A FREGEAN INTERPRETATION OF THE QUANTIFIERS

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Perhaps it should not be surprising that Frege's account of the quantifiers can be transformed almost literally into a perfectly adequate formal semantics. The inventor of the quantifiers should, after all, have known what he meant. But contemporary interpreters have chosen to ignore Frege's explanation preferring instead to follow the technically sound account of Tarski. In what follows I would like to suggest some reasons for the apparent superiority of Tarski's definition and then go on to defend Frege. I shall remark on the naturalness of Frege's reading and the awkwardness of Tarski's and then state a formally adequate definition of truth capturing Frege's intuitions.

In those places where it differs from the usual semantics, Frege's account offers the more plausible reading. (See Frege [2] [3], [4], and Dummett [1].) On both accounts constants refer to individuals and closed sentences to truth-values. They also offer essentially the same interpretation of predicates. In the usual semantics predicates refer to sets of or relations on individuals, and in Frege's they refer to what he calls «concepts», functions from individuals to truth-values. These readings are virtually identical in that to each set there corresponds its characteristic function and vice versa. The two theories differ, however, in their treatment of open sentences. On the usual account open and closed sentences are treated as the same in the sense that they are both taken as referring to the same sort of thing. Open sentences are a genuine kind of sentence. But on Frege's account open sentences are treated like predicates, referring to functions from individuals to truth values. Of the two, Frege's is, I think, more plausible. There is certainly a semantic distinction to be made between closed and open sentences. One stands alone and conveys information while the other does not. It is natural to try to explain the difference in terms of the kinds of objects referred to. Further, there is a straightforward sense in which open sentences can be associated with sets or, more generally, relations. Corresponding to an open sentence of n free variables is the set of n-tuples which if taken as the references of n constants substituted into the open sentence for the variables would render the closed sentence true. In the usual first order set theory open sentences refer to just such relations when employed in set abstracts. But despite this intuitive appeal, Frege's views are not easily reconciled with the needs of semantical theory and the recursive definition of truth.

It is required for the ordinary recursive definition of truth that semantic structure closely mirror syntactic structure. To each category of expressions in the syntax, there should correspond a category of semantical values, and to each formation rule there should be a corresponding function on semantic values. Further, given any formation rule and a whole expression produced by it, the corresponding function on semantic values should determine a unique value for the whole given values for its parts. In short, assigment of semantic value should constitute a homomorphism from syntactic to semantic structure. Recursive definitions of the general notion of semantic value exploit such structure by first assigning values to atomic expressions and then in the various recursive clauses defining the functions on values corresponding to formation rules. It is this tight correspondence to syntax by semantics that underlies the familiar theorems of substitutivity of identity and material equivalence; it is at least as fundamental to our conception of first order logic as these theorems. But in this general framework Frege's definition of the quantifiers seems, at first glance, unworkable. In the usual syntax for the quantifiers there is no distinction drawn between open and closed sentences. Formation rules are defined for both if for either. It follows then that there should be no difference in the kinds of values assigned to open and closed sentences. But Frege's account seems to require such a distinction, interpreting open sentences by concepts and closed sentences by truth-values. But as we shall see in the subsequent formal account, Frege's views can be squeezed into the necessary form by viewing truth-values as a degenerate type of concept. The general success of the usual Tarski definition lies in the fact that it does not distinguish between open and closed sentences in terms of values, manages to mirror syntax in semantics, and still provides some account of the difference between open and closed sentences.

Tarski's semantics is usually presented in one of two ways. (See Tarski [7] and [8] respectively.) In the first, open and closed sentences are assigned truth-values relative to an interpretation of their variables. It is in terms of this relativized concept of truth that semantics parallels syntax and values of parts determine those of wholes. A derivative concept of truth *simpliciter* is then defined as truth relative to all interpretations of variables. Interestingly enough, the non-relativized truth-value of the whole is not uniquely determined by those of its parts. Consider the case in which nothing is red and something but not everything is blue. Then both 'x is red' and 'x is blue' are F, but '$(\exists x)$ (x is red)' is F while '$(\exists x)$ (x is blue)' is T. Hence the need for the better behaved truth-values relative to variable assignments. Additional reference to the derivative concept of truth also makes it possible to distinguish semantically between open and closed sentences. Closed sentences turn out to be true *simpliciter* if they are true relative to at least one variable assignment whereas open sentences in general do not. This account, therefore, not only meets the technical requirements of the recursive definition, it also makes all the necessary conceptual distinctions. It does, however, suffer from the awkwardness, perhaps it should even be called implausibility, of assigning relative truth-values to closed sentences and ordinary truth-values to open sentences.

In the other usual development of Tarski's theory the variable assignments that satisfy sentences, both open and closed, are grouped into sets and literally assigned as semantic values to the sentences they satisfy. Though this interpretation of sentences is intuitively peculiar, formally its results are quite elegant. Only one notion of value needs to be recognized thus avoiding the relative/non-relative distinction, and the values of the parts uniquely determine that of the whole in straightforward ways. Thus, Montague, who particularly stresses the mirroring of syntax by semantics, uses this interpretation in some of his accounts of natural language. (See Montague [5] and [6].) Further, a sentence turns out to be what we would intuitively call true if it refers to the universal set of variable assignments and false if it refers to the empty set. Closed sentences are distinguished from open in that the former but generally not the latter refer either to the universal or the empty set. But intuitions must be bent a great deal to accept sets of variable assignments as a category of semantic value. They are certainly not suggested by any pre-analytic sense of reference nor are they even remotely a traditional metaphysical category. It can be plausibly argued that notions of reference can be expanded and adjusted for technical reasons or because of their satisfying consequences. Thus truth-values as semantic category have lost much of their strangeness and so, to some extent, have sets of variable assignments. But the closer theory conforms to intuition the better. Thus, with these remarks as justification, let us turn now to the formal details of a Fregean semantics for first order logic.

The basic syntactic categories of L are stipulated as follows:

(1) Constants: $C = \{a_1, \ldots, a_n, \ldots\}$
(2) Variables: $V = \{x_1, \ldots, x_n, \ldots\}$
(3) Terms: $T = C \cup V$
(4) Predicates: $P = \{Q^0_1, \ldots, Q^0_n, \ldots; Q^1_1, \ldots, Q^1_n, \ldots; \ldots\}$
(5) Logical Signs: $LS = \{(,), \forall, \wedge, \neg\}$

Let E, the set of *expressions* of L, be any finite string of symbols from $T \cup P \cup LS$. The set F of *formulas* of L is defined as

$\bigcap \{X$: (1) if $t_1, \ldots, t_n \varepsilon T$ and $P^n \varepsilon P$, then $P^n t_1 \ldots t_n \varepsilon X$;
(2) if $A, B \varepsilon X$, $v_n \varepsilon V$ then,

$(A \wedge B)$, $\neg (A)$, $(\forall v_n)(A) \varepsilon X\}$.

Notice that *V* as defined is in a natural order, which we now single out for attention. Let f be a 1 - 1 function mapping the set of natural numbers onto *V* such that $f(n) = v_n$. We now define the auxillary notion of the formula A *containing free variables in their natural order* $v_k, \ldots, v_m$ which we write briefly as $A[v_k, \ldots, v_m]$.

(1) If $A = P^n t_1 \ldots t_n$, then $A[v_k, \ldots, v_m]$ is $P^n t_1 \ldots t_n[v_k, \ldots, v_m]$ where $[v_k, \ldots, v_m]$ is f restricted to those natural numbers that have as value under f some variable $v = t_i$, for $i = 1, \ldots, n$.
(2) If $A = (B[v_g, \ldots, v_h] \wedge C[v_i, \ldots, v_j])$, then $A[v_k, \ldots, v_m] = (B \wedge C)[v_k, \ldots, v_m]$ where $[v_k, \ldots, v_m] = [v_g, \ldots, v_h] \cup [v_i, \ldots, v_j]$.
(3) If $A = \neg (B[v_i, \ldots, v_j])$, then $A[v_k, \ldots, v_m] = \neg (B)[v_i, \ldots, v_j]$.
(4) If $A = (\forall v_h)(B[_e, \ldots, v_j])$, then $A[v_k, \ldots, v_m] = (\forall v_h)(B)[v_e, \ldots, v_{h-1}, v_{h+1}, \ldots, v_j]$.

By a *model M* for *L* is meant any $<U, R>$ such that $U \neq \varphi$ and *R* is a function on $C \cup P \cup F$ meeting the following conditions. Let u with subscripts rang overe over *U*; c over *C*; A, B, and C over *F*; and P^n over members of *P* with superscript (degree) n.

(1) $R(c) \varepsilon U$;
(2) $R(P^n) \subseteq U x\{T, F\}$;
(3) if $A = P^n t_1 \ldots t_n[v_k, \ldots, v_m]$, then $R(A[v_k, \ldots, v_m]) = \{<u_k, \ldots, u_m, \alpha>$:
   (1) if for some $x_1, \ldots, x_n$, $<x_1, \ldots, x_n, T> \varepsilon R(P^n)$, and for all t_i, $i = 1, \ldots, n$,
      (a) if $t_i = v_j$, then $x_i = u_j$, for all $j = k, \ldots, m$,
      (b) if $t_i \varepsilon C$, then $x_i = R(t_i)$,
      then $\alpha = T$;
   (2) $\alpha = f$ otherwise};
(4) if $A = (B[v_g, \ldots, v_h] \wedge C(v_i, \ldots, v_j])$ then $R(A[v_k, \ldots, v_m]) = \{<u_k, \ldots, u_m, \alpha>$:

(1) if for some $x_g, \ldots, x_h, y_i, \ldots, y_j$,
$\{x_g, \ldots, x_h\} \cup \{y_i, \ldots, y_j\} = \{u_k, \ldots, u_m\}$, &
$<x_g, \ldots, x_h> \varepsilon R(B[v_g, \ldots, v_h])$, &
$<y_i, \ldots, y_j> \varepsilon R(C[v_i, \ldots, v_j])$,
then $\alpha = T$,
(2) $\alpha = F$ otherwise};

(5) if $A = \neg (B[v_k, \ldots, v_m])$, then $R(A[v_k, \ldots, v_m]) = U^k x \ldots x U^m x\{T, F\} - R(B[v_k, \ldots, v_m])$.

(6) if $A = (\forall v_h)(B[v_k, \ldots, v_m])$, then $R(A[v_k, \ldots, v_{h-1}, v_{h+1}, \ldots, v_m]) = \{<u_k, \ldots, u_{h-1}, u_{h+1}, \ldots, u_m, \alpha>:$
(a) if either $<u_k, \ldots, u_{h-1}, u_{h+1}, u_m, T> \varepsilon R(B[v_k, \ldots, v_m])$,
or for any $x \varepsilon U$,
$<u_k, \ldots, u_{h-1}, x, u_{h+1}, \ldots, u_m, T> \varepsilon R(B[v_k, \ldots, v_m])$,
then $\alpha = T$
(b) $\alpha = F$ otherwise

Clearly this semantics is straightforwardly equivalent to the usual one in the sense that models for one determine models for the other. Further all open sentences are assigned functions from n-tuples of U to $\{T, F\}$, and closed sentences are assigned truth-values.

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