The terminology, 'letters' and 'symbols', does not make it clear whether the predicate symbols involved are to be interpreted as predicate constants or predicate variables. It is preferable to use a terminology which indicates how the symbols are to be interpreted, that is, one should use 'predicate variables' or 'predicate constants', whichever the case may be.

In Hunter, [5], p. 141, predicate «symbols» are assigned some property or relation defined for objects in the domain of an interpretation, which, on further specification, becomes a set of ordered n-tuples of members of the domain of an interpretation. As with the assignments to an individual constant for a fixed domain, one can vary the interpretations of a predicate «symbol» by varying the assignments to it. Since these predicate «symbols» have no fixed assignment for a given domain and their assignments are allowed to range over all the sets of ordered n-tuples of members of the given domain, they are interpreted in the manner of predicate variables rather than predicate constants, in this semantics.

Results similar to (*) and (+) can be derived for predicate constants and variables, if there are both predicate constants and variables in the system. Let the predicate constants be symbolised using f', g', h', ... Let A(f') be provable. There is only one way in which f' could have been introduced into the proof of A(f') and that is that f' was introduced as part of a wff which was substituted for a schematic letter. However, the predicate variable f could have been introduced instead of f' and could have remained so for the rest of the proof, yielding a proof of A(f). Hence, if $\vdash$A(f') then $\vdash$A(f). Conversely, if f does not occur in A(f'), if $\vdash$A(f) then $\vdash$A(f'). Then we have:

(**) $\vdash$A(f') iff $\vdash$A(f), where f does not occur in A(f').

Because Predicate Calculus is complete, (**) will yield the validity-preserving rule:

(++) A(f') is logically valid iff A(f) is logically valid, where f does not occur in A(f').

Hence, one set of symbols, either those for predicate constants or those for predicate variables, is redundant, since the two sets of symbols are interchangeable in theses and in logically valid wffs. It seems from Tarski's semantics that the predicate constants should be eliminated as, in Hunter's account, predicate «symbols» are interpreted as predicate variables.

If one changes the semantics in such a way as to give the intuitive interpretation to predicate constants, that is, to assign to each predicate constant a unique set of ordered n-tuples from each domain, then, by a similar argument to that in § 2, the semantics becomes one for a first-order theory rather than for Predicate Calculus, because the semantics would have to contain characterising properties for the predicate constants. The case of a domainless semantics is dealt with as in § 2. Hence, unspecified predicate constants cannot be given their intuitive interpretation in any semantics for

Predicate Calculus and so should not be included in Predicate Calculus, with a semantics.

A similar case can be made for not including unspecified sentential constants in Predicate Calculus, with a semantics. Incidentally, this would also apply to the Sentential Calculus.

(ii) The same arguments used to eliminate unspecified individual, predicate and sentential constants in Predicate Calculus can be used for systems containing Predicate Calculus, such as first-order theories and Natural Deduction systems.

The admissible rules, (*) and (**) are obtainable for the above formal systems with the appropriate unspecified constants. The Predicate Calculus semantics is a part of the semantics for these systems and, using this part of the semantics, the validity-preserving rules, (+) and (++) are obtainable, since the remainder of the semantics is not relevant to the obtaining of these rules.

As argued before, when trying to give the intuitive interpretation to unspecified constants, one is forced into a semantics which contains properties used to characterise these constants, except for one type of first-order theory, for which an account will follow. For first-order theories, the semantics thus obtained would be the semantics of a first-order theory with all the unspecified constants specified sufficiently so that unique assignements have to be made for them in a semantics. So the semantics thus obtained would not be a semantics for the original first-order theory with unspecified constants but would be a semantics for a much stronger first-order theory with axioms characterising these constants. Such a semantics would not capture the arbitrariness of the unspecified constants in that it could only provide a specific interpretation for each constant.

There is also the case of unspecified individual constants in first-order theories which can only have one-element domains in their semantics, in which case the unspecified individual constants can only be assigned the one element and thus always have their intuitive interpretation. On the other hand, free individual variables would have to be interpreted as constants, since only one assignment can be made to them. But if free and bound individual variables are not separately symbolised, then one might as well leave the free individual variables in the formalisation to provide this technical simplifcation.

Hence, with the above exception, unspecified constants cannot be given their intuitive interpretation in any semantics for first-order theories or Natural Deduction systems and therefore unspecified constants should not be included in the formalisations of these systems, with a semantics.

For the rest of this section, we will note some points in relation to these systems.

In Predicate Calculus with Identity, which is taken to be a first-order theory, '=' is an example of a 2-place predicate constant and, for a given domain D in the semantics, '=' is assigned a particular set of ordered pairs with members from the domain D, i.e. $\{<d, d> / d \in D\}$. Also, '=' is axiomatically characterised by the two additional axioms (i) $x = x$ and (ii) $x = y \supset . A \supset B$, where B is obtained from A by substituting y for a particular argument-place of x in A, and where y is free for x in A.

A well-known example of unspecified individual constants being added to a first-order theory for the purpose of technical expedience is in the proof of the Skolem-Löwenheim Theorem. The individual constants are added so as to be able to prove that every consistent first-order theory has a consistent, complete and closed extension and to use this extension to obtain a countable model. (c.f. [7], pp. 160-164.) There is no need to interpret these individual constants and the notion of a first-order theory can be extended so as to include them.

In giving an account of first-order languages, one normally includes individual constants, predicate constants and function constants, but essentially these are there as a format, showing what sorts of constants can arise in a first-order theory. In a particular first-order theory, there is usually no need to include individual constants, say, if there are none axiomatically characterised and there is usually no need to add further individual constants if there are some specified ones.

The intuitive interpretation of a constant in the semantics of a first-order theory requires that the constant have a unique assignment using a given domain of a *normal* model. In non-normal models one would have to take into account the fact that several members, or even infinitely many members, of a domain may be identical with each other.

Tarski, in [10], and Tarski and Vaught, in [11], do not include any unspecified constants in their formalisations, all of which were first-order theories and not Predicate Calculus itself. Unspecified constants have been added by various authors, who have, as well, adapted his semantics to Predicate Calculus. So Tarski is not to blame for these adaptations to his original semantics.

In some Natural Deduction systems, individual constants are introduced by the Existential Instantiation rule. However, these constants are eliminated before a conclusion is reached and hence they do not appear in any thesis or deduction. Therefore, these constants do not have to be interpreted in the semantics of any of these Natural Deduction systems and they only form part of a method of proof. However, in Routley's Natural Deduction system, in [9], an expression, 'εxfx', read as 'any x which is f', is used instead of an individual constant and this expression can appear in theses and deductions. In this case, 'εxfx' would require an interpretation in the semantics, similar to that of a restricted variable.

§ 5. The Semantics of Predicate Calculus.

As a result of eliminating constants in Predicate Calculus, one can indicate improvements to Tarski's semantics. We will apply the improvements to Hunter's account in [5], pp. 141-149.

The main object of this section is to avoid the use of sequences in the semantics of Predicate Calculus by showing, without the use of sequences, that both free and bound individual variables can be interpreted in a simpler and more intuitive way.

As shown in § 2, an «individual constant» is interpreted as a free individual variable by assigning to it a member of the domain of an interpretation, and by allowing such assignments to range over all members of the domain, by varying the interpretations so as to keep the domain fixed. It is simpler to interpret free individual variables in this manner rather than by using sequences.

In the Tarski semantics, the use of sequences in interpreting a bound variable, say the k-th variable, is that the variable can be assigned a member of the domain using the k-th position in a sequence and that the assignment can be made to range over the domain by varying the member of the domain in the k-th position of the sequences. Such variation is required in determining whether a sequence satisfies a wff containing such a bound variable. Such a use of sequences can be replaced by using all the assignments in the domain for the k-th variable and by using truth-conditions for the particular quantifier involved.

In this semantics, one would need to replace the notion of «a sequence satisfying a wff for a given interpretation» by that of a wff being true for an interpretation with a given domain. Such an interpretation would be a set of assignments for all the (free) variables in a wff and an interpretation must be defined for a particular domain to allow the assignments to vary over their appropriate ranges. Once the assignments to all the (free) variables in a wff are made, it can be determined whether the wff is true or false for this interpretation. Using the notion of a wff being true for an interpretation with a given domain, one can define satisfiability and validity of a wff for a domain D and hence define satisfiability and logical validity of a wff. Note that this notion of a wff being true for an interpretation differs from Tarski's notion on p. 148 of [5].

Hence, the above method will enable one to eliminate sequences from the semantics. The sequences should be eliminated because, not only do they form a redundant concept, but they do allow different interpretations to be given for «individual constants» and free individual variables, as well as different methods of interpretation to be applied to the three types of (free) variables. It is better for the free individual, predicate and sentential variables to be interpreted by making assignments to the variables and by allowing these assignments to vary over the appropriate ranges. This is in keeping with the intuitive concept of a free variable, which would require a standard method of interpretation for the three types of free variables. It is by singling out the free individual variables for a different method of interpretation using sequences that Tarski arrives at his notion of a wff being true for an interpretation, which differs from the one we have introduced by applying the standard method of interpretation to all the free variables.

The semantics that one arrives at by making the above improvements on Tarski's semantics is similar to the semantics given in Church, [1], pp. 174-175 and to the semantics given in Leblanc, [6], p. 3. (1)

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