THE LOGICAL MODALITIES AND THE SYSTEM S5

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One important question in the philosophy of modal logic is how to interpret the modal operators in the various systems. In this paper I will examine an answer to this question with regard to one system, an answer that has been widely accepted: that in the system S5, the modal operators may be taken to express logical possibility and logical necessity. I will first argue that this answer, or «doctrine», as I shall call it, leads to a contradiction when interpreted in a way that initially appears plausible. Then I will discuss what I believe to be the main defense of this view, that given by Rudolf Carnap, and show that it contains several errors. I believe that these arguments, together with those that have already been given by Nino B. Cocchiarella, are sufficient to discredit the doctrine in question. (1)

1. *An S5 object language*

We begin by constructing an S5 object language. The symbols of the language are the logical symbols «∼», «v», and «□», a finite list of predicates, and a finite list of names. (The other logical symbols are defined as usual). The atomic sentences of the object language are those that are constructed solely out of names and predicates, with no logical symbols. Further sentences are generated from this

(1) For examples of adherence to the doctrine, see pp. 8-11 of [1]; pp. 22-23 of [9]; pp. 160-61 of [10]; and pp. 162 and 166 of [13]; for a qualified adherence, where the qualification anticipates the main point of this paper, see pp. 81-82 of [8]. For Carnap's defense of the doctrine, see [3] and Ch. 5 of [2]. For Cocchiarella's attack on it, see [5] and [6].

initial set by using the logical signs: if A and B are sentences of the object language, then so are $\sim A$, $(A \vee B)$, and $\Box A$. (2)

Certain sentences of the object language will be said to be «provable». A sentence of the object language is provable if and only if (iff) it is obtainable by uniform substitution in one of these axiom-schemata, or derivable by applications of these rules from sentences so obtainable:

AS0. A set of axiom-schemata which, together with R1, provide a complete basis for the non-modal propositional calculus.

AS1. $\Box(A \supset B) \supset (\Box A \supset \Box B)$
AS2. $\Box A \supset A$
AS3. $\sim\Box A \supset \Box\sim\Box A$
R1. If A is provable and $(A \supset B)$ is provable, then B is provable.
R2. If A is provable then $\Box A$ is provable.

(It is easily shown that these rules and axiom-schemata determine a system which is deductively equivalent to Lewis' system S5).

One difficulty with the doctrine that the S5 modal operators express the logical sense of the modal concepts is that it is vague: just what does it assert ? We can make it more specific by stating truth-conditions for sentences of our S5 object language. Now the property of being provable is intended to be a defined concept that represents our intuitive concept of logical truth or logical necessity. So it would seem that if the S5 necessity-operator expresses logical necessity, then the assertion, in an S5 object language, that a given sentence is necessarily true should itself be true if and only if the given sentence is provable. Thus we can give the doctrine a precise interpretation by adopting this truth-condition:

(2) Here and throughout the paper, «A», «B», and «A'» are used in the metalanguage as variables ranging over sentences of the object language, and the logical symbols when used in conjunction with the metavariables, are used autonymously. — My approach in the first part of this paper (sections 1 and 2) owes much to Chs 2, 3, and 4 of [11].

TC1. $\Box$A is true iff A is provable.

The other truth-conditions are the usual ones:

TC2. $\sim$A is true iff A is false.

TC3. (A v B) is true iff A is true or B is true (or both).

2. *A contradiction in the metalanguage*

We now have rules of truth for determining the truth-values of non-atomic sentences of the object language. But in addition, some sentences of the object language are provable. Since we want a sentence to be provable only if it is logically true, we do not want any sentences which are false according to our truth-conditions to be provable. Let us assume that this is in fact the case, and that any sentence which is provable is true.

Let A' be an atomic sentence of our object language, and consider the sentence:

(I) $(\Box\Box A') \vee (\Box\sim\Box A')$

(I) is provable. To simplify the proof, we will not bother to prove the S4 axiom-schema, $\Box A \supset \Box\Box A$, which, as is well known, is a theorem of S5; and steps in the proof that are justified by the non-modal propositional calculus are simply labeled «PC».

(1)	$\Box A' \vee \sim\Box A'$	PC
(2)	$\Box A' \supset \Box\Box A'$	S4 axiom-schema
(3)	$\sim A' \supset \Box\sim\Box A'$	AS3
(4)	$(\Box\Box A') \vee (\Box\sim\Box A')$	(1), (2), (3), PC

Since (I) is provable, it must be true, if we assume that all provable sentences are true. But now let us examine it in terms of our truth-conditions.

Consider first the left side of (I). $\Box$A' is true iff A' is provable. Now A' is atomic, by hypothesis. So if there were a proof of A', we could obtain from it a proof of $\sim$A' by substituting $\sim$A' for A' everywhere in the proof; if A' were provable, $\sim$A' would be too. But A' and $\sim$A' are not both true, given TC2; so assuming that only true sentences are provable, A' and ~A' are not both provable. Therefore A' itself is not provable, and so □A' is false. Consequently, (□□A') is false also. Thus the left side of (I) is false.

Next consider the right side. As we have seen, □A' is false; hence ~□A' is true. But it is not provable. For if it were, then, since A' is atomic, it would be possible to prove ~□(B v ~B), where B is any sentence, by substituting (B v ~B) everywhere for A' in a proof of ~□A'. But since (B v ~B) is provable, □(B v ~B) is true, and ~□(B v ~B) is false. So if ~□A' were provable, then a false sentence (~□(B v ~B)) would be provable, contrary to our assumption. So ~□A' is not provable. Consequently, (□~□A'), the right side of (I), is false.

Since both sides of (I) are false, (I) itself is false, given TC3. But as we have seen, (I) is provable. Our truth-conditions for sentences of the object language in conjunction with the assumption that all provable sentences are true thus lead us to the conclusion that (I) is both true and false.

We have reached a contradiction. We might resolve it by giving up the assumption that only true sentences are provable in S5; but in that case S5 loses all appeal as a system of logic. Better to jettison TC1, which asserts that □A is true iff A is provable. But TC1 constituted our attempt to give a precise meaning to the view that the S5 modal operators express the logical sense of the modal concepts. So much the worse, we conclude, for that doctrine.

3. One source of the doctrine

We thus have found some reason for believing that the doctrine we have been examing is false. What we have found is not entirely conclusive by itself; there may be some other interpretation of «expresses logical necessity» on which the S5 necessity-operator does express logical necessity. But Cocchiarella, basing his arguments on different considerations, has reached the same conclusion that we have. The evidence all seems to point in the same direction.

Nevertheless, some doubt may remain. It is likely to take this form: if the doctrine is false, how did it come to be so widely accepted ? Have no arguments been given in its favor ? In the remainder of the paper I will try to remove this source of doubt.

The main proponent of the view we are criticizing was, as we noted earlier, Rudolf Carnap. His best-known presentation of the view is probably that given in the final chapter of his book *Meaning and Necessity: A Study in Semantics and Modal Logic.* There we are told first (p. 173) that the necessity-operator is being used as a sign for logical necessity; then (p. 174) we are given a convention which an operator must satisfy if it is to express logical necessity, and told (pp. 174-75) that any operator that satisfies this convention will obey both the S4 and the S5 axioms. Next (pp. 182-84) semantical rules are stated which cause the necessity-operator to satisfy the previously stated convention; and finally (pp. 185-86) it is asserted, in effect, that every sentence that is provable in an S5 object language is L-true (logically true, valid) on these semantics, the very same semantics on which the necessity-operator expresses logical necessity. All of this would seem to show that the S5 modal operators do represent the logical sense of the modal concepts after all.

But though *Meaning and Necessity* contains what is probably the best-known presentation of the doctrine, Carnap does not there argue for it directly. He states his arguments instead in his contemporaneous paper, «Modalities and quantification.» So we will focus our attention on that paper, though the conclusions we reach will also apply to the *Meaning and Necessity* presentation.

In outline, the argument of «Modalities and quantification» is this: Carnap again begins by stating (36) (3) a convention

(3) Here and for the rest of the paper, numbers in parentheses refer to pages in [3]. In quoting from Carnap, I have sometimes altered his terminology somewhat; for example, he uses «N» rather than «$\Box$».

which an operator must satisfy if it is to express logical necessity. Then (38-43) he constructs a system of modal propositional logic, which he asserts (41n., 46) is equivalent to Lewis' system S5. Next (49-57) he constructs a system of modal functional logic, which he says (40) contains his modal propositional logic. Finally (54) he asserts that the necessity-operator in his modal functional logic satisfies the convention laid down initially, and so may properly be said to express logical necessity.

This pioneering article has not previously received any critical attention, to the best of my knowledge. As late as 1963 we find Robert Feys asserting that Carnap succeeded in «mak[ing] necessity correspond exclusively to L-truth,» and that Carnap's semantics for S5 «leads to a satisfactory proof of completeness of the system» (4). It is now easy to see, however, given the progress that has been made since 1946, that Carnap's article contains several errors. We will see that his quantified modal logic, though it does satisfy his criterion for an operator's expressing logical necessity, is not simply a quantified version of his modal propositional logic; and that the latter, though it is equivalent to S5, does not satisfy his criterion. Thus, I will argue, Carnap's results do not tend to show that the S5 necessity-operator expresses logical necessity.

4. *Carnap's criterion*

One of the chief merits of Carnap's paper is that he makes explicit what it means for a necessity-operator to express logical neccessity. He argues that we should regard an operator «□» as expressing logical necessity only if it satisfies the following condition:

C1-2. If «...» is L-true, «□(...)» is L-true; otherwise «□(...)» is L-false. (36).

(4) In [7], p. 292.

Here L-truth, now usually called validity, is the formally defined concept that corresponds to («explicates») the intuitive concept of logical truth. (There is a second convention, C1-3 (37), which Carnap sets up as a requirement for an operator's expressing logical necessity; but that further requirement will not concern us.)

The general nature of Carnap's argument for this criterion is that no factual information is needed to determine whether a given proposition is logically true. Hence the assertion that a given proposition is logically true is itself either logically true or logically false, according as the given proposition is logically true or not.

Carnap's convention seems to me to be an acceptable criterion for determining whether the necessity-operator in a given system expresses logical necessity. What I wish to discuss is whether Carnap's systems satisfy this criterion, and what relationships his systems bear to S5.

5. *Carnap's quantified modal logic*

Carnap calls the semantical part of his system of quantified modal logic «MFL» (modal functional logic) and the syntactical part «MFC» (modal functional calculus). The explicit claim that the necessity-operator represents logical necessity is made (54) with respect to MFL.

This claim is in fact justified, given MFL's satisfaction-condition for sentences governed by the necessity-operator and its definition of L-truth:

D9-5i. The range of □A is the universal range if the range of A is the universal range; otherwise the range of □A is the null range. (54)

D7-6a. A is L-true (in MFL) =Df. the range of A is the universal range. (54; 51)

It is easily seen, even without going into what Carnap means by «ranges», that D9-5i and D7-6a have the consequence that the necessity-operator of MFL satisfies condition C1-2. If A is L-true (has the universal range), then □A is L-true (has the universal range); otherwise □A is L-false (has the null range). Carnap has succeeded in constructing a system that meets his criterion for an operator's expressing logical necessity.

The question we want to ask, however, is: what does this show us about S5 ? Is MFL a quantificational form of S5 ?

We need to make this question more precise in order to answer it. So let us say that a quantificational system Q *contains* a propositional system P iff the result of uniformly substituting sentences of Q for sentences of P in an L-true sentence of P is always an L-true sentence of Q. (Roughly, Q contains P if every theorem of P is a theorem of Q.) We shall say that Q *is based on* P, on the other hand, iff the result of substituting sentences of P for those of Q in an L-true sentence of Q that contains no individual-variables (only constants) is always an L-true sentence of P. (Roughly, Q is based on P if every variable-free theorem of Q is a theorem of P.)

Now we may state our question more clearly, and we can see that it has two parts: does MFL contain S5 ? Is it based on S5 ? And the answer is that while MFL does, as Carnap claims (40, 46), contain S5, it is not based on S5. It is based on a propositional logic which is stronger than S5, a logic which has all the theorems of S5 and other theorems in addition (5).

This can be seen if we let A' be an atomic sentence of MFL. Since it is atomic, the range of A' is not the universal range. (In other words, its negation is satisfiable). But by D9-5i, since the range of A' is not universal, the range of □A' is the null range. Consequently, given the rule of ranges for negations (D5-7b, (50)), the range of ~□A' is the universal range, and so ~□A' is L-true in MFL according to D7-6a. But as we saw earlier, ~□A' is not provable in an S5 language if A' is atomic. So on any semantics on which S5 is sound and complete, ~□A' is not L-true. But it is L-true in MFL; therefore MFL is not based on S5.

The same point can be made by examining MFC. As Carnap

(5) The system in question is the one that has been called S13. For a characterization of it see [5] and [4].

asserts, sentences of the form $\sim\Box A$ are provable in MFC, «where A is a disjunction of *n* components ($n \geq 1$), each being an atomic sentence or a negation of an atomic sentence, but no atomic sentence occurring together with its negation» (57). Thus, letting $n = 1$ and letting A' be an atomic sentence, $\sim\Box A'$ is provable in MFC but not, of course, in S5.

So it is true that MFL-MFC contains S5, and that the necessity-operator in MFL-MFC expresses logical necessity. But we cannot conclude from this that the S5 necessity-operator expresses logical necessity by Carnap's criterion, because MFL-MFC is not based on S5. This means that it contains sentences which are L-true on non-quantificational grounds, but are not L-true in S5. If these sentences were L-true in S5, then S5's necessity-operator would express logical necessity; but since they are not, it does not.

6. *Carnap's modal propositional logic*

Carnap calls the syntactical part of his modal propositional logic «MPC» and the semantical part «MPL». MPC, with trivial modifications, is, as Carnap claims (46), deductively equivalent to the system S 5. So we may turn to the questions of whether MPC is sound and complete with respect to MPL, and whether MPL causes the necessity-operator to satisfy C1 - 2.

To do this, we must again examine the satisfaction-condition for sentences governed by the necessity-operator, and the definition of L-truth. The former is the same for MPL as it was for MFL, and is given by D9-5i, above. With regard to the latter, however, there is a complication. As Carnap explains, he «do[es] not define MPL as an independent system but only: The system *S* contains MPL» (40). Consequently, his defition of L-truth takes this form:

D4-1. A is *L-true by MPL* in *S* =Df. *S* contains MPL; A belongs to *S*; every sentence formed out of A in the following way is L-true in *S*: the ultimate MPL components of A (i.e., those sentences out of which A is built up with the help of connectives and «□», which themselves, however, are not thus built up out of other sentences) are replaced by any sentences of *S* (occurrences of the same component to be replaced by occurrences of the same sentence). (40-41)

This definition remains indeterminate unless «L-true in *S*» is defined. However, we know how Carnap intended that «L-true in *S*» be defined, since he remarks just before giving this definition that «L-truth is defined by the universality of the range» (40), as it is in D7-6a, above. So we may set up MPL as an independent system by replacing «L-true in *S*» in the definiens with «has the universal range»:

D4-1*. A is L-true in MPL =Df. every sentence formed out of A in the following way has the universal range: the ultimate MPL-components of A are replaced by any sentences of MPL.

This definition of L-truth does provide a semantics on which MFC (and consequently also S5) is sound and complete, as Carnap shows (45-46). But does it provide a semantics on which the necessity-operator satisfies condition C1-2 ?

No, it does not. Once more, let A' be an atomic sentence of MPL. Now if the necessity-operator is to satisfy C1-2 then, since A' is not L-true, □A' must be L-false. But □A' is L-false iff ~□A' is L-true; and ~□A' is not L-true in MPL. For if we replace A' in ~□A' with (B v ~B), we obtain ~□(B v ~B); and this sentence does not have the universal range. (It has the null range.) So it is not the case that every sentence formed out of ~□A' by replacing its ultimate components has the universal range; and so, according to D4-1*, ~□A' is not L-true in MPL. Therefore □A' is not L-false in MPL, even though A' itself is not L-true in MPL; hence MPL's necessity-operator does not satisfy C1-2. Thus these results of Carnap's, though they do provide a semantics on which S5 is sound and complete, do not tend to show that the S5 necessity-operator expresses logical necessity.

7. *Conclusion*

In the first part of this paper we explored one interpretation of the doctrine that the modal operators in the system S5 may be understood as expressing logical necessity and logical possibility, and found that it led to a contradiction. In the second part we examined a defense of the doctrine by one of its major proponents, and found it to be unsound. When we add to this the arguments which Cocchiarella has given against it, it seems quite certain that the doctrine is in error. (6)

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