ON PROING LINEARITY

Robert P. McArthur

1. In his review of my paper «The Makinson Completeness of Tense Logic» [LOGIQUE ET ANALYSE, 17 (Sept - Dec, 1974), pp. 453-460] (1) David Makinson notes that the proof offered for Lemma 4(b) [p. 458] is incorrect. The source of the mistake reveals something of the character of linear tense logics which has not been studied in the literature. In this paper I will report the error which Makinson discovered and provide a new proof for the Lemma. For those unfamiliar with linear tense logic or my earlier paper, a summary is provided in the next section.

2. A *tense logic* adds two sentential operators to the primitive signs for classical logic. These are 'G' (read: «It will always be the case that») and 'H' (read: «It has always been the case that»). Hence, in addition to the usual formation rules, a tense logic requires that 'GA' and 'HA' are well-formed formulas (wffs) if 'A' is a wff. For the purposes of this paper, the classical signs '$\supset$', '$\sim$', '(', and ')' (in their usual roles) will be deployed and a denumerable run of sentence letters will be presumed available. Henceforth, 'A', 'B', and 'C' will be used only to designate wffs, and '$\sim G \sim$' and '$\sim H \sim$' will be abbreviated by 'F' (read: «It will be the case that») and 'P' (read: «It has been the case that»), respectively.

The axiom schemata for the *linear* tense logic *CL* are as follows: (2)

(1) Makinson's review is forthcoming in ZENTRALBLATT FUR MATHEMATIK. It was through correspondance with Dr. Makinson that I learned of the error he discovered.

(2) Although known widely in the literature as *CL* this system is called TL^3 in the original paper. A few other alterations have been incorporated here to bring the notation and terminology in line with those used in my TENSE LOGIC (D. Reidel Publishing Co., 1976).

A1. A, wehere A is a tautology
A2. G(A ⊃ B) ⊃ (GA ⊃ GB)
A3. PGA ⊃ A
A4. GA ⊃ GGA
A5. (FA & FB) ⊃ (F(A & B) V (F(A & FB) V F(B & FA)))
A6. GA, where A is an axiom
A7. MI(A), where A is an axiom and MI(A) is the result of simultaneously replacing each occurrence of G in A by H and each occurrence of H by G.

Modus Ponens is the only rule of inference for *CL*.

The derivability of (wff) A from set (of wffs) S — S ⊢ A, for short — the provability of A — ⊢ A, for short — and the consistency of S are all understood for *CL* in the usual manner.

By a *truth-value assignment* φ for CL is meant any function from the sentence letters of *CL* to {1, 0} (the truth-values). By a *familiy of truth-value assignments* is meant any *indexing function* Ω from the positive integers to the truth-value assignments for *CL*. It is helpful intuitively to think of Ω as a set of truth-value assignements paired with indices, i.e., as having members of the sort $<\varphi, i>$, where φ is a truth-value assignement and i its (integer) index. Notice that whereas only one truth-value assignment is given each index, several indices may be given the same truth-value assignment. Thinking of indexed truth-value assignments as «world-states», this means the «world» can be the same on several occasions. (3)

Let R be any dyadic relation on the members of Ω which has the following properties:

P1. (∀x) (∀y) (∀z) ((R(x, y) & R(y, z)) ⊃ R(y, z))
P2. (∀x) (∀y) (∀z) ((R(x, y) & R(x, z)) ⊃ ((y = z) V (R(y, z) V R(z, y))))
P3. (∀x) (∀y) (∀z) ((R(y, x) & R(z, x)) ⊃ ((y = z) V (R(y, z) V R(z, y))))

(3) For details on the interpretation of the semantics for *CL* consult *TENSE LOGIC*, Ch. 1.

Truth and falsity are calculated for the wffs of *CL* by means of triples of the sort $<\Omega, <\varphi, i>, R>$, which are called *historical moments.* Pairs of the sort $<\Omega, R>$ are called *histories.* It is useful to think of R as the earlier/later-than relation on the «world-states» in Ω. Wff A is said to be *valid within the history* $<\Omega, R>$ iff A is true at every historical moment $<\Omega, <\varphi, i>, R>$ such that $<\varphi, i>$ is a member of Ω. A is said to be *CL-valid* iff A is valid within every history of *CL.* Satisfaction and entailment for CL are defined as usual.

The truth conditions for the wffs of *CL* are as follows:

(1) Where A is a sentence letter, A is true at $<\Omega, <\varphi, i>, R>$ iff $\varphi(A) = 1$;

(2) Where A is a negation $\sim B$, A is true at $<\Omega, <\varphi, i>, R>$ iff B is not true;

(3) Where A is a conditional $B \supset C$, A is true at $<\Omega, <\varphi, i>, R>$ iff either B is not true or C is;

(4) Where A is of the sort GB, A is true at $<\Omega, <\varphi, i>, R>$ iff B is true at *every* $<\Omega, <\mu, j>, R>$ such that $R(<\varphi, i>, <\mu, j>)$; and

(5) Where A is of the sort HB, A is true at $<\Omega, <\varphi, i>, R>$ iff B is true at *every* $<\Omega, <\mu, j>, R>$ such that $R(<\mu, j>, <\varphi, i>)$.

Showing that all of A1 — A7 are *CL*-valid is a straightforward matter and will not be dealt with here. As for *completeness* (demonstrating that any set of wffs which is *CL*-consistent is also *CL*-satisfiable), the earlier paper used an adaptation of Henkin's methods due to David Makinson to secure this result. As further background for the next section, the principal features of the proof will be reviewed. ([4])

([4]) See the original paper for the full proof. As pointed out there, the basic structure of this completeness proof for *CL* (and other tense logics) was originated by Makinson. See his «On Some Completeness Theorems in Modal Logic» ZEITSCHRIFT FUR MATHEMATISCHE LOGIK UND GRUNDLAGEN DER MATHEMATIK, Band 12 (1966), pp. 379-384.

Starting with a consistent set S of wffs of *CL*, it is expanded into a maximally consistent set S^∞. For each wff of the sort FA in S^∞, a set is formed consisting of A and all wffs B such that GB is a member of S^∞. The maximally consistent extension of such a set is called a *future attendant* of S^∞. For each wff of the sort PA in S^∞, a set is formed consisting of A and all wffs B such that HB is a member of S^∞. The maximally consistent extension of such a set is called a *past attendant* of S^∞.

Next formed is Ω_S, the least set containing S^∞, its future and past attendants, their past and future attendants, their future and past attendants, etc. Owing to the (at most) denumerability of Ω_S, its members can be ordered by the integers. S^∞ shall be dubbed S^∞_1, and the other members of Ω_S shall be referred to as S^∞_2, S^∞_3, etc.

A dyadic relation R_S is defined on the members of Ω_S as follows:

$R_S(S^\infty_i, S^\infty_j)$ iff, for any wff GA in S^∞_i, A is a member of S^∞_j.

For each S^∞_i in Ω_S there is a *corresponding* indexed truth-value assignment which is characterized as below:

$<\varphi, i>$ is the corresponding truth-value assignment for S^∞_i iff for every sentence letter A in S^∞_i, $\varphi(A) = 1$.

Note that the index accorded φ in this case is the integer assigned to S^∞_i in the ordering of Ω mentioned above.

Let Ω be the family of indexed truth-value assignements corresponding to the members of Ω_S, and let R be defined as $R(<\varphi, i>, <\mu, j>)$ iff $R_S(S^\infty_i, S^\infty_j)$, where $<\varphi, i>$ corresponds to S^∞_i and $<\mu, j>$ corresponds to S^∞_j.

By an induction on the complexity of a wff A, it is easily shown that A is a member of S^∞_1 iff A is true at $<\Omega, <\varphi, 1>, R>$, where $<\varphi, 1>$ is the corresponding (indexed) truth-value assignment for S^∞_1.

If it can be shown that $<\Omega_S, R_S>$, and hence $<\Omega, R>$, is appropriately constructed for *CL*, it follows that S, the initial set, is satisfiable in *CL*, thus guarenteeing the completeness of *CL*. But this is the controverted step.

3. The error in the orig'nal version of this completeness proof came at the point where the appropriateness of $<\Omega_S, R_S>$ was argued [Lemma 4(b)]. What is required here is proof that the relation R_S has all of the essential properties, *viz.* P1 - P3. Proof that R_S has P1 is routine, and causes no difficulty. But proof that R_S has properties P2 and P3 was, as Makinson has shown, significantly flawed.

The original argument [p. 458] took the following lines:

Suppose three sets in Ω_S, S^∞_i, S^∞_j, S^∞_k, are such that both $R_S(S^\infty_i, S^\infty_j)$ and $R_S(S^\infty_i, S^\infty_k)$. Further suppose that, for some A, FA & FB is a member of S^∞_i. Then depending upon which of F(A & B), F(A & FB), or F(B & FA) is also a member of S^∞_i — and one must surely be — either S^∞_j is just the same set as S^∞_k, or $R_S(S^\infty_j, S^\infty_k)$, or $R_S(S^\infty_k, S^\infty_j)$. Hence, R_S has property P2.

Makinson's counterexample supposed that A is a member of S^∞_j and that B is a member of a fourth set S^∞_m such that $R_S(S^\infty_j, S^\infty_m)$. Therefore, because of the transitivity of R_S (i.e. property P1), F(A & FB) can be a member of S^∞_i without the relation holding between S^∞_j and S^∞_k. Given the denumerability of Ω_S, the possibility envisioned by Makinson cannot be ruled out.

For a proof of the linearity of R_S, which is what the properties P2 and P3 confer, one must first show that the defining characteristic of R_S holds for 'H' as well as for 'G'. That is, it is necessary to establish the following thesis on R_S (for any two members S^∞_i and S^∞_j of Ω_S):

$R_S(S^\infty_i, S^\infty_j)$ iff, for every wff HA in S^∞_j A is a member of S^∞_i

As proof, first suppose that $R_S(S^\infty_i, S^\infty_j)$ and that HA belongs to S^∞_j, for any wff A. Then, for a *reductio*, further suppose A does not belong to S_i^∞. Given the maximal consistency of S_i^∞, it follows that ~A does belong to S_i^∞. By A3 and A7, FHA ⊃ A is an axiom of *CL*, and is, therefore, a member of S_i^∞. So, too, is its contrapositive by A1, i.e., ~A ⊃ ~FHA. Hence, again by the maximal consistency of S_i^∞, ~FHA belongs to S_i^∞. Since ~F~B ≡ GB, G~HA thus belongs to S_i^∞. But, by the definition of R_S and the assumption that $R_S(S_i^\infty, S_j^\infty)$, ~HA belongs to S_j^∞, which contradicts the maximal consistency of S_j^∞. On the other hand, suppose that, for every wff HA in S_j^∞, A belongs to S_i^∞. For a *reductio*, further assume that it is not the case that $R_S(S_i^\infty, S_j^\infty)$. Then there is a a wff GB in S_i^∞ such that B is not a member of S_j^∞, and, hence, such that ~B is a member. By the contrapositive of A3 — ~B ⊃ ~PGB — it follows that ~PGB is a member of S_j^∞. Hence H~GB is a member of S^∞. So, from the assumption, ~GB is a member of S_i^∞ which is a contradiction.

Using this additional fact about R_S, we can proceed to the main result — that R_S has P2. The proof is in 7 steps.

(1) Suppose, for three members S_i^∞, S_j^∞, and S_k^∞ of Ω_S, that both $R_S(S_i^\infty, S_j^\infty)$ and $R_S(S_i^\infty, S_k^\infty)$. Further suppose, for a *reductio*, that all of the following hold:

a. $S_j^\infty \neq S_k^\infty$

b. It is not the case that $R_S(S^\infty, S_k^\infty)$

c. It is not the case that $R_S(S_k^\infty, S_j^\infty)$

(2) From a, there is sure to be a wff C which is a member of S_j^∞ and is not a member of S_k^∞. Hence, ~C is a member of S_j^∞. From b., there is sure to be a wff GA which belongs to S_j^∞ such that A is not a member of S_k^∞ (this follows from the definition of R_S). Hence, ~A is a member of S_k^∞. From c., and

the proof given above, there is sure to be a wff HB which belongs to S_j^∞ such that B does not belong to S_k^∞. Hence, ~B is a member of S_k^∞.

(3) From step (2), ~A & (~B & ~C) is a member of S_k^∞, and GA & (HB & C) is a member of S_i^∞. Given that ⊢ A'⊃HFA' (by A3 and A1), HF(~A & (~B & ~C)) is a member of S_k^∞, and HF(GA & (HB & C)) is a member of S_j^∞. Hence, from the assumptions in step (1) on R_S, F(~A & (~B & ~C)) and F(GA (HB & C)) are both members of S_i^∞. Hence, by virtue of A5, one of the following must also be a member of S_j^∞:

d. F((~A & (~B & ~C)) & (GA & (HB & C)))
e. F((~A & (~B & ~C)) & F(GA & (HB & C)))
f. F(GA & (HB & C)) & F(~A & (~B & ~C)))

(4) Suppose d. is a member of S_j^∞. Then there is a member S_n^∞ of Ω_S such that (~A & (~B & ~C)) & (GA & (HB & C)) is a member of S_n^∞. But, given that C and ~C are conjuncts of this wff, this contradicts the maximal consistency of S_n^∞. Hence, d. cannot be a member of S_j^∞.

(5) Suppose e. is a member of S_i^∞. Then there is a member S_n^∞ of Ω_S such that (~A & (~B & ~C)) & F(GA & (HB & C)) is a member of S_n^∞. But given that ⊢ FHA' ⊃ A' (from A3 and A7) and that ⊢ F(A' & B') ⊃ (FA' & FB') (from A1, A2, and A6), it would follow that B and ~B both are members of S_n^∞, which contradicts its maximal consistency. Hence, e. cannot be a member of S_i^∞.

(6) Suppose f. is a member of S_n^∞. Then there is a member S_n^∞ of Ω_S such that (GA & (HB & C)) & F(~A & (~B & ~C)) is a member of S_n^∞. But given that ⊢ F(A' & B') ⊃ (FA' & FB'), it would follow that F~A (= ~GA) and GA both are members of S_n^∞ which contradicts its maximal consistency.

Hence, f. cannot be a member of S_i^∞.

(7) Supposing that any of d. — f. belongs to S_i^∞ leads to contradiction, hence at least one of a. — c. must be false. Hence, R_S has property P2.

By a similar argument, but using (PA & PB) ⊃ (P(A & B) V (P(A & PB) V P(B & PA))) (which follows from A5 and A7) in place of A5, it can be established that R_S also has P3. Hence, R_S is the appropriate relation for *CL*.

Colby College Robert P. McArthur