A RESOLUTION PRINCIPLE FOR A CLASS OF MANY-VALUED LOGICS*

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ABSTRACT

A class of languages is defined, the semantics for which includes an arbitrary number M of possible «truth-values» and an arbitrary «critical» value S. In addition to the standard connectives and quantifiers, the syntax includes a set of J operators. The J operators are monadic sentential operators; $J_k(E)$ intuitively means «E takes value k». A resolution principle for such languages is developed, and it is proved to be both sound and complete. Some applications are also discussed.

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0. Introduction

Systems of logic with non-standard semantics sometimes appear to be more useful, more intuitively applicable to real-life situations and the analysis of natural languages than do logics with the standard two-valued semantics. In recent years, a particular type of many-valued logic, called «fuzzy logic», has received attention in the computing science literature. (See [2] and the references contained there.)

Fuzzy logic uses conjunction, disjunction, negation, and universal and existential quantification (&, $\vee$, $\neg$, (x), and ($\exists$x), respectively). We will use E, E_1, E' etc. as meta-expressions and $x_1, x_2, \ldots$ as variables. The standard formation rules are used to build expressions of the language. For fuzzy logic, an interpretation over a domain D assigns real values over the closed interval [0, 1] to each atomic ground expression in the language, whereas for the normal two-valued case, the interpretation is restricted to the values 0 and 1. Letting T(E) designate the truth value of expression E, the interpretation for more complex fuzzy logic expressions is given by the following:

(i) $T(\neg E) = 1 - T(E)$
(ii) $T(E_1 \& E_2) = \min\{T(E_1), T(E_2)\}$
(iii) $T(E_1 \vee E_2) = \max\{T(E_1), T(E_2)\}$
(iv) $T((x_i)\, E\, (x_i)) = \inf\{T(E(x_i)) : x_i \in D\}$
(v) $T((\exists x_i)\, E\, (x_i)) = \sup\{T(E(x_i)) : x_i \in D\}$

The value .5 is usually taken to be a «critical value» for fuzzy logic. An interpretation I is said to satisfy an expression E just in case $T(E) \geq .5$ in I; I is said to falsify an expression E just in case $T(E) \leq .5$ in I. An expression E is said to be unsatisfiable just in case every interpretation falsifies E. Note that «unsatisfiable» in this sense is not equivalent to «not satisfiable». For example, if E is an atomic expression, then are interpretations in which E & $\neg$ E takes the value .5; hence the expression is satisfiable. However, for no interpretation does the expression take a value greater than .5, and hence it is unsatisfiable.

Using these notions, it is possible to obtain a complete resolution procedure for fuzzy logic. However, such a procedure is of very restricted practical interest; it has been shown by Lee ([2], p. 115) that «A set S of clauses is unsatisfiable in fuzzy logic if and only if it is unsatisfiable in two-valued logic.» In order to make automatic theorem proving for fuzzy logic of practical interest we need a method of incorporating information about the truth values of expressions into the language. We now turn to a limited solution to that problem.

Fuzzy logic allows expressions to take on any one of an infinite number of possible truth values. We will restrict our discussion here to the treatment of systems with a finite number of truth values. The values are assumed to be integers ranging from 1 through M. (Alternatively, these values may be assumed to be the subscripts of some non-integer values $v_1, \ldots, v_M$.) Fuzzy logic assumes a critical value of .5. For the logics discussed here, we will assume an arbitrary critical value S such that $1 < S \leq M$. The values from S through M are the intuitively «true» values-we will call them the designated values. The values from 1 to (but not including) S are the intuitively «false» values-we will call them the undesignated values.

In addition to negation, conjunction, disjunction, universal and existential quantification, we will introduce one-place sentence operators J_i, $1 \leq i \leq M$. Intuitively, $J_i(E)$ means «expression E takes value i». The formation rules are the same as for the standard case with the addition of: «If E is any well formed formula, then so is $J_i(E)$, for $1 \leq i \leq M$.» Thus in addition to the usual expressions, we will have expressions like the following:

$$J_1((x_1)\ (\exists x_2)Px_1x_2), \text{ and } (\exists x_1)(Qx_1\ \&\ J_2(Gx_1)).$$

Logics similar to the sort just described are treated in some detail by Rosser and Turquette in [3]. The major difference is that for Rosser and Turquette the «high» values are undesignated (false) while the «low» values are designated (true).

There are other differences which we will not mention, as our treatment below is self-contained.

In the following material, we will take a purely model theoretic approach. Sometimes reference will be made to «the two-valued case». The reader unfamiliar with this material is directed to the treatment by Kreisel and Krivine in [1]; our development parallels theirs to a considerable degree.

1. Sentential calculus

We will first treat the sentential calculus and later expand our treatment to the functional calculus of first order. We specify the syntax of our language as follows:

1. set of sentence parameters: $A = \{A_1, A_2, \dots\}$
2. monadic sentence operators:
   a. negation: $\neg$
   b. J operators: J_i, for integer i, $1 \leq i \leq M$
3. dyadic sentence operators:
   a. disjunction: $\vee$
   b. conjunction: &

We will assume the usual recursive definition of «formula».

The semantics for the language may be given by an elaboration of the notions «interpretation» and «valuation». An interpretation I is any function with domain A and range a subset (non-empty) of the integers $\{1, \dots, M\}$. Thus an intepretation assigns a value to each sentence parameter. Values are assigned to more complex expressions by the valuation function V, which is defined for all I as follows:

$$V(A_i, I) = I(A_i)$$
$$V(\neg E, I) = M + 1 - V(E, I)$$
$$V(E_1 \vee E_2, I) = \max\{V(E_1, I), V(E_2, I)\}$$
$$V(E_1 \,\&\, E_2, I) = \min\{V(E_1, I), V(E_2, I)\}$$
$$\begin{aligned} V(J_k(E), I) &= M, \text{ if } V(E, I) = k \\ &= 1, \text{ otherwise} \end{aligned}$$

For some chosen value S, $1 < S \leq M$, we say that the values greater than or equal to S but less than or equal to M are «designated»; the values greater than or equal to 1 but strictly less than S are «undesignated». If an expression E takes a designated value under interpretation I we say that I satisfies E; if E takes an undesignated value under I we say that I falsifies E. An expression is satisfiable (falsifiable) just in case there is some interpretation which satisfies (falsifies) the expression. We say that I satisfies a set of expressions if it satisfies every member of the set. A set of expressions is satisfiable just in case there is some interpretation which satisfies the set.

From the definition of V it is easy to verify the standard associative and commutative laws for $\vee$ and &, so parentheses will be omitted where no confusion will arise. We will make use of the following special notation:

$$\sum_{i=1}^{n} E_i =_{df} E_1 \vee \ldots \vee E_n$$

$$\prod_{i=1}^{n} E_i =_{df} E_1 \,\&\, \ldots \,\&\, E_n$$

$$\sim E =_{df} \sum_{i=1}^{S-1} J_i(E)$$

It is easy to verify that E is designated (undesignated) if and only if $\sim$ E is undesignated (designated). However, it should be noted that $\sim \sim$ E is not in general equivalent in value to E.

We now proceed to state and prove various theorems about the sentential calculus which will be of use later in our development.

Theorem 1: Let S be a set of formulas such that every finite subset of S is satisfiable. Then S is satisfiable.

Proof: Let $A_1, \ldots, A_n, \ldots$ be an enumeration of sentence parameters of the language. Suppose we have found a map f from the set $\{A_1, \ldots, A_k\}$ into the integers 1 through M such that every finite subset of S has an interpretation satisfying it in which $A_1, \ldots, A_k$ take the values $f(A_1), \ldots, f(A_k)$. We propose to extend the function f to A_{k+1}. We claim there must be some integer i between 1 and M such that by setting $f(A_{k+1}) = i$, all finite subsets of S have an interpretation satisfying the subset in which $A_1, \ldots, A_k$ take the value $f(A_1), \ldots, f(A_k)$ and A_{k+1} takes the value i; for suppose not. Then for each i there is a finite subset U_i of S such that there is no interpretation satisfying U_i in which $A_1, \ldots, A_{k+1}$ take the values $f(A_1), \ldots, f(A_k), i$. Consider the set $U = U_1 \cup \ldots \cup U_M$. Clearly U is a finite subset of S, and by construction there is no interpretation satisfying U in which $A_1, \ldots, A_k$ take the values assigned by f. But this contradicts our assumption, and hence there must be an i to extend f to A_{k+1}. By employing some choice procedure at each step (e.g., take the least value i such that the required interpretations exist), we may define by recursion on n an interpretation I satisfying every finite subset of S in which $A_1, \ldots, A_n$ take the values assigned by f. Now, it is easy to see that I satisfies S as follows: Consider an arbitrary formula E in S; to see that I satisfies E, we simply take n large enough so that all sentence letters in E occur in the list $A_1, \ldots, A_n$. Q.E.D.

Theorem 2: If every interpretation satisfies one of the formulas of a set S of formulas, then there are formulas $E_1, \ldots, E_n$ in S such that $E_1 \vee \ldots \vee E_n$ is satisfied by every interpretation.

Proof: Suppose the hypothesis of the theorem is true but that such a set does not exist. Then for each finite subset $\{E_1, \ldots, E_n\}$ of S there is some interpretation which falsifies every member of it and therefore satisfies the set $\{\sim E_1, \ldots, \sim E_n\}$. Let S' be the set $\{\sim E : E \in S\}$. Then every finite subset of S' has an interpretation which satisfies it, and so by Theorem 1, there is an interpretation which satisfies S'. But this contradicts the assumption that every interpretation satisfies some formula of S. Q.E.D.

Theorem 3: If every interpretation falsifies one of the formulas of a set S of formulas, then there are formulas $E_1, \ldots, E_n$ in S such that $E_1 \,\&\, \ldots \,\&\, E_n$ is falsified by every interpretation.

Proof: Consider the set S' consisting of formulas $\sim E$ for E in S. Every interpretation satisfies one of the formulas of S'. Thus by Theorem 2, there are formulas $\sim E_1, \ldots, \sim E_n$ such that

$\sim E_1 \vee \ldots \vee \sim E_n$ is satisfied by every interpretation. Thus every interpretation falsifies at least one of $E_1, \ldots, E_n$. Hence every interpretation falsifies $E_1 \& \ldots \& E_n$. Q.E.D.

In the following material we will make some use of the empty string, which will be designated by *e*. We will stipulate that for any E, $E \vee e = e \vee E = E$. We assume that whenever a contraction of this type is possible, it is done. In terms of our semantics, the empty string is always assigned the value 1.

We will also make use of various notions analogous to the two-valued conjunctive normal form (CNF). Consider an expression in the following form:

$$\prod_{i=1}^{n} \sum_{j=1}^{m_i} E_{ij}$$

Each of the expressions $\sum_{j=1}^{m_i} E_{ij}$ is called a «clause». The formulas E_{ij} may be of various types:

(i) A_p
(ii) $\neg A_p$
(iii) $J_k(A_p)$
(iv) $J_{k_1}(\ldots J_{k_t}(A_p) \ldots)$
(v) e

If the E_{ij} are all of types (i)-(iv), the formula will be said to be 1-CNF. If the E_j are all of types (i)-(iii) and (v), the formula will be said to be 2-CNF. If the E_{ij} are all of types (iii) and (v), the formula will be said to be J-CNF.

Theorem 4: To any formula E, there corresponds a 1-CNF formula E' such that for all I, $V(E, I) = V(E', I)$.

Proof: The proof is similar to the standard proof for two-valued CNF. It is easy to verify the following equivalences:

(a) $E_1 \vee E_2$ eq. $E_2 \vee E_1$, and $E_1 \& E_2$ eq. $E_2 \& E_1$
(b) $\neg \neg E$ eq. E
(c) $\neg (E_1 \vee E_2)$ eq. $\neg E_1 \& \neg E_2$
(d) $\neg (E_1 \& E_2)$ eq. $\neg E_1 \vee \neg E_2$
(e) $\neg J_k(E)$ eq. $\sum_{i=k} J_i(E)$

(f) $E_1 \vee (E_2 \mathbin{\&} E_3)$ eq. $(E_1 \vee E_2) \mathbin{\&} (E_1 \vee E_3)$

(g) $J_k(\neg E)$ eq. $J_{M+1-k}(E)$

(h) $J_k(E_1 \mathbin{\&} E_2)$ eq. $[J_k(E_1) \mathbin{\&} \sum_{i=k}^{M} J_i(E_2)] \vee [J_k(E_2) \mathbin{\&} \sum_{i=k+1}^{M} J_i(E_1)]$

(i) $J_k(E_1 \vee E_2)$ eq. $[J_k(E_1) \mathbin{\&} \sum_{i=1}^{k} J_i(E_2)] \vee [J_k(E_2) \mathbin{\&} \sum_{i=1}^{k-1} J_i(E_1)]$

The proof is then by induction on the complexity of E. Q.E.D.

Theorem 5: To any 1-CNF formula E there corresponds a 2-CNF formula E' such that for all I, $V(E, I) = V(E', I)$.

Proof: We must show how to remove expressions of the form $J_{k_1}(\ldots J_{k_t}(A_p) \ldots)$ from the given expression. We obtain the following equivalence by nothing that the J operators may take only the values 1 and M:

$$J_k(J_r(E)) \text{ eq. } J_r(E), \text{ if } k = M$$
$$\text{eq. } \sum_{i=r} J_i(E), \text{ if } k = 1$$
$$\text{eq. } e, \text{ otherwise}$$

By iterated applications of this equivalence, each occurrence of the type $J_{k_1}(\ldots J_{k_t}(A_p) \ldots)$ may be replaced by a disjunction of formulas of the types $J_m(A_p)$ and *e*. Q.E.D.

Theorem 6: To any 2-CNF formula E there corresponds a J-CNF formula E' such that for any interpretation I, E is satisfied by I if and only if E' is satisfied by I.

Proof: We form E' by replacing each E_{ij} in E of the form A_p by $\sum_{i=S}^{M} J_i(A_p)$, and each E_{ij} in E of the form $\neg A_p$ by $\sum_{i=1}^{M+1-S} J_i(A_p)$.

Suppose we have an interpretation I that satisfies E. Then since semantically speaking conjunction is a minimizing function, I must satisfy every clause of E, and hence we need consider only a single arbitrary clause. Further, since semantically speaking disjunction is a maximizing function, we need consider only the case in which the value of the clause is the value of one of the transformed E_{ij}. In short, it is sufficient to show that the value assigned to E_{ij} is not less than S if and only if the value assigned to its transform is not less than S. Now, suppose A_p takes the value k, where $k \geq S$. Then $J_k(A_p)$ takes value M. Since $J_k(A_p)$ occurs in the transform $\sum_{i=S}^{M} J_i(A_p)$, and since disjunction is a maximizing function, the transform takes value M. On the other hand, suppose the transform $\sum_{i=S}^{M} J_i(A_p)$ takes a value not less than S. Then at least one of the $J_k(A_p)$ must take a value not less than S, where $k \geq S$. But this means that $J_k(A_p)$ takes value M, and hence A_p takes value k, which is not less than S. For the other transform, suppose $\neg A_p$ takes a value not less than S, say k. Then A_p takes the value $M + 1 - k$. Hence $J_{M+1-k}(A_p)$ takes value M. Since $M + 1 - S \geq M + 1 - k \geq 1$, $\sum_{i=1}^{M+1-S} J_i(A_p)$ takes value M. On the other hand, suppose the transform is satisfied. Then for some k between 1 and $M + 1 - S$, $J_k(A_p)$ is satisfied. Hence A_p takes value k, which means $\neg A_p$ takes value $M + 1 - k$. But $M \geq M + 1 - k \geq S$. Q.E.D.

Theorem 7: To any formula E there corresponds a J-CNF formula E' such that for any I, E is satisfied by I if and only if E' is satisfied by I.

Proof: The proof is immediate from Theorems 4-7. Q.E.D.

We now go on to expand our treatment to cover the first-order predicate calculus with functions but without identity. The intuitive semantics behind the connectives so far introduced will remain the same, although some of the technical details will change.

2. First-order functional calculus

We will use the same sentential connectives introduced above, but this should not give rise to confusion. Additional symbols are also introduced. The syntax of the language may be specified as follows:

1. sets of n-ary relations, for $n = 1, 2, \ldots$:
   $R_n = \{P^n_1, \ldots, P^n_j, \ldots\}$
2. sets of n-ary functions, for $n = 0, 1, \ldots$:
   $F_n = \{f^n_1, \ldots, f^n_j, \ldots\}$
3. set of variables:
   $V = \{x_1, \ldots, x_j, \ldots\}$
4. monadic sentence operators:
   a. negation: $\neg$
   b. J operators: J_i, for integer i, $1 \leq i \leq M$
   c. universal quantifiers: (x_i), for $i = 1, 2, \ldots$
   d. existential quantifiers: $(\exists x_i)$, for $i = 1, 2, \ldots$
5. dyadic sentence operators:
   a. disjunction: $\vee$
   b. conjunction: &

We denote the set of all relations by $R = \bigcup_n R_n$, and the set of all functions by $F = \bigcup_n F_n$. We assume the standard definition of «term» and «atomic formula». We will sometimes treat atomic formulas as sentence parameters of the sentential calculus. We denote the set of all terms by T and the set of all atomic formulas by At. We assume the standard definitions of «formula», «free variable», and «bound variable». As usual a formula is said to be closed if it contains no free variables. We will also assume for convenience that formulas contain no vacuous or overlapping quantifiers. We use the terms «formula» and «expression» interchangeably, and we will continue to make use of the special notation introduced above.

Central to our account of semantics is a slightly different notion of interpretation than that used above; it is also slightly different from that used for the standard two-valued functional calculus. Instead of assigning to each n-place relation a set of n-tuples from the domain, for each integer between 1 and M we assign such a subset to the relation. Formally, an interpretation I consists of the following:

1. a non-empty domain D of objects

2. for each f^n_i, an n-ary function f^n_i from D^n into D; this yields a map from T into D in the usual way
3. for each P^n_i, an n-ary function P^n_i from D^n into the set of integers $\{1, \ldots, M\}$

Our valuation function is also slightly different from the standard one. Consider the set G consisting of all functions g from V into D. For a given interpretation, the valuation function maps an expression and an integer from 1 through M into a subset of G. For the formal elaboration, we first derive a map from At x $\{1, \ldots, M\}$ into the power set of G as follows:

$$I(P^j_i t_1 \ldots t_j, k) = \{g \in G: \mathit{P}^j_i(gt_1, \ldots, gt_j) = k\}$$

where $t_1, \ldots, t_j$ are any terms, and where gt_p, $1 \leq p \leq j$, is the value taken by the function derived in the obvious way from the f^n_m when the variables take the values assigned by g. For any variable x_i, $gx_i = g(x_i)$; and more generally for terms $t_1, \ldots, t_n$ and function f^n_m, $gf^n_m(t_1, \ldots, t_n) = \mathit{f}^n_m(gt_1, \ldots, gt_n)$. For any expression E, any integer k between 1 and M, and any given interpretation I, the valuation function V is defined to be a subset of G as follows:

$g \in V(E, k, I)$ iff $g \in I(E, k)$, for E atomic
$g \in V(\neg E, k, I)$ iff $g \in V(E, M + 1 - k, I)$
$g \in V(J_i(E), k, I)$ iff $k = 1$ and $g \notin V(E, i, I)$, or
  $k = M$ and $g \in V(E, i, I)$
$g \in V(E_1 \vee E_2, k, I)$ iff $g \in V(E_1, k, I)$ and $g \in V(E_2, j, I)$
  for some $j \leq k$, or
  $g \in V(E_2, k, I)$ and $g \in V(E_1, j, I)$ for some $j \leq k$
$g \in V(E_1 \& E_2, k, I)$ iff $g \in V(E_1, k, I)$ and $g \in V(E_2, j, I)$
  for some $j \leq k$, or
$g \in V(E_2, k, I)$ and $g \in V(E_1, j, I)$ for some $j \gtrsim k$
$g \in V((\exists x_i)E, k, I)$ iff for every function g' differing from g
  for at most the argument x_i, there is no $j > k$ such that
  $g' \in V(E, j, I)$; and for some such function g', $g' \in V(E, k, I)$

$g \in V((x_i)E, k, I)$ iff for every function g' differing from g for at most the argument x_i, there is no $j < k$ such that $g' \in V(E, j, I)$; and for some such g', $g' \in V(E, k, I)$

Several important facts are easily verified from this definition. For any interpretation I, for every formula E and every function g in G, there is one and only one integer k between 1 and M such that $g \in V(E, k, I)$. Further, in any interpretation, given any expression E and value k, whether or not a given function g is in $V(E, k, I)$ depends only on the values which g takes over the free variables in E. That is, suppose $x_1, \ldots, x_n$ are all the free variables in E; then for any two functions in G, say g and g', if g and g' agree over the arguments $x_1, \ldots, x_n$, then for any k, $g \in V(E, k, I)$ iff $g' \in V(E, k, I)$. Thus if E is a closed formula then for some k between 1 and M, $V(E, k, I) = G$, and for all $j \neq k$, $V(E, j, I)$ is empty. We will say that E takes the value k under I (or that I assigns E the value k) just in case $V(E, k, I) = G$.

As before, we assume some specific critical value S to be given, such that S is an integer greater than 1 but less than or equal to M. The values greater than or equal to S are said to be «designated», while those less than S are said to be «undesignated». If E takes a designated value under I, we say that E is satisfied by I; if E takes an undesignated value under I, we say that E is falsified by I. A set of expressions is satisfied by I just in case every expression in the set is satisfied by I. If at least one expression in a set is falsified by I, we say that the whole set is falsified by I. An expression is satisfiable just in case there is some I which satisfies it; an expression is falsifiable just in case there is some I which falsifies it. No expression can be both satisfied and falsified by the same interpretation.

An expression is said to be «prenex» («a prenex expression», «in prenex form») if it consists of a sequence (possibly empty) of quantifiers whose scope is a quantifier-free expression. The quantifier-free portion is called the «matrix». If all the quantifiers are universal (existential) the expression is said to be universal (existential).

Theorem 8: To every expression E, there corresponds a prenex expression E' such that for all I, k, and g, $g \in V(E, k, I)$ iff $g \in V(E', k, I)$.

Proof: Since V (E, k, I) depends only on the free variables in E, we may assume that the set of free variables and the set of bound variables are disjoint; further, we may assume that the bound variables are pairwise distinct. From the definition of the valuation function V, above, it may be shown that all of the standard transformations used in obtaining prenex formulas for the two-valued case remain valid. The only complication is the distribution of the J operators through the quantifiers. Let E be any expression with the free variable x_i, and let E(i/j) be just like E only with all occurrences of x_i replaced by x_j, where x_j is some variable that does not occur in E. Then from the definition of V, the transformation for $J_k((\exists x_i)E)$ is

$$(\exists x_i)J_k(E) \,\&\, \neg\, (\exists x_j) \sum_{n=k+1}^{M} J_n(E(i/j))$$

Similarly, the transform for $J_k((x_i)E)$ is

$$(\exists x_i)J_k(E) \,\&\, \neg\, (\exists x_j) \sum_{n=1}^{k-1} J_n(E(i/j))$$

It may be shown by induction on the comlexity of E that E can be transformed into a formula in which no quantifier occurs within the scope of a J operator. The remainder of the proof is then parallel to that for the two-valued case, by induction on the complexity of the resulting formula. Q.E.D.

Theorem 9: To each prenex formula E there corresponds a prenex formula E' whose language does not differ from that of E except perhaps for the addition of a single function symbol, such that the sequence of quantifiers for E' is exactly like that of E except for the removal of the left-most existential quantifier (if there is one), such that the free variables of E and E' are the same, and such that:

(a) For any I, k, and g, if $g \in V(E', k, I)$ then for some $j \geq k$, $g \in V(E, j, I)$; and

b) Each interpretation I over the language of E can be extended to an interpretation I' (with the same domain) over the language of E' such that for all g and k, $g \in V(E, k, I)$ iff $g \in V(E', k, I')$.

Proof: By induction on the number of quantifiers in E. Suppose there are none. Then set E' to E and clearly both (a) and (b) are satisfied.

Suppose $E = (x_i)E_1$. Then by the induction hypothesis, there is a formula E_1' which satisfies the theorem with respect to E_1. We set $E' = (x_i)E_1'$. We prove that (a) is satisfied by contradiction. Suppose for some g, I, k, and $j < k$, $g \in V(E', k, I)$ but $g \in V(E, j, I)$. Then for some g' differing from g for at most the argument x_i, $g' \in V(E_1, j, I)$. But we know that for some r, $g' \in V(E_1', r, I)$, and by the induction hypothesis, $r \leq j$. Hence, $r < k$. This contradicts the fact that $g \in V(E', k, I)$, for by definition of V there can be no such g' such that for $r < k$, $g' \in V(E_1', r, I)$. The fact that (b) is satisfied follows directly from the definition of V and the induction hypothesis.

Suppose $E = (\exists x_i)E_1$. Let $x_1, \ldots, x_n$ be the free variables in E and let f^n_j be a function symbol not occurring in E. We form E' by replacing every free occurrence of x_i in E_1 by the term $f^n_j(x_1, \ldots, x_n)$. For (a), suppose $g \in V(E', k, I)$; we know that for some r, $g \in V(E, r, I)$. Let g' take the same values as g except $g'(x_i) = f^n_j(g(x_1), \ldots, g(x_n))$. Then clearly $g' \in V(E_1, k, I)$. Since g' differs from g for at most the argument x_i, by the definition of V, r must be greater than or equal to k. For (b), let some interpretation I for E be given. To extend I for E', it is necessary to interpret the function f^n_j. Intuitively, for each g, if $g(x_j) = d_j$, $1 \leq j \leq n$, we will assign $f^n_j(d_1, \ldots, d_n)$ the value d for which the value of E_1 is a maximum when $g(x_i) = d$. To do this, let $g \in G$ be given. Let G' be the set of functions g' differing from g for at most the argument x_i. (Note: As usual, we are assuming $g \in G'$.) We make the following definitions:

$$k^* = \max_{g' \in G'} \{k\colon g' \in V(E_1, k, I)\}$$

$$D' = \{g'(x_i): g' \in G' \text{ and } g' \in V(E_1, k^*, I)\}$$

Clearly by construction D' is not empty. Let d be an arbitrary member of D'. Then we set $f^n_j(g(x_1), \ldots, g(x_n)) = d$. By employing this procedure for each $g \in G$, we obtain a complete definition of f^n_j. (Actually, we only consider those g which differ over the free variables of E_1.) The required interpretation I' is just I supplemented by the interpretation of f^n_j. Now, if $g \in V(E, k, I)$, then k is the maximum value for whch there is a g' differing from g for at most the argument x_i such that $g' \in V(E_1, k, I)$. But x_i is not free in E', and the interpretation of f^n_j is such as to make E_1 take on its maximum value for the function g. Hence $g \in V(E', k, I')$. On the other hand, suppose $g \in V(E', k, I')$. Form g' just like g except $g'(x_i) = f^n_j(g(x_1), \ldots, g(x_n))$. Clearly $g' \in V(E_1, k, I)$. Also, by construction of f^n_j, k is the maximum value j for which there is some such $g' \in V(E_1, j, I)$. Hence $g \in V(E, k, I)$. Q.E.D.

Theorem 10: To each prenex formula E, there corresponds a universal prenex formula E' whose language does not differ from that of E except perhaps for the addition of a finite number of function symbols, such that:

(a) For any I, k, and g, if $g \in V(E', k, I)$ then for some $j \geq k$, $g \in V(E, j, I)$; and
(b) Each interpretation I over the language of E can be extended to an interpretation I' (with the same domain) over the language of E' such that for all g and all k, $g \in V(E, k, I)$ iff $g \in V(E', k, I')$.

Proof: By iterated applications of Theorem 9. Q.E.D.

A canonical interpretation of a set of formulas is an interpretation whose domain is the set of terms built up from the functions and variables occurring in the set, such that the function symbols are given their canonical values as functions on the terms. Such interpretations will be of great importance for the material to follow. We will now prove that we need only consider canonical interpretations to answer questions about satisfiability.

Theorem 11: Let E be a closed universal prenex formula. If there is an interpretation I that assigns E the value k, then there is a canonical interpretation I' that assigns E the value j, where $j \geq k$.

Proof: Let the interpretation I be given, and let g^* be an arbitrary member of G. The domain of I' will be the terms, and the functions receive their canonical interpretation. The interpretation for the relation symbol P^m_n is the function $\mathcal{P}^m_n$ which we define as follows: $\mathcal{P}^m_n(t_1, \ldots, t_m) = k$ iff $\mathcal{P}^m_n(g^*t_1, \ldots, g^*t_m) = k$, where $\mathcal{P}^m_n$ is the interpretation of P^m_n under I. Let G_T be the set of functions g_T mapping the variables into the terms. We define the map R from G_T into G by setting $R(g_T) = g$, where $g(x_i) = g^*g_T(x_i)$ for all variables x_i. We now prove a series of lemmas to aid in proving the main theorem.

Lemma 11.1: Let $g_T \in G_T$, and suppose $R(g_T) = g$. Then for any term t, $gt = g^*g_Tt$.

Proof: By induction on the complexity of t. If t is a variable, then the condition is immediate from the definition of R. Suppose t is of the form $f^m_n(t_1, \ldots, t_m)$. Then

$$\begin{aligned} gt &= \mathfrak{f}^m_n(gt_1, \ldots, gt_m) \\ &= \mathfrak{f}^m_n(g^*g_Tt_1, \ldots, g^*g_Tt_m), \text{ by induction} \\ &= g^*f^m_n(g_Tt_1, \ldots, g_Tt_m) \\ &= g^*g_Tf^m_n(t_1, \ldots, t_m) \\ &= g^*g_Tt \end{aligned}$$

Lemma 11.2: Let $g_T \in G_T$ and suppose $R(g_T) = g$. Let E be any formula with no quantifiers. Then for any k, if $g_T \in V(E, k, I')$ then $g \in V(E, k, I)$.

Proof: By induction on the complexity of E. If E is atomic, the result is immediate from the definition of I' and Lemma 11.1. The induction for &, $\vee$, and $\neg$ is straightforward and is omitted here. We give the proof for the J operators. Suppose

$E = J_m(E_1)$. If $g_T \in V(J_m(E_1), k, I')$ then either $k = 1$ or $k = M$. Suppose $k = M$. Then $g_T \in V(E_1, m, I')$ and hence by induction, $g \in V(E_1, m, I)$, which means that $g \in V(J_m(E_1), M, I)$. Suppose $k = 1$. Then $g_T \notin V(E_1, m, I')$. Thus for some $n \neq m$, $g_T \in V(E_1, n, I')$. But then by induction, $g \in V(E_1, n, I)$, and hence $g \notin V(E_1, m, I)$. Thus $g \in V(J_m(E_1), 1, I)$. Q.E.D.

Lemma 11.3: Let $g_T \in G_T$, and suppose $R(g_T) = g$. Let E be any formula with no quantifiers. Then for any k, if $g \in V(E, k, I)$, then $g_T \in V(E, k, I')$.

Proof: Suppose $g \in V(E, k, I)$ but $g_T \notin V(E, k, I')$. Then for some $j \neq k$, $g_T \in V(E, j, I')$. Then by Lemma 11.2, $g \in V(E, j, I)$, which contradicts our assumption that $g \in V(E, k, I)$. Q.E.D.

Lemma 11.4: Let E be any universal prenex formula and suppose $g \in V(E, k, I)$ and $R(g_T) = g$. Then for some $j \geq k$, $g_T \in V(E, j, I')$.

Proof: By induction on the number of quantifiers in E. If E has no quantifiers, then the result follows immediately from Lemma 11.3. Suppose $E = (x_i)E_1$, where E_1 is a universal prenex formula. Further suppose $g \in V(E, k, I)$ and $R(g_T) = g$. Suppose contrary to the theorem that $g_T \in V(E, j, I')$ and that $j < k$. This means that for some g_T' differing from g_T for at most the argument x_i, $g_T' \in V(E_1, j, I')$. By the induction hypothesis, $g' \in V(E_1, p, I)$, where $p \leq j$ and $g' = R(g_T')$. Hence $p < k$. But by definition of R, g' differs from g for at most the argument x_i, and this contradicts the assumption that $g \in V(E, k, I)$. Q.E.D.

Proof of main theorem continued: If I assigns E the value k, then $V(E, k, I) = G$. Since E is closed, there is some j such that $V(E, j, I') = G_T$. By Lemma 11.4, $j \geq k$. Q.E.D.

Corollary 11.1: A set of closed universal prenex formulas is satisfiable iff there is a canonical interpretation which satisfies the set.

It is sometimes valuable to treat quantifier-free expressions as if they were expressions in sentential calculus. For this purpose, atomic formuas play the role of sentence parameters, and the connectives &, $\vee$, $\neg$, and the J operators are semantically interpreted as in the sentential calculus. We will say that a quantifier-free expression is sententially satisfiable (falsifiable) just in case there is some sentential interpretation which satisfies (falsifies) the expression when it is treated as an expression of the sentential calculus.

Theorem 12: Having a canonical interpretation I of a language is equivalent to having an interpretation I' of the sentential calculus on the set of atomic formulas of the language.

Proof: For each relation P^n_i, I gives a function $\mathcal{P}^n_i$ from T^n into the set $\{1, \ldots, M\}$. For each atomic expression $P^n_i t_1 \ldots t_n$, we set $I'(P^n_i t_1 \ldots t_n) = \mathcal{P}^n_i(t_1, \ldots, t_n)$. Similarly, given I', we can obtain I. Q.E.D.

Theorem 13: Let $E(x_1, \ldots, x_m)$ be a quantifier-free formula with free variables $x_1, \ldots, x_m$, and let $t_1, \ldots, t_m$ be any terms of the language. Let g be any function from the set of variables into the set of terms such that $g(x_i) = t_i$, $1 \leq i \leq m$. Then in a given canonical interpretation I, $g \in V(E(x_1, \ldots, x_m), k, I)$ if and only if in the corresponding interpretation I' of the sentential calculus on the atomic formulae, $V(E(t_1, \ldots, t^m), I') = k$.

Proof: By induction on the complexity of E. The case for E atomic is trivial because of the relationship between I and I'. (See the proof of Theorem 12.) The induction over the connectives is straightforward, so we will do only the case for conjunction here. Suppose $E = E_1 \& E_2$. Then $g \in V(E_1(x_1, \ldots, x_m) \& E_2(x_1, \ldots, x_m), k, I)$ iff. $g \in V(E_1(x_1, \ldots, x_m), k, I)$ and for some $j \geq k$, g is also an element of $V(E_2(x_1, \ldots, x_m), j, I)$; or $g \in V(E_2(x_1, \ldots, x_m), k, I)$ and for some $j \geq k$, $g \in V(E_1(x_1, \ldots, x_m), j, I)$. But by the induction hypothesis, this is true iff: $V(E_1(t_1, \ldots, t_m), I') = k$ and for some $j \geq k$, $V(E_2(t_1, \ldots, t_m), I') = j$; or $V(E_2(t_1, \ldots, t_m), I') = k$ and for some $j \geqq k$, $V(E_1(t_1, \ldots, t_m), I') = j$. But this is true iff $V(E_1(t_1, \ldots, t_m) \& E_2(t_1, \ldots, t_m), I') = k$. Q.E.D.

We now prove a very important theorem which is the multi-valued analogue of the standard Herbrand result.

Theorem 14: Let E $(x_1, \ldots, x_n)$ be a quantifier-free formula with free variables $x_1, \ldots, x_n$. Then $(x_1) \ldots (x_n)E(x_1, \ldots, x_n)$ is unsatisfiable iff there are terms $t^i_1, \ldots, t^i_n$, $1 \leq i \leq k$, of the language of E such that the formula

$E(t^1_1, \ldots, t^1_m) \& \ldots \& E(t^k_1, \ldots, t^k_m)$

is sententially unsatisfiable.

Proof: Suppose such a set of terms exists. Then by the definition of V, every canonical interpretation fails to satisfy $(x_1) \ldots (x_n)E(x_1, \ldots, x_n)$. Then by Theorem 11, the formula is unsatisfiable.

On the other hand, suppose the formula is unsatisfiable. Then there is no canonical interpretation that satisfies the formula. But then by the definition of V, for every canonical interpretation I, there are terms $t_1, \ldots, t_n$ such that for $g(x_i) = t_i$, $1 \leq i \leq n$, $g \in V(E(x_1, \ldots, x_n), j, I)$, where $j < S$. But by Theorems 12 and 13, for every sentential interpretation I' on the atomic formulas, there is a series of terms $t_1, \ldots, t_n$ such that $V(E(t_1, \ldots, t_n), I') = j$, where $j < S$. But then by Theorem 3, there are terms $t^i_1, \ldots, t^i_n$, $1 \leq i \leq k$, such that the formula

$E(t^1_1, \ldots, t^1_n) \& \ldots \& E(t^k_1, \ldots, t^k_n)$

is sententially unsatisfiable. Q.E.D.

We say that a formula is (universal, existential) prenex J-CNF if it is (universal, existential) prenex and the quantifier-free part is J-CNF.

Theorem 15: To each closed prenex expression E, there corresponds a closed prenex J-CNF expression E' such that in any interpretation, E is satisfied if and only if E' is satisfied.

Proof: To form E', we simply treat the matrix of E as an expression in the sentential calculus over atomic formulas. The result then follows from Theorems 7, 11, 12 and 13. Q.E.D.

3. Resolution

The resolution principle for the logics just described will be quite similar to two-valued resolution. We will assume that the principle is to be applied only to formulas that are universal prenex J-CNF. The resolution principle seeks to determine whether or not a given expression is satisfiable. If there is an empty clause in the expression (a clause consisting only of the empty string), then we know immediately that the expression is unsatisfiable. Hence in all practical cases, the expression with which we begin will not have an empty clause. Each clause will be a disjunction of expressions of the form $J_k(E)$, where E is some atomic formula; such an expression will be called a «J-literal». A «literal» will be for us an atomic expression.

Central to resolution procedures is the notion of «substitution». There is a close relationship between substitution and our g functions, but this point will not be elaborated here. We may use exactly the same notion of substitution for many-valued logics as that used for the two-valued case. We also employ the same notion of «unifying» a set of literals. The standard unification algorithm may be used, and all of the usual notation and theorems associated with it may be assumed unchanged, since the algorithm is based solely on syntactic considerations which remain unchanged by the introduction of the J-operators. We will sometimes make use of the identity substitution; this substitution simply maps each variable into itself without change.

Two J-literals, $J_{k_1}(E_1)$ and $J_{k_2}(E_2)$ will be said to be «complementary» just in case the literals E_1 and E_2 are identical but $k_1 \neq k_2$. Intuitively, a pair of complementary J-literals simply make contradctory semantic assignments to a given atomic expression. Thus in any canonical interpretation, both could not be satisfied, and hence their conjunction could never be satisfied. Two sets of J-literals $\{J_{p_1}(E_{11}), \ldots, J_{p_n}(E_{1n})\}$ and $\{J_{q_1}(E_{21}), \ldots, J_{q_m}(E_{2m})\}$ are said to be complementary just in case for all i and j, $E_{1_i} = E_{1_j} = E_{2_i} = E_{2_j}$, and $p_i \neq q_j$.

Let E_1 and E_2 be two clauses composed of J-literals. Their many-valued resolvent (henceforth, mv-resolvent) E_3 is obtained as follows:

(i) Let $x_{2_1}, \ldots, x_{2_n}$ be the variables occurring in E_2 and let x_i be the variable in E_1 with the highest index. Then Θ is the substitution $\{(x_{i+1}, x_{2_1}), \ldots, (x_{i+n}, x_{2_n})\}$. (None of the variables in $E_2\Theta$ occurs in E_1, and vice versa.)

(ii) Let C_1 and C_2 be the sets of J-literals occurring in E_1 and E_2, respectively. Suppose there is a pair of sets of J-literals $K = \{K_1, \ldots, K_p\}$ and $L = \{L_1, \ldots, L_q\}$ such that:

(a) $K \subseteq C_1$ and $L \subseteq C_2$;

(b) the set $\{K_1', \ldots, K_p', L_1, \ldots, L_q'\}$ is unifiable where K_i' and L_i' are the literals contained in the J-literals K_i and L_i, respectively; and

(c) if λ is a unifying substitution, then $K\lambda$ and $L\lambda$ are complementary.

Let λ_0 be the chosen simplest unifying substitution so that $K\lambda_0$ and $L\Theta\lambda_0$ are sets of complementary J-literals. Then E_3 is the disjunction of the J-literals:

$$(C_1 - K)\lambda_0 \cup (C_2 - L)\Theta\lambda_0$$

The many-valued resolution principle may now be stated as follows:

Many-valued resolution principle: From any two clauses of J-literals, E_1 and E_2, infer an mv-resolvent of E_1 and E_2.

Let C be a set of clauses of J-literals. Then MVR(C) is the set consisting of the members of C together with the mv-resolvents of all pairs of members of C. We define $MVR^n(C)$ recursively as follows:

(i) $MVR^0(C) = C$

(ii) $MVR^n(C) = MVR(MVR^{n-1}(C))$, for $n > 0$

Let C be a set of clauses and let E be a clause of J-literals. Then the least value n such that $E \in MVR^n(C)$ gives the number of applications of the mv-resolution principle required to obtain E from the members of C.

We now proceed to state and prove a series of theorems which establish the desirable properties of mv-resolution.

In obtaining the resolvent of two clauses, we make use of two substitutions, λ_0 and Θ. However, λ_0 is applied alone to one of the clauses, while the composite substitution $\Theta\lambda_0$ is applied to the other. It is more convenient to think in terms of the two substitutions λ_0 and $\Theta\lambda_0$. Thus in the following material whenever reference is made to the substitutions used in obtaining a resolvent, we shall mean λ_0 and $\Theta\lambda_0$.

Theorem 16: If E_3 is an mv-resolvent of E_1 and E_2, and λ_1 and λ_2 are the substitutions employed for E_1 and E_2, respectively; then for any substitution λ, if an interpretation I sententially satisfies both $E_1\lambda_1\lambda$ and $E_2\lambda_2\lambda$, then I sententially satisfies $E_3\lambda$.

Proof: First suppose $E_3 = e$. Then $E_1\lambda_1$ is of the form

$J_{p_1}(E) \vee \ldots \vee J_{p_n}(E)$

and $E_2\lambda_2$ is of the form

$J_{q_1}(E) \vee \ldots \vee J_{q_m}(E)$

where for all i and j, $p_i \neq q_j$. But then $E_1\lambda_1\lambda$ is just

$J_{p_1}(E\lambda) \vee \ldots \vee J_{p_n}(E\lambda)$

and $E_2\lambda_2\lambda$ is just

$J_{q_1}(E\lambda) \vee \ldots \vee J_{q_m}(E\lambda)$

Then there is no I which sententially satisfies both $E_1\lambda_1\lambda$ and $E_2\lambda_2\lambda$, and thus the theorem follows vacuously.

On the other hand, suppose $E_3 \neq e$. Then $E_1\lambda_1$ is of the form $E_4 \vee E_5$ and $E_2\lambda_2$ is of the form $E_6 \vee E_7$, where E_5 and E_7 are disjunctions of complementary J-literals and either $E_4 \neq e$ or $E_6 \neq e$. If I sententially satisfies both $E_1\lambda_1\lambda$ and $E_2\lambda_2\lambda$ then I sententially satisfies either $E_4\lambda$ or $E_6\lambda$, since I cannot sententially both $E_5\lambda$ and $E_7\lambda$. Hence I sententially satisfies $E_5\lambda \vee E_7\lambda$; that is, I sententially satisfies $E_3\lambda$. Q.E.D.

We will inroduce a little more special notation. Let K be a set of substitutions $\{\lambda_1, \ldots, \lambda_m\}$, and let C be any set of clauses $\{E_1, \ldots, E_n\}$. Then K(C) is the set of clauses $\{E_i\lambda_j : E_i \in C$ and $\lambda_j \in K\}$.

Theorem 17: Let C be any set of clauses of J-literals, let K_1 be the set consisting of the identity substitution and the set of all substitutions employed in forming MVR(C), and let K_2 be any non-empty set of substitutions. Denote $K_2(K_1(C))$ by C_1

and $K_2(MVR(C))$ by C_2. Then for any interpretation I, if I sententially satisfies C_1, I sententially satisfies C_2 as well.

Proof: Suppose the theorem is false. Then there is an interpretation I, an expression E in MVR(C), and a substitution λ_2 in K_2 such that I does not sententially satisfy $E\lambda_2$, but I does sententially satisfy C_2. Now, E cannot be a member of C, for then $E\lambda_2$ would be a member of C_2 and hence I would satisfy $E\lambda_2$, contrary to our assumption. Thus E must be a resolvent of two clauses E_1 and E_2, both in C. Let λ_1' and λ_1'' be the substitutions employed in obtaining E from E_1 and E_2. Then $E_1\lambda_1'\lambda_2 \in C_2$ and $E_2\lambda_1''\lambda_2 \in C_2$, and hence both are sententially satisfied by I. But then by Theorem 16, I sententially satisfies $E\lambda_2$. But this contradicts our original assumption that I does not sententially satisfy $E\lambda_2$. Q.E.D.

Theorem 18: If C is any set of clauses of J-literals such that for some n, $e \in MVR^n(C)$, then there is a non-empty finite set of substitutions K such that K(C) is sententially unsatisfiable.

Proof: By induction on n. The case for $n = 0$ is trivial; take K to consist of only the identity substitution. For the induction step, suppose for $p > 0$, $e \in MVR^p(C)$. But $MVR^p(C)$ is just $MVR^{p-1}(MVR(C))$, and so by the induction hypothesis, there is a set of substitutions K_1 such that $K_1(MVR(C))$ is sententially unsatisfiable. Let K_2 be the set consisting of the identity substitution and all the substitutions employed in going from C to MVR(C). Then by Theorem 17, $K_1(K_2(C))$ is sententially unsatisfiable. We then take K to be all substitutions of the form $\lambda_2\lambda_1$, where $\lambda_1 \in K_1$ and $\lambda_2 \in K_2$. Q.E.D.

Some substitutions are rather trivial in the sense that they can be «undone». Such substitutions are only a «renaming» of variables, maintaining distinctness. An example is the substitution Θ mentioned in (i) of the definition of resolvent. We say that such substitutions are «reversible» since for any such substitution λ, there is another λ' such that $E\lambda\lambda' = E$. It is not difficult to show that the composition of any finite number of reversible substitutions is a reversible substitution. Further, it may easily be shown that if E_3 is an mv-resolvent of E_1 and E_2, then for any reversible substitutions λ_1 and λ_2, there is a reversible substitution λ_3 such that $E_3\lambda_3$ is an mv-resolvent of $E_1\lambda_1$ and $E_2\lambda_2$. We will make use of these facts in the next theorem.

Theorem 19: Let C be a set of clauses composed of J-literals. If C is sententially unsatisfiable, then for some n, $e \in MVR^n(C)$, and it is possible to so obtain e making only reversible substitutions.

Proof: If $e \in C$, then the proof is trivial. Hence we assume $e \notin C$. We define the number r(C) to be the number of J-literals appearing in C minus the number of clauses in C. The proof is then by induction on r(C).

If $r(C) = 0$, then each clause in C is composed of a single J-literal. The only way C can be sententially unsatisfiable is if there are two clauses of the form $J_p(E)$ and $J_q(E)$, where $p \neq q$. (Otherwise, for each J-literal, $J_k(E_i)$, we could assign the literal E_i the value k and thus sententially satisfy C.) But applying the mv-resolution principle to these two clauses would yield the empty string e, using only the substitution Θ and its inverse. Thus $e \in MVR^1(C)$.

For the induction step, suppose $r(C) = n + 1$, for $n \geq 0$. Then there is at least one clause E in C which is of the form $E_1 \vee E_2$, where $E_1 \neq e$ and $E_2 \neq e$. Let $C' = C - \{E\}$. We break C into two parts, $C_1 = C' \cup \{E_1\}$ and $C_2 = C' \cup \{E_2\}$. It is easy to verify that both C_1 and C_2 must be sententially unsatisfiable since C is sententially unsatisfiable. Further, $r(C_1) < n + 1$ and $r(C_2) < n + 1$. Hence by induction, for some n_1, $e \in MVR^{n1}(C_1)$ and for some n_2, $e \in MVR^{n2}(C_2)$; further, in both cases, e may be so obtained making only reversible substitutions. But then noting that $C = C' \cup \{E_1 \vee E_2\}$, either $E_2\lambda \in MVR^{n1}(C)$, for λ a reversible substitution, or $e \in MVR^{n1}(C)$. If the latter is the case, then the proof is complete, so suppose the former is the case. Then since $C' \subseteq MVR^{n1}(C)$, $e \in MVR^{n1+n2}(C)$. Q.E.D.

Theorem 20: Let λ_1 and λ_2 be substitutions. If E_3 is an mv-resolvent of $E_1\lambda_1$ and $E_2\lambda_2$, then there is a substitution λ' and an expression E_3' such that E_3' is an mv-resolvent of E_1 and E_2 and $E_3 = E_3'\lambda'$.

Proof: Suppose $x_{11}, \ldots, x_{1m}$ are all the variables of E_1 and $x_{21}, \ldots, x_{2n}$ are all the variables of E_2. We may assume that λ_1 has a substitution for each variable in E_1, λ_2 has a substitution for each variable in E_2, and that neither has substitutions for any other variables. If not, we can always add identity substitutions over the needed variables and eliminate the unused substitutions. Each substitution may then be represented as follows:

$$\lambda_1 = \{(t_{11}, x_{11}), \ldots, (t_{1m}, x_{1m})\}$$
$$\lambda_2 = \{(t_{21}, x_{21}), \ldots, (t_{2n}, x_{2n})\}$$

where the t's are terms. We may represent E_1 by $E_1' \vee E_1''$ and E_2 by $E_2' \vee E_2''$. We know there is a substitution λ_0 and a change of variables substitution Θ such that $E_3 = E_1'\lambda_1\lambda_0 \vee E_2'\lambda_2\Theta\lambda_0$, and $E_1''\lambda_1\lambda_0$ and $E_2''\lambda_2\Theta\lambda_0$ are disjunctions of complementary sets of J-literals. We define two substitutions as follows:

$$\Theta' = \{(x_{1\ m+1}, x_{21}), \ldots, (x_{1\ m+n}, x_{2n})\}$$
$$\lambda_2' = \{(t_{21}, x_{1\ m+1}), \ldots, (t_{2n}, x_{1\ m+n})\}$$

Then $E_2\Theta'\lambda_2'\Theta = E_2\lambda_2\Theta$. We know λ_0 is composed of two sets of substitutions, λ_0' and λ_0''. The substitutions in λ_0' are for variables in $E_1\lambda_1$ and those in λ_0'' are for variables not in $E_1\lambda_1$. Therefore, $E_1\lambda_1\lambda_0' = E_1\lambda_1\lambda_0$ and $E_2\lambda_2\Theta\lambda_0'' = E_2\lambda_2\Theta\lambda_0$. **Let** $\lambda_1'' = \lambda_1\lambda_0'$ and let $\lambda_2'' = \lambda_2'\Theta\lambda_0''$. Then λ_1'' and λ_2'' do not make substitutions ove any of the same variables. Consider $\lambda^* = \lambda_1'' \cup \lambda_2''$.

$$E_1\lambda^* = E_1\lambda_1'' = E_1\lambda_1\lambda_0' = E_1\lambda_1\lambda_0$$
$$E_2\Theta'\lambda^* = E_2\Theta'\lambda_2'' = E_2\Theta'\lambda_2'\Theta\lambda^{0''} = E_2\lambda_2\Theta\lambda_0'' = E_2\lambda_2\Theta\lambda_0$$

Thus we know that λ^* is a unifying substitution for E_1'' and $E_2''\Theta'$. Let λ^*_0 be the chosen simplest such substitution. Then we know $E_3' = E_1'\lambda^*_0 \vee E_2'\Theta'\lambda^*_0$ is an mv-resolvent of E_1 and E_2. Further, since λ^*_0 is the simplest substitution, there is a substitution λ' such that $\lambda^*_0\lambda' = \lambda^*$. But then $E_3'\lambda' = E_3$. Q.E.D.

Theorem 21: Let K be any non-empty set of substitutions. If $E \in MVR^n(K(C))$, for some integer n and set of clauses C, then there is a substitution λ' and an expression E' such that $E' \in MVR^n(C)$ and $E = E'\lambda'$.

Proof: By induction on n. If $n = 0$, the case is trivial. For the induction step, suppose $E \in MVR^{n+1}(K(C))$, where $n \geq 0$. If $E \in MVR^n(K(C))$, the desired result is an immediate consequence of the induction hypothesis. Thus suppose E is a resolvent of E_1 and E_2, both members of $MVR^n(K(C))$. By the induction hypothesis, there are expressions E_1' and E_2' and substitutions λ_1' and λ_2' such that $E_1 = E_1'\lambda_1'$ and $E_2 = E_2'\lambda_2'$, where E_1' and E_2' are both members of $MVR^n(C)$. Then by Theorem 20, there is an expression E' and a substitution λ' such that E' is an mv-resolvent of E_1' and E_2' and $E'\lambda' = E$. Thus $E' \in MVR^{n+1}(C)$. Q.E.D.

Theorem 22: Let K be any non-empty set of substitutions and C any set of clauses of J-literals. If $e \in MVR^n(K(C))$, for some n, then $e \in MVR^n(C)$.

Proof: If $e \in MVR^n(K(C))$, then by Theorem 21, there is an expression E' and a substitution λ' such that $E'\lambda' = e$ and such that $E' \in MVR^n(C)$. But no substitution can make a non-empty clause into an empty one, so $E' = e$. Q.E.D.

Theorem 23: Let E be any closed expression, and let E' be the universal prenex J conjunctive normal form of E. Let C be the set of clauses obtained from E'. Then E is unsatisfiable if and only if for some n, $e \in MVR^n(C)$.

Proof: First suppose E is unsatisfiable. Then by Theorems 8, 10, and 15, E' is unsatisfiable. By Theorem 14, there is a finite non-empty set of substitutions K such that K(C) is sententially unsatisfiable. Then by Theorem 19, for some n, $e \in MVR^n(K(C))$. By Theorem 22, $e \in MVR^n(C)$.

On the other hand, suppose $e \in MVR^n(C)$ for some n. By Theorem 18, there is a non-empty finite set of substitutions K such that K(C) is sententially unsatisfiable. By Theorem 14, E' is unsatisfiable. By Theorems 8, 10, and 15, E is unsatisfiable. Q.E.D.

Theorem 23 establishes the soundness and completeness of the mv-resolution principle. Note that no specific values for M or S were used at any stage of our development. Thus the procedure works for the entire class of languages under consideration.

4. Applications

Like the two-valued resolution principle, the mv-resolution principle is designed to test whether or not a given expression is satisfiable. But because of the J operators, the mv-resolution principle can be applied to a very wide class of problems. Let E and E' be any two expessions and k and k' be any two subsets of {1, ..., M}. Then we may test assertions of the type «If E takes a value from k, then E' must take a value from k'». We test such assertions by applying the mv-resolution principle to the expression

$$\sum_{i \in k} J_i(E) \ \& \sum_{j \notin k'} J_j(E')$$

More complicated assertions may also be tested. For example, consider «E' must always take a value greater than or equal to that taken by E». This statement could be tested by applying mv-resolution to the formula

$$\sum_{i=2}^{M} (J_i(E) \ \& \sum_{j=1}^{i-1} J_j(E'))$$

Instead of «greater than or equal to» we could have any decideable condition relating the semantic values of E and E'.

By treating sets of expressions as some function of their components, where that function is expressible in the syntax, we may test statements asserting relations between a set of expressions and a single expression, or assertions of relationships among several sets of expressions. For example, «If the expressions in A are satisfied, then so is expression E» would be tested by testing the formula

$$\sum_{i=S}^{M} J_i(\prod_{E' \in A} E') \ \& \sum_{j=1}^{S-1} J_j(E)$$

As another example, the statement «If the largest value of any formula in A_1 is k_1 and the smallest value of any formula in A_2 is k_2, then some formula in A_3 must take value k_3» could be tested by testing the formula

$$J_{k_1}(\sum_{E_1 \in A_1} E_1) \; \& \; J_{k_2}(\prod_{E_2 \in A_2} E_2) \; \& \; \sum_{i \neq k_3} \sum_{E_3 \in A_3} J_i(E_3)$$

The «trick» in all of these cases is to determine under what conditions the given assertion would be false; these conditions are then expressed by a formula in the syntax, and that formula is tested for satisfiability using the mv-resolution principle. If the formula is unsatisfiable, the given assertion is correct; otherwise it is incorrect.

One may reasonably ask if there are any circumstances which require a multi-valued analysis. It is well known that probability assertions do not fit into the types of languages here discussed. However, another possible candidate is the analysis of decideability statements. For a given proposition, we may ask if it is decideable and true, decideable and false, or undecideable. Thus it initially seems that a three-valued analysis may be appropriate. We could assign 1 to «decideable and false», 2 to «undecideable», and 3 to «decideable and true». However, consider $P \vee \neg P$. We know that regardless of the value of P, this expression is always true, and hence should be decideable and true. But on our proposed analysis, if P takes value 2 (undecideable) then $P \vee \neg P$ takes value 2 (undecideable). Hence the analysis does not seem to work for the decideability of statements. (All alternative assignements of values to the three categories «decideable and false», «undecideable», and «decideable and true» face similar difficulties.)

If we shift our attention from statements to machines, the above approach can be fruitful. Suppose we have a set of «elementary decision machines» (edm's), which we will designate by D_i^j. Each machine D_i^j has j input lines; the i is merely an index. For a given set of inputs, machine D_i^j may do one of three things: (i) stop and output 0; (ii) run on forever with no output; or (iii) stop and output 1. Each edm may be thought to correspond to some condition on (or assertion about) the inputs; output 0 corresponds to «decideable and false», failure to stop corresponds to «undecideable», and output 1 corresponds to «decideable and true». Thus we sharply distinguish between no output and output of 0.

Suppose we may connect our edm's by threshold-like elements which correspond to &, $\vee$, and $\neg$. The &-gate has two inputs; it outputs 0 if at least one 0 input is received and outputs 1 if two 1 inputs are received. The $\vee$-gate has two inputs; it outputs 1 if at least one 1 input is received and outputs 0 if two 0 inputs are received. The $\neg$ -gate has a single input; it outputs 1 if a 0 input is received and outputs 0 if a 1 input is received. In all other cases, the gates have no output. Consider circuits composed of edm's and such gates such that the edm's receive input only from sources external to the circuit. Such circuits may be analyzed using the approach suggested above. We would assign «output 0» the value 1, «no output» the value 2, and «output 1» the value 3. Again consider a circuit whose equation is of the form $P \vee \neg P$. If the component P has no output, then the entire circuit would have no output. This is in exact correspondence with our logical analysis. Such circuits may be a useful approach to pattern recognition based on local feature analyzers.

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