A NEW BASIS FOR CLASSICAL PROPOSITIONAL CALCULUS

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«If what is wanted is perspicuity and naturalness in the presentation of arguments», says Professor Kneale, «the best set of axioms for use with the principles of substitution and detachment is that given by Hilbert and Bernays in 1934.» (1)

Those axioms are (2):

1. CpCqp
2. CCpCpqCpq
3. CCpqCCqrCpr
4. CKpqp
5. CKpqq
6. CCpqCCprCpKqr
7. CpApq
8. CqApq
9. CCprCCqrCApqr
10. CEpqCpq
11. CEpqCqp
12. CCpqCCqpEpq
13. CCpqCNqNp
14. CpNNp
15. CNNpp.

Though naturalness is admittedly a foggy concept, it stands to reason that the easier to learn a set of axioms is, the more

(1) William & Martha KNEALE: The Development of Logic, Oxford, 1962, p. 527.

(2) David HILBERT & Paul BERNAYS: Grundlagen der Mathematik, 2nd ed., vol. 1, Berlin, Heidelberg & New York, 1968, p. 67. I use the Polish notation because of its convenience for deriving theorems, though of course it is much less natural than the infix notation. When A, B and C are well-formed formulae, or designations of such formulae, A(B/C) denotes the result of substituting C for B through A, A-B the result of detaching B from A, A-B-C the result of detaching C from A-B.

natural it is. Now from some experience in teaching elementary logic I feel sure that Hilbert and Bernay's set would be the best for the purpose but for the fact that it contains exported forms. Indeed it is only by being first taught the Law of Exportation that most students come to see the truth of axioms 1 and 2, and axioms 3, 6, 9, 12 and 13 are less easily grasped than the equivalent non-exported forms. It was therefore a pleasant surprise to discover that axioms 1 and 2 may be replaced by the Law of Exportation, viz.

1*. CCKpqrCpCqr,

and, consequently, axiomes 3, 6, 9, 12 and 13 by the equivalent non-exported forms, viz.
3*. CKCpqCqrCpr
6*. CKCpqCprCpKqr
9*. CKCprCqrCApqr
12*. CKCpqCqpEpq
13*. CKCpqNqNp.

Consistency and completeness of the new set will be proved by deriving from it 1, 2, 3, 6, 9, 12 and 13. We can get 1, 3, 6, 9, 12 and 13 without much ado, simply by substituting in the Law of Exportation and detaching 4, 3*, 6*, 9*, 12* and 13* respectively. But deriving 2 is a rather longish affair: I have been unable to find a shorter derivation than the following one.

20.	CpCqq	1*(r/q) — 5
21.	Cpp	20(p/CpNNp, q/p) — 14
22.	CpKpp	6(q/p, r/p) — 21 — 21
23.	CKpqKqp	6(p/Kpq, r/p) — 5 — 4
24.	CCpqCKprq	3(p/Kpr, q/p, r/q) — 4(q/r)
25.	CCpqCKrpq	3(p/Krp, q/p, r/q) — 5(p/r, q/p)
26.	CKpKqrq	25(p/Kqr, r/p) — 4(p/q, q/r)
27.	CKpKqrr	25(p/Kqr, q/r, r/p) — 5(p/q, q/r)
28.	CKpKqrKpq	6(p/KpKqr, q/p, r/q) — 4(q/Kqr) — 26
29.	CKpKqrKKpqr	6(p/KpKqr, q/Kpq) — 28 — 27
30.	CKCpqCrpCrq	3(p/KCpqCrp, q/KCrpCpq, r/Crq) — 23(p/Cpq, q/Crp) — 3*(p/r, q/p, r/q)

31.	CCpqCCrpCrq	1*(p/Cpq, q/Crp, r/Crq) — 30
32.	CCpKqrCpKrq	31(p/Kqr, q/Krq, r/p) — 23(p/q, q/r)
33.	CKCpqCprCpKrq	3(p/KCpqCpr, q/CpKqr, r/CpKrq) — 6* — 32
34.	CCpqCCprCpKrq	1*(p/Cpq, q/Cpr, r/CpKrq) — 33
35.	CCKpqrCKpqKrq	34(p/Kpq) — 5
36.	CCKpqrCKpqKpr	6(p/Kpq, q/p) — 4
37.	CCpqCKprKqr	3(p/Cpq, q/CKprq, r/CKprKqr) — 24 — 35(q/r, r/q)
38.	CCpqCKrpKrq	3(p/Cpq, q/CKrpq, r/CKrpKrq) — 25 — 36(p/r, q/p, r/q)
39.	CKpqKpNNq	38(p/q, q/NNq, r/p) — 14(p/q)
40.	CKCpNqqNp	3(p/KCpNqq, q/KCpNqNNq, r/Np) — 39(p/CpNq) — 13*(q/Nq)
41.	CCpqCNqNp	1*(p/Cpq, q/Nq, r/Np) — 13*
42.	CKCNpNqqp	3(p/KCNpNqq, q/NNp, r/p) — 40(p/Np) — 15
43.	CKCpqrKCNqNpr	37(p/Cpq, q/CNqNp) — 41
44.	CKCpqpq	3(p/KCpqp, q/KCNqNpp, r/q) — 43(r/p) — 42(p/q, q/p)
45.	CCpqCCKqrsCKprs	3(p/Cpq, q/CKprKqr, r/CCKqrsCKprs) — 37 — 3(p/Kpr, q/Kqr, r/s)
46.	CKKCpCqrpqr	45(p/KCpCqrp, q/Cqr, r/q, s/r) — 44(q/Cqr) — 44(p/q, q/r)
47.	CKCpCqrKpqr	3(p/KCpCqrKpq, q/KKCpCqrpq) — 29(p/CpCqr, q/p, r/q) — 46
48.	CCpCqrCKpqr	1*(p/CpCqr, q/Kpq) — 47
49.	CCKppqCpq	3(q/Kpp, r/q) — 22
2.	CCpCpqCpq	3(p/CpCpq, q/CKppq, r/Cpq) — 48(q/p, r/q) — 49

Since Hilbert and Bernays'set is consistent and complete (3), s
is mine.

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(3) HILBERT & BERNAYS, op. cit., pp. 65-67.