THE CLASS OF POSSIBILITIES

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The purpose of this paper is to propose an axiomatic system which be capable to refer to a maximum possible universe of discourse.

Such an entity is a universe class, and the deductive system pertaining to it belongs in the class calculus, which is limited. However, more refined logical instruments, like the functional and the relational calculuses, are reflected in the class context since propositional forms and general propositions as well as relationships form classes. Moreover, functions represent classes, and general propositions refer to them, while the domains and fields of relations are classes; relations, in turn, may be established as defining attributes of classes. As for elementary propositions, which also constitute classes, we may define the classes which are implicity connoted in them.

Our domain embraces both particulars and classes, including the unit-classes of each particular and class, the classes being both members and subclasses of this impure universal class. And the formal analytic system that we apply to this universe belongs to the universe as an element.

Now we shall assume that be definition of the universe as a maximum whole, that which embraces everything, is such that establishes it as a universe of posibilities, or the class of possibilities.

Possibility or consistency is the attribute of that universe and its components, the universe being the class of all classes which are not included in complementary classes, and of all the members of such consistent classes. The definition of class and of complement — made in terms of classes — are omitted here; we just will mention that our undefinable elements are 'particular', 'collection', 'quality', and 'whole'. The universe is thus a possibility or consistency, and so are its members, either particulars (what we may decide to call particulars) or classes. As we shall see, the specific possibilities, as such, are identified with the universe — The central proposition of this paper is: 'Every possibility, as a possibility, is all the possibilities'.

Impossibility or inconistency (or nonpossibility or nonconsistency) is the attribute of the class which is included in complementary classes, and whose extension is empty. This is the class complement of the universe, or the null-class. The number class of the null-class is zero, while its intension connotes its self-contradiction. But the extension of a possible class which we would call unrealized at present, say the class of 'centaurs' or the class of 'dinosaurs', is not empty, consisting of the collection that would be actual if the class were realized, according to a division of the universe into a subclass of realized possibilities and a complementary subclass of non-realized possibilities. A non-realized possible class is of an undertermined numerosity, not belonging to a number class. (The class of 'the women present' in a room where there are men only is of undetermined numerosity if we refer to an unrealized class, and it is empty if we connote a realized class; it would be then an unrealized realized class, that is, the null-class).

The null-class, of round squares or four-footed bipeds, is included in every class, and as a member it is also null. If it were counted in a class together with other members, the class would be equivalent to the subclass of those other members. The unit-class to which it belonged would be the null-class itself, and so the unit-class of such a unit-class in an infinite regress.

The class of possibilities is composed both by necessary consistencies, or tautologies, and by nonnecessary consistencies. And it is infinite. On the other hand, the null-class is always present.

Although our general approach to the class calculus is extensional, we cannot dispense with intensions, as they are needed to denote those classes which are not possible to define by enumeration, including the infinite ones. The intensional approach shall be used also in an attempt to solve an important problem posed by our system.

That problem is this. The universe of possibilities includes all possible classes. A class and its complement may be both possible, but their conjunction is impossible. Now, there may be possible classes whose presence would seem to preclude that of another which, by itself, is also possible. These are the cases where a class is alternatively considered as included in complementary classes, an oddity that has to be accepted if we are going to refer to *all* the possibilities. The demonstration that there is a form where this is the case without contradicting our definition of consistency is the proof that the universe assumed is in fact the class of *all* the possibilities.

Contradictions

Before proceeding to state our formal system it is necessary to deal with the contradictions pertaining to impure and infinite classes like our universe.

First, let us declare that in our system the assertion that a contradictory class is replaceable by itself renders a non-contradictory class. Thus, while the class of 'the round triangles' is inconsistent, the class intensionally defined as 'the substitutability of the round triangles for the round triangles' $(0 = 0)$ is possible whereas 'round triangles-round triangles', which is 'round triangles' $(0 \times 0 = 0)$, is inconsistent. Furthermore, a paradoxical proposition designates the null-class; the class that could be defined from the proposition 'this proposition is false' is the class of 'true-not-true propositions', which is inconsisent.

Now, our universe is impure, that is, composed of classes of different orders or types. Since it by definition includes all (possible) classes, it includes classes which are members of themselves, as well as those which are not. The universe figures among the classes which are members of themselves — which form a class which is of course within the universe —. But the class of all classes which are not self-contained belongs to the null-class because it is paradoxical, as pointed out by Bertrand Russell at the beginning of the 20th century (1). Its complement is the universe class (as the complement of the class of round squares — null-class — is not the class of all that is not a round square, including — say — four-legged bipeds, but the universe class, which is the complement of the null-class). But in the universal class there remains the class of all not self-contained classes *save* the including class; its complement is the class of all self-contained classes plus the excepted class, the complement of the class of all self-contained classes being the (unit-)class whose only member is the modified class.

We will except from our possible universe such classes that are self-contradictory as manifestations of the null-class. We do not want to make the theory of logical types a general requisite because we still want to have possible 'impure' classes since our universe embraces all possibilities and it is, therefore, an impure class itself, the possibilities belonging to different orders. And among the impure classes we have self-contained ones, like 'all classes of more than one member', whose order is infinite. (A formal circumvention of Russell's paradox like that of Zermelo's set theory would be technically superfluous in our system).

Another contradiction, also pointed out by Russell (2), is more closely related to our immediate problem. Georg Cantor (3) demonstrated that every (pure) class has more subclasses than it has members, so that a class of n members has 2^n subclasses. But the universe, being the class of 'everything', must embrace its subclasses as members, together with its other members, the subclasses being, then, not more numerous than the members, which is contrary to what has just taken as proved. The conclusion would seem to be that the universe as such is as impossible as the null-class, everything being a nonbeing.

(1) See Bertrand RUSSELL, *The Principles of Mathematics*, London, 1903.
(2) *Op. cit.*
(3) *Jahresbericht der deutschen Mathematiker-Vereinigung*, 1 (1892).

Formally, an infinity may be derived from nothingness. Starting with the empty class we immediately get the class of that class, which together with its only member forms a class of two members. And so on, combining terms, to the infinite. But we said that the class of the null-class is the null-class; so that such an infinity is actually reduced to the null-class alone, and from this process a universe cannot be obtained. If we start with a finite number of terms, even one, the same process leads to an infinite universe. And from this or any infinite collection of primitive terms we still get to an infinite universe.

Upholding the premise that there is an infinity of particulars, based on the prelogical consideration of a solipsistic world where there is the private concept of an infinite series of cardinal numbers, we may approach the universe of possibilities by ascending the scale of logical types and forming impure classes of higher and higher orders, the only whole class of each type being of an order one degree higher than the order of its highest order member. Then there are other classes of the same order but these are the subclasses of the whole class which, together with it and with the previous classes, subclasses and finally particulars, form the class of the immediately higher order. As is obvious, each whole class, save that of the first order, is impure, and among its members are both pure and impure classes, the order of particulars being zero. We could start the process wherever we choose, but since we want to have the whole universe we must start with the particulars, whatever they may be. The process is the same if we start with a limited number of particulars — the partial universe defined by the whole of particulars and classes in that specific line of subclassification suffering the effect of the paradox we are considering. If each of the partial universes is infinite, its number of members is equal to the number of members of the universe of possibilities according to the definition of the infinity that the whole of an infinite collection may be put into one-to-one correspondence with some of its proper parts. So we will go back to our universe understanding that the problem of its numerosity is applied to the partial universes which also include their subclasses as members.

In the process of subclassification the null-class is discarded in each step since although it is a formal subclass it cannot be a member. Likewise we disregard the class of the null-class, the class of the class of the null-class and so on. And we consider as a given class its subclass which together with the null-class forms it. By this procedure and by that of disregarding the repeated versions of the recurrent classes, we arrive at the conclusion that the number of members of each whole class of different orders is

$$2^n - 2^m + n$$

where *n* is the number of members of the anterior order's whole class, and *m*, the number of members of the whole class of the order previous to the previous one.

This is equal to the members *n* of a given class, of whatever order, plus its subclasses 2^n, minus the null-class, minus the repeated versions of recurrent classes. Obviously these repeated versions are the subclasses of the class previous to the class of *n* terms except the null-class — which has had to be removed in the similar previous step of the process together with the repeated classes left on removing the null one —, since those subclasses are taken again upon the members *m* of the class previous to the class of *n* terms in the process. The subtraction of the null-class and the exception of such a substraction for the previous step cancel out each other, and we are left just with -2^m, the original equation, before simplification, being (x is the number of members of the considered class):

$$x = n + 2^n - 1 - (2^m - 1)$$

which is:

$$x = n + 2^n - 1 - 2^m + 1$$

which reduces to our formula:

$$x = 2^n - 2^m + n$$

Now, in the process toward the universe, *n* and *m* are infinite, and according to the arithmetic of transfinite cardinals, we have that 2^n being larger than 2^m and, naturally, than *n*, the sum is equal 2^n. Since *n* was the sum total of an infinity of terms among which are the subclasses of an infinite class, *n* has to be larger than $\aleph_0$. As the process is the same in previous steps, we cannot determine the value of *n*, but we know it is a transfinite number larger than $\aleph_0$. To remaind us of this, let us call it N. Then 2^n is 2^N, or a number larger than N.

A correction has to be made to the original paradox to the effect that the number of subclasses of the universe which are added to the other members as to form the whole class is reduced in those terms which are repeated versions of the elements taken more than once over, plus the null-class, which does not belong to the universe. But this correction does not affect the formulation of the paradox because we are dealing on transfinite numbers. For a class of 2^N terms, the subclasses are 2^{2^N}. The number that is subtracted from this, is lesser; therefore the number remains 2^{2^N}; and this added to 2^N, the number of previous terms, is still 2^{2^N}, or equal to the number of formal subclasses, which fact is still contrary to our assumption that this number has to be larger than the number of members.

Applying our formula, where again M is a transfinite number larger than $\aleph_0$ and lesser than N, we get $2^N - 2^M + N$, or 2^N. And on the basis of 2^N members, we get our prior result, 2^{2^N}, from $2^{2^N} - 2^N + 2^N$.

But it seems that we cannot talk about getting a sum total of members equal to the number of subclasses in any stage of the process, since, as we just saw, the number keeps growing, passing from N to 2^N to 2^{2^N} and so on again to the infinite. In any level the number of terms, say Y, renders the number of subclasses 2^Y which, by the arithmetic of transfinite cardinals, is the new number of terms, and larger than the anterior. It would appear that the number of members of the universe is actually both equal and lesser than the number of subclasses, which is in intself the paradox.

In this endless process toward the totality, the number of members of the universe, including its legitimate subclasses, should be found in the theoretical limit of the infinite series which constituted the limit of the infinite series of infinite series — what could be called the ultra-infinite number. Since this is not a definite infinite number N from which a larger number 2^N could be derived, there is no contradiction in the fact that, for the universe of possibilities and for all those partial universes theoretically constructed in the same manner applied to it, there is a sum total of members of an ultra-infinite number $\aleph$ whose own infinite process makes it unincreasable. Therefore, $\aleph$ is equal to (say) $2^{\aleph}$, and their sum is also equal to $\aleph$. Hence we shall conclude that to ultra-infinite classes the consideration that classes have more subclasses than members does not apply, and that their number is equal, and their addition equal to any of them.

The perpetually expanding universe includes the fact that it is a member of itself, and the classes which are members of themselves are of an infinite order since among its members there is at least a class (itself) which is a class that contains at least itself which contains itself, to the infinite. The universe is also a class member of a class (say, the class of classes of more than one member) which is obviously contained in the universe, taking likewise the order to the infinite. These infinities are again absorbed into the ultra-infinite order of the universe of possibilities, which is finally accepted.

The system of possibilities

Once admitted the universe as a total whole, let us continue with the main argument. Our *formal* domain will be of classes.

The classes of the formal universe are designated by capital letters, save the universe class which is designated by the numeral one, '1', and the null-class, designated by zero, '0'. The minus sign '—' before a letter or either numeral designates the complement. Letters may be 'stroked' — |A|.

On dealing with a domain of classes (particular members of classes being implicit), our formal system is expressed in the class calculus, as already stated. But in order to establish the uniqueness of the universe and at the same time retaining the diversity of its elements, we shall interpret our system as a two-valued algebra — like the propositional calculus — which nevertheless keeps the characteristics of the class calculus. For this we will provide certain rules.

As a class calculus, the operations of the system are those traditionally agreed upon — negation (designated by the minus sign, '—', preplaced), disjunction (designated by the addition sign '+') and conjunction (designated by the multiplication sign 'x' or by a dot '.' or by the mere apposition of letters or parentheses). The connectives are those expressed by the equal sign '=' (substitutability) and by the implication sign '⊃' (detachment). Parentheses are used as well as parenthetical dots, and so the quantifiers, designated as the 'universal' and the 'existential'. Commas are employed in sequences.

The rules of formation are those permitting the use of the operatives and connectives. We shall emphazise that a *formula* is an expression in accord with the formation rules, an expression where there are no operatives nor connectives, or where there is an operative between two expressions (which may contain subordinate operatives or connectives or both), or where there is a connective between two expressions (which may contain subordinate connectives or operatives or both). A formula may thus depend on a connective or on an operative holding between complex formulas, constituting an ampler one. All the formal expressions of the system are formulas (but not vice versa), and such formulas are laws of the system. The laws, which are connective formulas, in turn are members of the universal class together with the other possibilities. A logical expression as such is a member of the class of those expressions, but, as already mentioned, what it expresses is defined as a class, and this class is also a member of the domain. Thus, the (true) formula stating that a class multiplied by the universal class is equal to the class itself expresses the class of classes-being-multiplied-by-the-universe-being-always-the-classes-themselves.

An additional rule of formation contributes to provide for the fact that, in the same manner that any impossible class *is* the null-class, every possible class *is* the universe class of possibilities. A possibility as such, either characterized as merely a possibility or as a specific possible being, is the same as all the possibilities as a whole, but still conserves its formal individuality, like that of the true propositions in the propositional calculus, where they are all substitutable with the same truth value. In our system, the equation between a particular possible class and the universe is effected by definition or by inference, and besides these two cases we establish it only when the fact of being a possibility enters explicitly into the connotation of the class (when referring to a particular, the case is that of the explicit connotation of a class to which it belonged including the fact that the class — that could be a unit-class — is possible). In other words, if we have a class A, even though it be possible we do not say that it is the possible universe, 1, save when equation A = 1 is somehow established by definition or produced by inference. But we could in a way equate A with 1 if the intension of A includes the fact that it is possible. Of course we can redefine each possible class as such a class, and then equate it with 1. Our additional rule of formation shall be called rule of intension, and according to it it is permitted to effect formally such redefinition of every possible class so that the quality of its being possible be a part of its connotation. Intensionally partial classes (not the universe) so redefined are here represented by a stroked letter, for instance, |A|, which means that A, whatever partial possible clas sit may be, including 'any possible class (in which case it carries the strokes automatically), is possible. (The universe class carries the same connotation but we dispense with the strokes since '1' designates possibiltiy as such). Then |A| may be equated with 1, which is expressed by a postulate. he establishment of |A| can be made as a hypothesis within the train of deduction. The formal value of the intension rule rests upon the feature that, although |A| is not the same as A, it is derived from it and occupies its place in the formulas.

The rules of transformation are mainly: substitution, based on the connective '=', and detachment, based on the implication '⊃'. There is a third rule, the so-called principle of application whereby an expression relative to any element of a class give rise to a like expression relative to each element of the class; since by application we equate a general expression with the logical sum of the specific elements (in the sense that each specific element is equated with the general term — this term is either that specific element or that or that), application may be considered a special case of substitution, through the definition of applicability.

Substitution permits the replacement of a term by another when the relation of substitutability holds between them; we also say that those terms are equal between themselves. Detachment permits the establishment of a term when once a term to which it has the relation of implication is established; this relationship, for instance, $A \supset B$, holds whenever $AB = A$ or $A + B = B$ by theorems which will not be demonstrated here, and it holds for $— A + B = 1$ by definition. Also, by a theorem, $A \supset A$.

Before enumerating our postulates we are going to make a definitory assumption which will have the character of a preformal hypothesis of the system. This is that a particular may be expanded so that it can be equated with 1 whenever it is described as a member of 1 or of an intensionally possible class |A|. This is pertinent because the membership relation of a particular to a class is a necessary derivation of its description; on the other hand, that a class (A) is a member of some other class (1) is not necessarily a factor of its definition; that is why we may talk about a class which is a member of 1 (besides the fact that it is included in 1) and not to consider the class as 1, not until we explicitly establish it as $|A|$ and, by the aforesaid postulate (which will be set forth presently), we equate it with 1.

As a two-valued algebra, the system may be given an expression in matrix form by using tables which represent the basic relationships between 1 and 0 and of those terms which are equated with either one. The given postulates are sixteen:

1) $(\exists 1, 0)\ 1 = {}^{-}0$
2) $(A)\ (\exists 0)\ A + 0 = A$
3) $(A)\ (\exists 1)\ A \times 1 = A$
4) $(A)\ (\exists -A, 1)\ A + -A = 1$
5) $(A)\ (\exists -A, 0)\ A \times -A = 0$
6) $(A, B)\ (\exists C)\ A + B = C$
7) $(A, B)\ (\exists C)\ A \times B = C$
8) $(A)\ A = A$
9) $(A, B)\ A + B = B + A$
10) $(A, B)\ A \times B = B \times A$
11) $(A, B, C)\ A + (B \times C) = (A + B) \times (A + C)$
12) $(A, B, C)\ A \times (B + C) = (A \times B) + (A \times C)$
13) $(|A|)\ (\exists 1)\ |A| = 1$
14) $(A)\ (\exists 1)\ (A = A) = 1$
15) $(A)\ (\exists -A, 0)\ (A = -A) = 0$
16) $(A, B)\ (A = B) = ((A \supset B) \times (B \supset A))$

I will omit in this paper the demonstrations of the theorems. A list of the most relevant ones follows (the quantifiers are understood):

1T) 1 is unique.
2T) 0 is unique.
3T) $A + A = A$
4T) $AA = A$
5T) $A + 1 = 1$
6T) $A \times 0 = 0$
7T) $A + (AB) = A$

8T) A (A + B) = A
9T) AB + A — B = A
10T) (A + B (A + — B) = A
11T) The complement is unique.
12T) B = — A . ⊃ . A = — B
13T) A = — (— A)
22T) — (A + B) = — A — B
23T) — (AB) = — A + — B
24T) (A + B) + C = A + (B + C)
25T) (AB)C = A(BC)
26T) A + B = B . ⊃ . AB = A
27T) AB = A . ⊃ . A + B = B
28T) A = B . ⊃ . B = A
29T) A = B . B = C . ⊃ . A = C
30T) A = C . B = C . ⊃ . A = B
31T) 0 = — 1
32T) — (A = A) = 0
33T) — (A = — A) = 1
34T) (A = 1) ⊃ ((A = 1) = 1)
35T) (A = 0) ⊃ ((A = 0) = 1)
36T) (|A| = 1) = 1
37T) (1 = 1) = 1
38T) (0 = 0) = 1
41T) (1 = 0) = 0
45T) 1 = 1
46T) 0 = 0
47T) 1 + 1 = 1
48T) 0 + 0 = 0
49T) 1 × 1 = 1
50T) 0 × 0 = 0
51T) 1 + 0 = 1
52T) 1 × 0 = 0
53T) |A| + |— A| = 1
54T) |A| × |— A| = 1
55T) |A| = |— A|

The ruth-tables, or matrices, express the formulas of the system according to all different combinations of their constituent parts as to their equalization (or not equalization) with 1, by which all the formulas can be said to be either 1 or 0. The complement of a resulting class equal 1 is 0. It must be kept in mind that if a class A is intensionally equated with 1, its complement — A may or may not be equal 0, according to whether A is or is not, respectively, extensionaaly 1 before the equalization; — A may well be also equated with 1. But the resluting class is no longer, in the first case, A, but a class Z, or, to express the intensional rule, |A|, whose complement is 0, whether we also designate it by — Z or just as — 1. If, like A, — A admits of being equated with 1, or being designated by |— A|, the resulting class, be it designated by |— A| or (say) by W, will have as its complement the class 0, whether we designate it also by — W or just as — 1.

In the tables, the indacted classes A, B, C, ... will be the resulting classes 1 or 0 from, respectively, classes equated with 1 and the complement of classes equated with 1. The only exceptions are the tables for classes specifically indicated as 1 or 0.

Any formula that for every combination of values is always equatable with 1 is a tautology. The tables facilitate the inspection of such a feature. As an example let us consider formula A + B:

A	B	A + B
1	1	1
1	0	1
0	1	1
0	0	0

A + B is not a tautology since for case A = 0, B = 0 it is not equal 1, but equal 0. However, formula A + B *may* represent a possibility, as is the case in the first three combinations. A tautology is a necessary possibility; and its complement, a (necessary) inconsistency. A possible non-tautology is a not necessary possibility, but the complement of the possibility as such is an inconsistency.

A + — A, for instance, is a tautology:

A	—A	A + —A
1	0	1
0	1	1

For tautology 1 + 0 there is only one case:

1	0	1 + 0
1	0	1

Now, A — A is not a tautology since it is equal 0. What is more, it is always an inconsistency as it is always equated with 0:

A	—A	A — A
1	0	0
0	1	0

But A — A = 0 is a tautology:

A	—A	0	A — A	A — A
1	0	0	0	1
0	1	0	0	1

Likewise A + — A = 1 is a tautology:

A	—A	A + —A	1	A + —A = 1
1	0	1	1	1
0	1	1	1	1

Or (A + — A = 1) = 1 always, and the equalization with 1 can be then carried out *ad infinitum.*

Proofs of consistency and completeness

Now we assert that our system is formally consistent and complete. In this paper we are going to spare the reader the demonstrations of the two propositions, which demonstrations are essentially the same as those for the propositional calculus. But for the sake of completeness and clarification we are going to relate the course of the demonstrations.

Formal consistency means that no assertion of the postulates is equal 0. Or that a formula A and its complement — A are not both found among the postulates and theorems.

Completeness of the system means that every necessarily consistent formula (tautology) belongs in the system as a law (as a postulate or as a theorem). If either a formula 1 or 0 is to be a law, then formula 1 is a law.

The consistency proof is based on the demonstration that all the postulates are tautologies and that the property of being tautologous is hereditary, that is, transmissible to formulas derived, through the rules of transformation, from the postulates — to the theorems. As the laws are all tautologies (1), no law is 0, and the conjunction of laws is always 1.

But the tautological property is used now as a formal attribute of the system. Since the proof comes prior to the consideration of a tautology as a necessary consistency, it cannot be at this stage ascertained that, the laws being tautologous, the system is consistent.

We then show that if the system were inconsistent *any* formula would be derivable. The opposite situation would be that if the system is consistent, at least one formula B constructed according to the rules of formation is not derivable from the postulates. Since the postulates are tautologies and the tautological property is hereditary, a formula not a tautology would not be a law of the system. Hence, the exhibition of such a formula formally proves the consistency of the system.

After proving that assuming A and —A as laws, any formula B is derivable, we proceed to exhibit a formula which is not derivable, that is, that is not a tautology. This formula is, for example, AB = A. Its table follows:

A	B	AB	AB = A
1	1	1	1
1	0	0	0
0	1	0	1
0	0	0	1

Therefore, the system is formally consistent.

It must be clarified now that the proofs of tautology for postulates 6 and 7 include an element of definition. Postulate 6 is: A + B = C, and postulate 7 is: AB = C. What we do in regard of those two postulates is to establish by definition the cases of equality.

For postulate 6 (A + B = C) the table is:

A	B	A + B	C	A + B = C
1	1	1	1	1
1	0	1	1	1
0	1	1	1	1
0	0	0	0	1

where we arrange the column of the C accordingly. Similarly, the arranged table for postulate 7 (AB = C) is:

A	B	AB	C	AB = C
1	1	1	1	1
1	0	0	0	1
0	1	0	0	1
0	0	0	0	1

Turning back to formula AB = A, we must notice that a possible non-tautology is a class of the system, as is also every tautology, whereas an inconsistency is 0. Formula AB = A is not a tautology, but it represents a (non-necessary) consistency when we restrict the cases to those where AB is equal to A. It can also represent an inconsistency if we restrict the cases for hypotheses A = 1, B = 0. Then AB =0, and since A = 1, we get 0 = 1, but (1 = 0) = 0 (41T); or, (AB = A) = 0.

As for completeness, it is expressed by the fact that all tautologies there are are either postulates or theorems (laws) of the system.

The proof is based on the truth-valuedness of our system, whereby a tautology (1) must derive from the postulates whether they are 1 or 0, and they are 1 as shown in the consistency proof.

The proof of possibility

Having shown that the system of possibilities is both consistent and complete, we proceed now to prove that being a system of all the possibilities, our universe, although A — A = 0, admits of mutually incompatible possible classes, and even of intensional classes which are alternatively considered as the conjunction of a class with complementary classes. These conjoined (possible) classes are classes which are not taken as the universe and whose presence would seem to preclude that of another class (the other conjoined class) which, by itself, is also possible, although their possibilities are not formally connoted but indicated by the feature that the classes are members of the universe.

(The conjunction could be with mutually exclusive classes not complementary, but this case is reduced to complementary classes by taking one of the exclusive classes and its complement which includes the other exclusive class).

The demonstration that such reciprocally contradictory classes are all within the universe is the proof that the universe is in fact the universe of possibilities. This is so because possible intensional classes not the universe are either reciprocally contradictory or not; those not reciprocally contradictory are readily taken as members of the universe; and then the reciprocally contradictory classes exhaust the possibilities.

The proof is tantamount to showing that there is a form wherein the case presents itself that a formal class is alternatively considered as included in complementary (possible) classes and yet is in fact possible ,since if it were not possible, the two reciprocally contradictory classes would be null, as $A0 = 0$ (6T).

An instance where the case is absent is that of the possible class of (say) men and the possible class of non-men; if the former is A, the latter is $-A$. The principles $A + -A = 1$ and $A-A = 0$ hold. But neither $A = 1$ or $-A = 1$ nor $A = 0$ or $-A = 0$.

If we applied the intensional formation rule and explicitly stated that the class of men is possible and the class of (possible) non-men is possible, we respectively would get $|A|$ and $|-A|$. By 13, $|A| = 1$ and $|-A| = 1$. Therefore, by substitution, $|A| + |-A| = 1$ (53T) and $|A| \; -A = 1$ (54T),
stitution, $A + |-A| = 1$ (53T) and $|A| \, |-A| = 1$ (54T),
and $|A| = |-A|$ (55T).

However, if we stated of a class A that A = 1, then, of necessity, — A = 0 (31T). Then we could have |A|, but not |— A|. Conversely, if A = 0, then — A = 1.

(In the case where both |A| and |— A| hold, althoug they derive respectively from A and — A, if we wanted to designate them by non stroked letters these could not be A and — A but completely different symbols, say Y and Z, for a completely new class, namely 1).

Now let us examine the other case. We have defined an inconsistent class as that which is (intensionally) included in complementary classes. That is the null-class, of no extension. But in our system of possibilities, a (possible) class may be alternatively considered as included in a possible class and in the possible complement of the latter, rendering two conjoined classes each one of them being possible in itself. Yet the two instances are in the same system and the class should be considered as null, and, consequently, the two conjoined classes must be considered null as well; on the other hand, our assertion is that the class (and each of the conjoined classes) is not null, which involves a contradiction.

Let us consider the case of a class H of which we do not connote that is possible but which we accept as a member of the universe through its not being self-contradictory. Then we consider it as totally included (as either a minor or a coincident class) in another possible class B without running into any contradiction in this strict instance. On the other hand we consider H as totally included (also as either a minor or a coincident class) in the possible class — B, without encountering any contradiction in this other strict instance.

The respectively resulting intensional classes are HB and H — B, and we say that HB = H and H — B = H. Now the case may present itself that while H is totally included in (say) B, in the other alternative H might be partially included in — B. The formula for the second alternative would not be H — B = H but A — B = A with the proviso that AH = A. But if H is partially included in — B we may restrict our study to the part included in — B, and to consider it a class A which is totally included (according to separate intensions) both in B and —B. The resulting respective classes are AB and A—B, and we say that AB = A and A—B = A.

Extensionally, both classes AB and A—B are the class A. Intensionally, the defining attributes of B and —B alternatively become, together with a prior property of A, a defining attribute of A.

Now, the premises are that A, B, and —B are members of the universe, and that AB = A and A—B = A. And we want to conclude that AB and A—B are members of the universe. Obviously, the premises involve a contradiction — formulas AB = A and A—B = A express the definition of inconsistency for A, and we say that A is a member of the universe (not inconsistent).

In fact, since the two formulas belong in the same system and each is alternatively a precondition for the other, there is a pertinent substitution which renders them null — from AB = A, in reference to A—B = A, we get A—BB = A, or A0 = A, or 0 = A; from A—B = A, in reference to AB = A, we get AB—B = A, or A0 = A, or 0 = A. Indeed, the precondition of AB = A is A—B = 0, and that of A—B = A is AB = 0; or, A = 0.

On the other hand, since AB = A and A—B = A, and A = 0, we get AB = 0 and A—B = 0, which is contrary to our objective.

By elimination, the only non-contradictory manner in which A may be considered — but only in an indirect way — as a member of the universe (B and —B being also members of the universe) while upholding AB = A and A—B = A, is to apply our formation rule for classes intensionally possible and *connote* that A, B, and —B are possible. The new constructed formulas are |A|, |B|, |—B|, |A||B| = |A|, |A||—B| = |A|, formally derived from our hypotheses.

In this manner, a possible class |A| can be considered as included in possible classes |B| and |—B| which represent, respectively, complementary classes B and —B, without contradiction since |A| = 1 (13) (and |B| = 1 and |—B| = 1), and $|A||B| = 1$ and $|A| - B| = 1$ (54T), or $1 \times 1 = 1$ (49T). This result, $|A||B| = 1$, $|A||-B| = 1$, renders our thesis.

Of course, $|A|$ is not included in complementary classes, as $|B| = |-B|$ (55T), but the formulas $AB = A$, $A - B = A$ are alternatively valid even though they belong in the same system because in reference to such system (of possibilities) they ultimately become $|A||B| = |A|$, $|A||-B| = |A|$, or $1 \times 1 = 1$ (49T), or $1 = 1$ (45T); and $(1 = 1) = 1$ (37T).

By substitution we get the same result — $|A||B| = |A| |-B||B|$, or $|A||B = |A|$, or $|A||B| = 1$, or $i = 1$, and $|A||B| = |A||B||-B|$, or $|A||-B| = |A|$, or $|A| |-B| = 1$, or $1 = 1$, since $|B||-B| = 1$, as opposed to $B - B = 0$. While $A + -A = 1$ (4) and $|A| + |-A| = 1$ (53T); whereas $A - A = 0$ (5), $|A||-A| = 1$ (54T).

The letter within the strokes is a notational reminder of the specific class upon which, by hypothesis, we applied the definition of a possible class. But we thereby transcend the specific class and arrive at 1, where the embracement of all the specially connoted possibilities, no matter how reciprocally contradictory, is not inconsistent. (In the universe of possibilities, that all men are mortal is as possible as that all men are not mortal).

The possible universe

It should be noticed that the inclusion of all the member possibilities in 1, when we apply our definition, is coincidental; that is, every special possibility as such, in virtue of postulate 13 and the general hypothesis on particulars, is the whole possible universe which turns out to be formally the only possible class just as the null-class is all the specially noted null classes.

Every formally special possibility is thus all the possibilities since what is possible encloses any consistent thing.

A possibility as such, either realized or not, is the potential being of any consistent realization. In fact, when dealing on specific formal classes which, being members of the universe, are not *connoted* as possible, we refer in the last instance to possibilities. But in this case their being members of the universe of possibilities is *denoted* by the definition of the universe, whereby it comprises both possible classes connoted as such and just classes in whose definition there is no inconsistency. The latter are also possible but are not connoted as such since in their definition there is no mention of their not being included in complementary classes or their being possible.

Of course, even though we treat some classes in this manner, we have been talking about possibilities all along, under a certain formal guise. On removing this guise through our intensional rule, in accord with the definition of the universe its ultra-infinite plurality is immersed into the one.

As the possible universe comprises everything, its presence as the unique absolute entity would seem to be the chief concern (explicit or implicit) of philosophy, whose final answer, without prejudice of its other inquiries, would be the notion that the absolute being is there.

Corollary

In our universe, any possible class is both a subclass and a member of the system. Among them we may find such special systems that, by their formal complexity, cannot be subject to a finitistic proof of consistency. Considered apart, we do not know whether they are consistent, but we know that, if they are so, they are included in the universe of possibilities. However, if they admit of an infinite proof of consistency by a demonstration based on their metalanguage, whose consistency is in turn based on its metasystem and so on to the infinite, the system of possibilities provides a definitive proof of consistency although not finitistic. If there were reason to presume that an evidence of inconsistency would not present itself in the infinite process of reference to metalanguages, there is a resolution of the proof in the ultimate reference to the universe of possibilities since it constitutes the final system and its infinity is of the terminal ultra-infinite kind.