A NORMAL FORM FOR ALGEBRAIC CONSTRUCTIONS II

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This note continues [8], which introduced word-constructions.

In section 1 we give an improved definition of word-constructions; it is essentially the same as that in [8], but the new formulation is more model-theoretic. We also introduce an associated functor, which seems to be important.

In section 2 we prove the main properties of word-constructions: the uniform reduction theorem (Theorem 11), the effectivity theorem (Theorem 13), the preservation theorem (Theorem 14), and some categorical properties. Section 2.2 contains a small correction to the Theorem of [8].

In section 3 we prove a normal form theorem (Theorem 22) which characterises those functors which are naturally isomorphic to the associated functors of existential or positive existential word-constructions. The functors in question are simply those which preserve filtered limits. The proof of the normal form theorem is by way of left Kan extensions. Less technically, this section is about the precise overlap between newer (categorical) and older (syntactic) approaches to the foundations of algebra.

Finally in section 4 we relate our work to other people's. The local functors of Feferman [4] turn out to be associated functors of existential word-constructions, so that the preservation theorem for these functors is simply a weak form of the preservation theorem for word-constructions. (At least this holds good up to the level of generality which we have pursued in this paper — for example we have ignored singular cardinals.) The relationship to some functors defined by Eklof [2], [3] proves more complex. We also relate word-constructions to Gaifman's single-valued operations [7]. Answering two questions of Gaifman, we show that there are first-order operations which are not expressible as word-constructions.

I thank Paul Eklof, Solomon Feferman, Haim Gaifman and P.J. Higgins for some valuable correspondence and preprints. I also thank Paul Henrard and the Université Catholique de Louvain for organising the very pleasant conference at which some of these results were first presented.

1: Word-constructions and their associated functors

1.1. Generators and relations

Let Σ, Ω be similarity types of structures, with corresponding languages $L_{\infty\varkappa}(\Sigma)$ etc. By a *construction from* Σ *to* Ω, we mean a map which takes some or all Σ-structures to Ω-structures. Word-constructions are a formalisation of the intuitive notion of *constructions by uniformly definable generators and relations.* We shall define word-constructions in section 1.3; sections 1.1 and 1.2 will cover some preliminaries from logic.

By a *presentation* (more precisely, an Ω-*presentation*) we mean an ordered triple $\langle\Omega', X, \Phi\rangle$ where

Ω' is a similarity type extending Ω;

X is a set of closed terms of $L(\Omega')$ the *generators*);

Φ is a set of atomic sentences of $L(\Omega')$ (the *relations*).

The presentation $\langle\Omega', X, \Phi\rangle$ defines an Ω-structure B = df $\langle\Omega', X, \Phi\rangle$, possibly empty, as follows. Let $\overline{X}$ be the closure of X under the function symbols of Ω; individual constants count as O-ary function symbols. Define a binary relation $\sim$ on $\overline{X}$ by

$$\sigma \sim \tau \text{ iff } \Phi \models \sigma = \tau \tag{1}$$

for each $\sigma, \tau \in \overline{X}$. Then $\sim$ is clearly an equivalence relation; let $\tau^{\sim}$ be the equivalence class of τ, and let $|B|$ be the set of all equivalence classes $\tau^{\sim}$ ($\tau \in \overline{X}$).

For each n-ary function symbol F of Ω, define F_B: $|B|^n \rightarrow |B|$ by

$$F_B(\tilde{\tau_1}, \ldots, \tilde{\tau_n}) = (F\tau_1 \ldots \tau_n)^{\sim}; \tag{2}$$

for each n-ary relation symbol R of Ω, define $R_B \subseteq |B|^n$ by

$$<\tilde{\tau_1}, \ldots, \tilde{\tau_n}> \in R_B \text{ iff } \Phi \models R\tau_1 \ldots \tau_n. \tag{3}$$

It is easy to verify that F_B and R_B are well-defined.

B is defined to be the Ω-structure with domain $|B|$, functions F_B and relations R_B. We say that B is the Ω-structure df $<\Omega', X, \Phi>$ *presented* by the presentation $<\Omega' X, \Phi>$.

LEMMA 1. *With the above definitions, if* $\varphi(v_1, \ldots, v_n)$ *is an atomic formula of* $L(\Omega)$, *and* $\tau_1, \ldots, \tau_n \in X$, *then*

$$B \models \varphi\, [\tilde{\tau_1}, \ldots, \tilde{\tau_n}] \textit{ iff } \Phi \models \varphi\, (\tau_1, \ldots, \tau_n). \tag{4}$$

Proof. We consider first the case where φ is an equation, and we use induction on the number of occurrences of function symbols in φ. If φ is "$v_1 = v_2$", then (4) is (1). If φ is "$v_1 = F(\sigma(v_2, \ldots, v_n))$", then by induction hypothesis we have

$$B \models (\sigma\, (v_2, \ldots, v_n) = v_{n+1})\, [\tilde{\tau_1}, \ldots, \tilde{\tau_n}, \sigma\, (\tau_2, \ldots, \tau_n)^{\sim}]$$
$$\text{iff } \Phi \models \sigma\, (\tau_2, \ldots, \tau_n) = \sigma\, (\tau_2, \ldots, \tau_n). \tag{5}$$

Since the right-hand side of (5) holds, so does the left. Therefore

$$B \models \varphi\, [\tilde{\tau_1}, \ldots, \tilde{\tau_n}]$$
$$\text{iff } B \models v_1 = F\,(v_{n+1}))\, [\tilde{\tau_1}, \ldots, \tilde{\tau_n}, \sigma(\tau_2, \ldots, \tau_n)^{\sim}]$$
$$\text{iff } \tilde{\tau_1} = F_B\, \sigma\, (\tau_2, \ldots, \tau_n)^{\sim}$$

$$\text{iff} \quad \tau_1^{\sim} = F(\sigma(\tau_2, \ldots, \tau_n))^{\sim} \qquad \text{by (2)}$$

$$\text{iff} \quad \Phi \models \varphi(\tau_1, \ldots, \tau_n) \qquad \text{by (1)}.$$

The remaining cases are similar.

If φ is of form $Rv_1 \ldots v_k$, then (4) is (3). Finally if φ has form $R\sigma_1 \ldots \sigma_k$ we consider the formulae $Rv_1 \ldots v_k$, $v_1 = \sigma(\vec{v'_1})$, $\ldots$, $v_k = \sigma(\vec{v'_k})$.

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The property of Lemma 1 characterises df $\langle\Omega', X, \Phi\rangle$ up to isomorphism. We shall need this fact in a strong form, which requires a broader setting, as follows.

By a *homomorphism* $f\colon A \to B$ of Ω-structures, we mean a map $f\colon |A| \to |B|$ which preserves atomic formulae from A to B. In later sections Ω may be a many-sorted similarity type, in which case we require homomorphisms to respect the sorts. If T is a theory in $L(\Omega)$, we write (Ω, T)-*Str* for the category whose objects are the Ω-structures which are models of T, and whose morphisms are the homomorphisms between these models. In particular Ω-*Str* is (Ω, O)-*Str*, the category of all Ω-structures; O-*Str* is *Set*, the category of sets. Note that we allow the empty structure throughout, provided Ω has no individual constants.

By a *strict universal Horn sentence* we mean a sentence of form $\forall v_1 \ldots \forall v_n [\varphi_1 \wedge \ldots \wedge \varphi_k \to \psi]$ where $\varphi_1, \ldots, \varphi_k, \psi$ are atomic. Putting $k = O$, this includes universally quantified equations. If T is a set of strict universal Horn sentences of $L(\Omega)$, then (Ω, T)-*Str* is called a *quasivariety*. A theorem of Mal'cev ([12] Theorem 3, p. 419) characterises the quasivarieties (Ω, T)-*Str* as the full subcategories of Ω-Str which are closed under isomorphism, substructures and reduced products (including the trivial product 1). In particular every quasivariety is left complete; by the adjoint functor theorem quasivarieties are right complete as well.

Let (Ω, T)-*Str* be a quasivariety. The Ω-presentation $\langle\Omega', X, \Phi\rangle$ defines an Ω-structure $B = \mathrm{df}_T\langle\Omega', X, \Phi\rangle$ by exactly the same definition as above, except that (1), (3) respectively are replaced by

$$\sigma \sim \tau \quad \text{iff} \quad \Phi, T \models \sigma = \tau \tag{1$_T$}$$

$$\langle\tau_1^{\sim}, \ldots, \tau_n^{\sim}\rangle \in R_B \quad \text{iff} \quad \Phi, T \models R\tau_1 \ldots \tau_n. \tag{3$_T$}$$

We define an $\langle\Omega', X, \Phi\rangle$-*structure* in the category (Ω, T)-*Str* to be a pair $(A, {}^*)$ where A is an object of (Ω, T)-*Str* and * is a surjective map $^*: \overline{X} \to |A|$ such that for every atomic formula $\varphi(v_1, \ldots, v_n)$ of $L(\Omega)$ and terms $\tau_1, \ldots, \tau_n \in \overline{X}$,

$$\Phi, T \models \varphi(\tau_1, \ldots, \tau_n) \Rightarrow A \models \varphi[\tau_1^*, \ldots, \tau_n^*]. \tag{6}$$

THEOREM 2 (Dyck's Theorem). *Let* (Ω, T)-*Str be a quasivariety,* $\langle\Omega', X, \Phi\rangle$ *an* Ω-*presentation,* $B = \mathrm{df}_T\langle\Omega', X, \Phi\rangle$ *and* $\sim: \overline{X} \to |B|$ *as above. Then*

a. $(B, \sim)$ *is an* $\langle\Omega', X, \Phi\rangle$-*structure in* (Ω, T)-*Str;*
b. *for every* $\langle\Omega', X, \Phi\rangle$-*structure* $(A, {}^*)$ *in* (Ω, T)-*Str, there is a unique homomorphism* $f: B \to A$ *such that* $^* = f\sim$; *f is surjective.*

Proof. a. We may assume that T is a set of strict universal Horn sentences. Let $\overline{\Phi}$ be the set of all atomic sentences ψ of $L(\Omega')$ such that $\Phi, T \models \psi$. Then $B = \mathrm{df}\langle\Omega', X, \overline{\Phi}\rangle$, so that $(B, \sim)$ satisfies (6) by Lemma 1. We must show also that $B \models T$. Let $\forall v_1 [\varphi(v_1) \to \psi(v_1)]$ be a sentence of T. Since $\sim$ is onto $|B|$, we can verify that B is a model of this sentence by noting that for each $\tau \in \overline{X}$,

$$B \models \varphi[\tau^{\sim}] \Rightarrow \Phi, T \models \varphi(\tau) \Rightarrow \Phi, T \models \psi(\tau) \Rightarrow B \models \psi[\tau^{\sim}] \tag{7}$$

by Lemma 1. A similar argument applies to all sentences of T.

b. $\sim$ is onto $|B|$, and $\sigma^{\sim} = \tau^{\sim}$ implies $\Phi, T \models \sigma = \tau$ by $(1)_T$, hence $\sigma^* = \tau^*$ by (6). Hence the condition $^* = f\sim$ defines a unique map f: $|B| \to |A|$. f is surjective since * is. We must show that f is a homomorphism. Suppose $F_B(\tau_1^{\sim}, \ldots, \tau_n^{\sim}) = \sigma^{\sim}$; then $\Phi, T \models F\tau_1 \ldots \tau_n = \sigma$ by (2) and $(1)_{T}$, so $F_A(\tau_1^*, \ldots, \tau_n^*) = \sigma^*$ by (6), and hence $F_A(f(\tau_1^{\sim}), \ldots, f(\tau_n^{\sim})) = f(\sigma^{\sim})$. The argument for relation symbols is similar.

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Dyck's Theorem characterises $(B, \sim)$ up to isomorphism as the initial $\langle\Omega', X, \Phi\rangle$-structure in (Ω, T)-*Str*. More important for us, Dyck's Theorem describes a canonical construction for right limits in the category (Ω, T)-*Str*, as follows.

EXAMPLE 3: *construction of right limits in quasivarieties.*

Let (Ω, T)-*Str* be a quasivariety, *D* a small category and F: $D \to (\Omega, T)$-*Str* a functor. (We identify diagrams with functors.)

Let Ω' be Ω with an added individual constant $c_{d,a}$ for each object d of *D* and each element $a \in |Fd|$. Let X be the set of these added constants. Let Φ be the union of the positive diagrams in $L(\Omega')$ of the structures Fd, together with all the sentences

$$c_{d,a} = c_{e,Fy(a)} \qquad (y : d \to e \text{ a morphism of } D).$$

Since (Ω, T)-*Str* is closed under substructures, Dyck's Theorem asserts that $(df_T\langle\Omega', X, \Phi\rangle, \sim) = \underrightarrow{\text{Lim}}\ F$. (We write $\underrightarrow{\text{Lim}}$ for limit cones, and lim for limit objects.) We shall use this in the proofs of Theorems 17 and 18 below.

In future we shall allow Ω to be many-sorted; in this case the set X of generators in an Ω-presentation must be replaced by a family $(X^s)_{s \text{ a sort of } \Omega}$. The results of this section then remain true, after some insignificant alterations.

1.2 *Infinitary quasivarieties*

Besides allowing the similarity types Ω to be many-sorted, we may also extend the results of section 1.1 by allowing Ω to contain function and relation symbols of infinite arity. We define the *length* of Ω to be the least cardinal $\varkappa$ such that every function or relation symbol of Ω has arity $< \varkappa$.

Let $\varkappa$ be a regular cardinal and Ω a similarity type of length $\leq \varkappa$. By a *$\varkappa$-strict universal Horn sentence* of $L(\Omega)$, we mean a sentence of form $\forall \{v : v \in I\} [\bigwedge \Phi \rightarrow \psi]$, where I is a set of $< \varkappa$ variables, Φ is a set of $< \varkappa$ atomic formulae of $L(\Omega)$, and ψ is an atomic formula of $L(\Omega)$. Categories of form (Ω, T)-*Str*, where T is a set of $\varkappa$-strict universal Horn sentences of $L(\Omega)$, will be called *$\varkappa$-quasivarieties*. The results of section 1.1 hold for $\varkappa$-quasivarieties, after the appropriate notational changes.

With $\varkappa$ and Ω as above, let Φ be a set of atomic sentences of $L(\Omega)$ and T a set of $\varkappa$-strict universal Horn sentences of $L(\Omega)$. Consider the infinitary natural deduction calculus $\mathcal{C}$ which has:

Axioms (i) $\sigma = \sigma$ where σ is a closed term of $L(\Omega)$

(ii) φ where φ is a sentence in Φ

Rules (iii)
$$\frac{\varphi(\sigma_\alpha)_{\mu < \alpha} \qquad \sigma_\alpha = \tau_\alpha \ (\alpha < \mu)}{\varphi(\tau_\alpha)_{\alpha < \mu}}$$

(iv)
$$\frac{\sigma = \tau}{\tau = \sigma}$$

(v)
$$\frac{\varphi_\alpha \ (\alpha < \mu)}{\psi}$$

where $\bigwedge_{\alpha < \mu} \varphi_\alpha \rightarrow \psi$ is an instance of a sentence $\in$ T

We say $\Phi, T \vdash \psi$ if there is a proof of ψ in $\mathcal{C}$.

THEOREM 4. *Let* $\varkappa, \Phi, T$ *be as above. Then for every atomic sentence* ψ *of* $L(\Omega)$, $\Phi, T \vdash \psi$ *iff* $\Phi, T \models \psi$.

Proof. Let X be the set of all individual constants of Ω. We may define $B' = df'_T < \Omega, X, \Phi >$ just as $df_T < \Omega, X, \Phi >$, but with $\vdash$ in place of $\models$ in $(1)_T$, $(3)_T$. Replacing semantics by syntax in the proof of Lemma 1, we can show that for every atomic formula $\varphi(\vec{v})$ of $L(\Omega)$ and sequence $\vec{\tau}$ of terms in X,

$$B' \models \varphi[\vec{\tau}^{\sim}] \text{ iff } \Phi, T \vdash \varphi(\vec{\tau}). \tag{8}$$

The syntactic version of Theorem 2(a) shows $B' \models T$. Hence if $\Phi, T \models \psi$, then $B' \models \psi$, so $\Phi, T \vdash \psi$ by (8). The converse is proved as usual, by induction on the length of proofs.

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COROLLARY 5 (Strong compactness: cf. Słomiński [17] IV (2.10)) *In the above situation, if* $\Phi, T \models \psi$ *then there are* $\Phi_0 \subseteq \Phi$ *and* $T_0 \subseteq T$, *both of cardinality* $< \varkappa$, *such that* $\Phi_0, T_0 \models \psi$.

Proof. Every proof in the calculus $\mathcal{C}$ has $< \varkappa$ nodes.

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1.3. *Word-constructions*

Let $\varkappa$ be a regular cardinal. Let Σ, Ω be similarity types of length $\leq \varkappa$, possibly many-sorted. A *$\varkappa$-word-construction* Γ from Σ to Ω is defined to be an ordered quadruple $< \Sigma, \Omega, (\Gamma^s)_{s \text{ a sort of } \Omega}, \Gamma^{At} >$ such that there is a similarity type Ω' of length $\leqq \varkappa$ which extends Ω, and

(1) for each sort s of Ω, Γ^s is a function whose domain is

a set of terms of $L(\Omega')$ with variables from $L(\Sigma)$; for each term $\tau \in \mathrm{dom}\,\Gamma^s$, Γ^s_τ is a formula of $L_{\infty\varkappa}(\Sigma)$ whose free variables are among those in τ;

(2) Γ^{At} is a function whose domain is a set of atomic formulae of $L(\Omega')$ with variables from $L(\Sigma)$; for each atomic formula $\varphi \in \mathrm{dom}\,\Gamma^{At}$, Γ^{At}_φ is a formula of $L_{\infty\varkappa}(\Sigma)$ whose free variables all occur in φ.

If Γ is a $\varkappa$-word-construction as above, and A is a Σ-structure, then we define the Ω-structure $\Gamma(A)$ to be $\mathrm{df}\langle\Omega'', (X^s)_{s \text{ a sort of } \Omega'}, \Phi\rangle$ where

$$\left.\begin{array}{l} \Omega'' \text{ is } \Omega' \text{ with the elements of } |A| \text{ added as new individual constants,} \\ X^s = \{\tau(\vec{a}) : \tau \in \mathrm{dom}\,\Gamma^s,\ A \models \Gamma^s_\tau[\vec{a}]\}, \\ \Phi = \{\varphi(\vec{a}) : \varphi \in \mathrm{dom}\,\Gamma^{At}_\varphi,\ A \models \Gamma^{At}_\varphi[\vec{a}]\}. \end{array}\right\} \quad (9)$$

Γ thus defines a construction $A \mapsto \Gamma(A)$ from Σ to Ω; for brevity we may also refer to this construction as Γ.

Word-construction means: $\varkappa$-word-construction for some regular cardinal $\varkappa$. Note that if $\varkappa, \lambda$ are regular cardinals and $\varkappa < \lambda$, then every $\varkappa$-word-construction is also a λ-word-construction.

EXAMPLE 6. Let F be the function which takes each integral domain R to its field of fractions F(R). We express F as an ω-word-construction as follows. Σ and Ω shall both be one-sorted similarity types, with function symbols $+$ and $\cdot$ of arity 2, and O of arity O. Ω' shall have the extra 2-ary function symbol $\langle,\rangle$. Labelling the one sort in Ω as sort O, we put

$$\Gamma^0_{\langle v_0, v_1\rangle} \equiv_{df} v_1 \neq O$$

$$\Gamma^{At}_{\langle v_0, v_1\rangle = \langle v_2, v_3\rangle} \equiv_{df} v_0 . v_3 = v_1 . v_2 \wedge v_1 \neq O \wedge v_3 \neq O$$

$$\Gamma^{At}_{\langle v_0, v_1\rangle + \langle v_2, v_3\rangle = \langle v_4, v_5\rangle} \equiv_{df} (v_0 . v_3 + v_2 . v_1) . v_5 = (v_1 . v_3) . v_4 \wedge v_1 \neq O \wedge v_3 \neq O \wedge v_5 \neq O$$

$$\Gamma^{At}_{\langle v_0, v_1\rangle . \langle v_2, v_3\rangle = \langle v_4, v_5\rangle} \equiv_{df} (v_0 . v_2) . v_5 = (v_1 . v_3) . v_4 \wedge v_1 \neq O \wedge v_3 \neq O \wedge v_5 \neq O$$

$$\Gamma^{At}_{\langle v_0, v_1\rangle = O} \equiv_{df} v_0 = O.$$

It is often better to construe the field of fractions construction as a map taking each integral domain R to the morphism

$$i : R \rightarrow F(R) \tag{10}$$

where i is the canonical embedding. This morphism can be construed in turn as a two-sorted structure: sort O carries F(R), sort 1 carries R, and i is a function from sort 1 to sort O. Γ is easily amended so that $\Gamma(R)$ is the morphism (10); it suffices to add to Ω' the function symbols $+^*$, $.^*$, O^* of sort 1, and the function symbol j from sort 1 to sort O, together with the formulae

$$\Gamma^1_{v_0} \equiv_{df} v_0 = v_0$$

$$\Gamma^{At}_{v_0 +^* v_1 = v_2} \equiv_{df} v_0 + v_1 = v_2$$

$$\Gamma^{At}_{v_0 .^* v_1 = v_2} \equiv_{df} v_0 . v_1 = v_2$$

$$\Gamma^{At}_{v_0 = O^*} \equiv_{df} v_0 = O$$

$$\Gamma^{At}_{j(v_0) = \langle v_1, v_2\rangle} \equiv_{df} v_0 . v_2 = v_1 \wedge v_2 \neq O.$$

Hodges [9] gives other worked examples of word-constructions, including the construction taking each ordered field to its real closure, and the construction taking each valued field to its henselisation.

1.4 *The associated functor*

Every word-construction Γ from Σ to Ω gives rise to functors, as follows.

If A is a subcategory of Σ-*Str*, we define A_Γ to be the subcategory of A whose objects are those of A, and whose morphisms are those morphisms $f: A \to B$ of A which preserve all the formulae Γ^s_τ and Γ^{At}_φ from A to B.

Let $f: A \to B$ be a morphism of A_Γ. Write Ω''_A, X^s_A, Φ_A for the items of (9) occurring in the definition of $\Gamma(A)$; likewise Ω''_B, X^s_B, Φ_B for $\Gamma(B)$. For each $\tau \in \mathrm{dom}\,\Gamma^s$ and $\vec{a}$ in $|A|$, we have

$$A \models \Gamma^s_\tau[\vec{a}] \;\Rightarrow\; B \models \Gamma^s_\tau[f\vec{a}]$$

so that if $\tau(\vec{a}) \in X^s_A$, then $\tau(f\vec{a}) \in X^s_B$. Thus f induces a map $\bar{f}: X^s_A \to X^s_B$ viz. $\bar{f}(\tau(\vec{a})) = \tau(f\vec{a})$ for each τ, $\vec{a}$ as above. Similarly f induces a map $\bar{f}: \Phi_A \to \Phi_B$, viz. $\bar{f}(\varphi(\vec{a})) = \varphi(f\vec{a})$ for each $\varphi \in \mathrm{dom}\,\Gamma^{At}$ and $\vec{a}$ in $|A|$. If $\sigma, \tau \in X^s_A$ and $\sigma \sim_A \tau$,

then $\Phi_A \models \sigma = \tau$; hence $\Phi_B \models \bar{f}(\sigma) = \bar{f}(\tau)$ and so $\bar{f}(\sigma) \sim_B \bar{f}(\tau)$.
Define $\Gamma(f) : |\Gamma(A)| \to |\Gamma(B)|$ by

$$\Gamma(f)\,(\tau^{\sim A}) = (\bar{f}\tau)^{\sim B}.$$

By Lemma 1, $\Gamma(f)$ is a homomorphism, $\Gamma(f) : \Gamma(A) \to \Gamma(B)$.

If f: $A \to B$ and g: $B \to C$ are morphisms of A_Γ, then the composed map gf: $A \to C$ is also a morphism of A_Γ, and we have

$$\Gamma(gf)\,(\tau^{\sim A}) = (\overline{gf}\tau)^{\sim C} = (\bar{g}(\bar{f}\tau))^{\sim C}$$
$$= \Gamma(g)\,(\bar{f}\tau)^{\sim B} = \Gamma(g)\,.\,\Gamma(f)\tau^{\sim A}$$

Also if 1_A is the identity morphism, $1_A : A \to A$, then clearly $\Gamma(1_A) = 1_{\Gamma(A)}$.

Hence the maps $A \mapsto \Gamma(A)$, $f \mapsto \Gamma(f)$ form a functor from Σ-Str_Γ to Ω-Str; we call this the *associated functor* of Γ. When there is no danger of confusion, we shall call this functor itself Γ. The functor Γ: $A_\Gamma \to \Omega$-Str defined above is simply the restriction to A_Γ of the associated functor.

The associated functor yields a neat device for showing that certain constructions cannot be expressed as word-constructions.

THEOREM 7. Let Γ be a word-construction from Σ to Ω, such that for each Σ-structure A, $\Gamma(A)$ contains a canonical copy of A. Then the associated functor $\Gamma : \Sigma$-$Str_\Gamma \to \Omega$-Str is faithful.

Proof. Suppose f: $A \to B$ is a morphism of Σ-Str_Γ. By canonicality we have a commutative diagram

$$\begin{array}{ccc} |A| & \xrightarrow{\quad f \quad} & |B| \\ \eta_A \Big\downarrow & & \Big\downarrow \eta_B \\ |\Gamma(A)| & \xrightarrow[\Gamma(f)]{} & |\Gamma(B)| \end{array}$$

where η_A, η_B are injections. Hence f is recoverable from $\Gamma(f)$ as $\eta_B^{-1} . \Gamma(f) . \eta_A$

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Automorphisms preserve all formulae; hence if Γ is as in Theorem 7 and A is a Σ-structure, then the associated functor Γ embeds the automorphism group of A in that of $\Gamma(A)$.

EXAMPLE 8. There is no word-construction Γ such that for each field A, $\Gamma(A)$ is the embedding $e_A: A \to \overline{A}$ of A into its algebraic closure $\overline{A}$. For suppose otherwise, and let A be the field $\mathbb{Q}(X, Y, Z)$, where X, Y, Z are independent transcendentals. A has an automorphism of order 3; by Theorem 7, so has $\Gamma(A)$ and hence so has $\overline{A}$. But a well-known theorem of Artin and Schreier says that no algebraically closed field can have an automorphism of finite order > 2.

A similar but slightly more careful argument shows that there is no word-construction Γ such that for each field A, $\Gamma(A)$ is a field isomorphic to the algebraic closure of A.

How is A_Γ related to A?

EXAMPLE 9. There is no functor sending each group G to its centre Z(G). Nevertheless there is such an ω-word-construction Γ; the generator part of Γ is

$\Gamma^0_{v_0} \equiv_{df} \forall v_1\ v_0 . v_1 = v_1 . v_0.$

Gp_Γ has as morphisms the group homomorphisms f: $G \rightarrow H$ such that $f(Z(G)) \subseteq Z(H)$.

We call a formula φ of $L_{\infty\varkappa}(\Sigma)$ *positive existential* ($\exists^+_1$ for short) if φ is in the smallest set of formulae of $L_{\infty\varkappa}(\Sigma)$ containing the atomic formulae and closed under conjunction, disjunction and existential quantification; φ is called *existential* ($\exists_1$) if it lies in the smallest set containing the atomic and negated atomic formulae and closed under conjunction, disjunction and existential quantification.

The word-construction Γ is said to be $\exists^+_1$ (resp. $\exists_1$) if all the formulae Γ^0_τ, Γ^{At}_φ are $\exists^+_1$ (resp. $\exists_1$) formulae.

If A is a subcategory of Σ-*Str*, then we write A_e for the subcategory of A whose objects are those of A and whose morphisms are those morphisms of A which are embeddings.

THEOREM 10. *Let Γ be a word-construction from Σ to Ω, and A a subcategory of Σ-Str. Then:*

a. *if Γ is $\exists^+_1$, then $A_\Gamma = A$;*

b. *if Γ is $\exists_1$, then $(A_e)_\Gamma = A_e$*

Proof. Homomorphisms preserve positive existential formulae, while embeddings preserve existential formulae.

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What is perhaps more striking is that a partial converse of Theorem 10 is also true, up to natural isomorphism of functors, for a wide range of A including Σ-*Str* itself. This follows at once from the Normal Form theorem to be proved below (Theorem 22).

2: *Properties of word-constructions*

2.1 *Reduction of definable properties*

The following theorem describes the central property of word-constructions.

THEOREM 11 (Uniform reduction theorem). *Let $\varkappa$ be a regular cardinal and Γ a $\varkappa$-word-construction from Σ to Ω. Then for every sentence φ of $L_{\infty\varkappa}(\Omega)$ there is a sentence φ^{Γ} of $L_{\infty\varkappa}(\Sigma)$ such that for every Σ-structure A,*

$$\Gamma(A) \models \varphi \quad \textit{iff} \quad A \models \varphi^{\Gamma} .$$

Proof. Let Ω' be as in section 1.3. For simplicity assume Σ, Ω, Ω' are one-sorted, with the single sort O. Let Tm be the smallest set of terms of $L(\Omega')$ which contains every $\tau \in \mathrm{dom}\ \Gamma^0$ and is closed under the function symbols of Ω and under change of variables from $L_{\infty\varkappa}(\Sigma)$. $L_{\infty\varkappa}(\Sigma)$ is assumed to have variables v_α $(\alpha < \varkappa)$; we introduce new double-indexed variables v^{α}_{β} during the proof. If we write $A \models \psi[\vec{a}]$, we imply that any free occurrence of v^{α}_{β} in ψ is matched by a term a^{α}_{β} of the sequence $\vec{a}$. We write $FV(\tau)$, $FV(\varphi)$ for the set of variables occurring free in τ, φ.

For each formula φ of $L_{\infty\varkappa}(\Omega)$ and each map $w\colon FV(\varphi) \rightarrow Tm$ we shall define a formula φ^{Γ}_{w} of $L_{\infty\varkappa}(\Sigma)$ so that

(1) every free variable of φ^{Γ}_{w} is of form v^{α}_{β} where $v_{\alpha} \in FV(\varphi)$ and $v_{\beta} \in FV(w(v_{\alpha}))$;

(2) for each Σ-structure A and each appropriately indexed $\vec{a}$ in $|A|$, define $\vec{b}$ if possible as a sequence of elements of $|\Gamma(A)|$ whose α-th term ($v_{\alpha} \in FV(\varphi)$) is $\tau(a^{\alpha}_{\beta_1}, \ldots)^{\sim}$, where $w(v_{\alpha})$ is $\tau(v_{\beta_1}, \ldots)$; then

$$A \models \varphi^{\Gamma}_{w}[\vec{a}] \text{ iff: } \vec{b} \text{ is well-defined and } \Gamma(A) \models \varphi[\vec{b}].$$

The definition is by induction on the complexity of φ, simultaneously for all w: $FV(\varphi) \to Tm$.

First, let $Ex_{\Gamma}(w)$ be the formula

$$\bigwedge_{v_{\alpha} \in \text{dom } w} \bigvee_{\substack{\sigma \text{ term of } L(\Omega), \\ w(v_{\alpha}) = \sigma(\tau_{\gamma}(v_{\gamma}))_{\gamma}, \\ \tau_{\gamma} \in \text{dom } \Gamma^0 \text{ for each } \gamma}} \bigwedge_{\gamma} \Gamma^0_{\tau_{\gamma}}(\vec{v}^{\alpha}_{\gamma})$$

where $\vec{v}^{\alpha}_{\gamma}$ is the sequence of variables got from the sequence $\vec{v}_{\gamma}$ by replacing each v_{δ} by the double-indexed variable v^{α}_{δ}. Then

$A \models Ex_\Gamma(w)[\vec{a}]$ iff for each $v_\alpha \in \mathrm{dom}\ w$, if $w(v_\alpha)$ is a term

$$\tau(v_{\beta_1}, \ldots), \text{ then } \tau(a^\alpha_{\beta_1}, \ldots)^\sim \in |\Gamma(A)|$$

$$\text{iff } \vec{b} \text{ is well-defined.}$$

Case a: φ atomic. We write φ_w for the formula derived from φ by replacing each free occurrence of v_α in φ by an occurrence of $w(v_\alpha)$ in which every occurrence of v_β is replaced by the double-indexed variable v^α_β. We write Fm for the set of all formulae got from formulae $\in \mathrm{dom}\ \Gamma^{At}$ by substituting arbitrary double-indexed variables for the free variables.

We put

$$\varphi^\Gamma_w \equiv_{df} Ex_\Gamma(w) \wedge$$

$$\bigvee_{\substack{Fm \supseteq \{\psi_\alpha(\vec{u}_\alpha) : \alpha < \delta\} \models \varphi_w, \\ \delta < \varkappa, \\ \psi_\alpha \in \mathrm{dom}\ \Gamma^{At} \text{ for all } \alpha < \delta}} \exists_{\vec{u}} \bigwedge_{\alpha < \delta} \Gamma^{At}_{\psi_\alpha}(\vec{u}_\alpha)$$

where $\Gamma^{At}_{\psi_\alpha}(\vec{u}_\alpha)$ is $\Gamma^{At}_{\psi_\alpha}$ with the variables $\vec{u}_\alpha$ substituted for the corresponding variables of ψ_α and $\vec{u}$ consists of all the free variables of the $\Gamma^{At}_{\psi_\alpha}(\vec{u}_\alpha)$ which are not free in φ_w. Note that we can restrict to $\delta < \varkappa$ by Corollary 5.

Case b: φ is $\neg\psi$. We put

$$\varphi^{\Gamma}_{w} \equiv_{df} \mathrm{Ex}_{\Gamma}(w) \wedge \neg (\psi^{\Gamma}_{w}).$$

Case c: φ *is* $\bigwedge \Phi$. We put

$$\varphi^{\Gamma}_{w} \equiv_{df} \bigwedge_{\psi \in \Phi} \psi^{\Gamma}_{w} .$$

Case d: ψ *is* $\exists \{v_\alpha : v_\alpha \in I\} \psi$. We put

$$\varphi^{\Gamma}_{w} \equiv_{df} \bigvee_{x:\, I \to Tm} \exists \vec{u}_x (\psi^{\Gamma}_{w \cup x})$$

where $\vec{u}_x$ consists of the variables v^{α}_{β} free in $\psi^{\Gamma}_{w \cup x}$ with $v_\alpha \in I$.

This completes the definition of φ^{Γ}_{w} as required. The theorem follows, when φ is a sentence, by taking φ^{Γ} to be φ^{Γ}_{0}.

Note that φ^{Γ}_{w} in the proof above has very much the same structure as φ; in particular φ^{Γ}_{w} is always a formula of $L_{\infty\varkappa}(\Sigma)$. We infer that if Γ is a $\varkappa$-word-construction from Σ to Ω, and A, A' are Σ-structures such that $A \equiv_{\infty\varkappa} A'$, then $\Gamma(A) \equiv_{\infty\varkappa} \Gamma(A')$. We shall sharpen this in section 2.4, by calculating a bound on the complexity of φ^{Γ}_{w} in terms of that of φ.

2.2 *Effectivity*

Jensen and Karp [10] introduced Prim functions, partly as a tool for studying the syntax and semantics of infinitary languages. These functions are exactly what we need for classifying the maps involved in word-constructions. When we say that a map $b \mapsto f(b)$ is Prim $(\vec{X})$, we mean that there is a Prim $(\vec{X})$ function F such that $F(b) = f(b)$ for all relevant b; we extend [10] by allowing $\vec{X}$ to contain O-ary functions.

A relation is said to be Prim $(\vec{X})$ when its characteristic function is. Jensen and Karp [10] list various Prim functions and relations; we shall need a few more, and we leave most of the checking to the reader. The functions and relations in the following lemma are by no means the only ones we shall use, but they are typical.

LEMMA 12. *Let* $\varkappa$ *be a regular cardinal, and* Ω *a similarity type of length* $\leqq \varkappa$. *Define:*

$R_0(A, \varphi, \vec{a}) \equiv_{df}$ *A is an* Ω*-structure,* φ *a formula of* $L_{\infty\varkappa}(\Omega)$, $\vec{a}$ *a valuation for* φ *in* A, *and* $A \models \varphi(\vec{a})$.

$F_1(X) \equiv_{df}$ *the closure of the set* X *of terms under the function symbols of* Ω.

$F_2(\Phi, X) \equiv_{df}$ *the set of all atomic sentences* $r(\vec{\tau})$ *where* r *is equality or a relation symbol occurring in the formulae* Φ *of* $L_{\infty\varkappa}(\Omega)$, *and* $\vec{\tau}$ *is a sequence of terms from* X.

$R_3(\Phi, T, \psi) \equiv_{df}$ $\Phi \cup \{\psi\}$ *is a set of atomic sentences of* $L(\Omega)$, T *is a* $\varkappa$*-strict universal Horn theory in* $L_{\infty\varkappa}(\Omega)$, *and* $\Phi, T \models \psi$.

Then R_0, F_1, F_2, R_3 *are* Prim $(\varkappa, \Omega, \mathscr{P}_{<\varkappa})$.

Proof. R_0 is a standard exercise. For F_1 we define

$$G(f, y) = \{\text{terms } f(\vec{\tau}) : \vec{\tau} \text{ a sequence of terms } \in y\},$$
$$H(x, \alpha) = x \cup \bigcup \{G(f, H(x, \beta)) : f \text{ a function symbol of } \Omega, \beta \in \alpha\},$$
$$F_1(X) = H(X, \varkappa).$$

F_2 follows readily. For R_3 we use the fact (= Theorem 4) that $\Phi, T \models \psi$ iff ψ has a proof P from Φ, T in the calculus ℓ. P is then a partially ordered set of atomic sentences from the set $z = F_2(\Phi \cup T \cup \{\psi\}, \{\tau \in TC(\Phi \cup T \cup \{\psi\}) : \tau \text{ is a term of } L_{\infty\varkappa}(\Omega)\})$, indexed, say, by ordinals $< \varkappa$. We therefore have

$\Phi, T \models \psi$ iff $(\exists P \in \mathscr{P}_{<\varkappa}((z \times \varkappa) \times (z \times \varkappa)))$ [P is a proof of $\Phi, T \vdash \psi$].

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THEOREM 13 (Effectivity theorem). *Let $\varkappa$ be a regular cardinal, and Γ a $\varkappa$-word-construction from Σ to Ω. Then both maps* $A \mapsto \Gamma(A)$ *and* $\varphi \mapsto \varphi^{\Gamma}$ *are* Prim $(\varkappa, \Gamma, \mathscr{P}_{<\varkappa})$.

Proof. This is a matter of coding up the definitions of sections 1.1, 1.3 and 2.1; we sketch a possible route. First, the map $\varphi \mapsto (\varphi_w^{\Gamma})_w$ of Theorem 11 is defined by a primitive recursion on the complexity of φ, and (using Lemma 12) all the clauses are Prim $(\varkappa, \Sigma, \Omega, \mathscr{P}_{<\varkappa})$. It is perhaps convenient to translate the double-indexed variables into single-indexed ones by means of a Prim pairing function on ordinals.

It follows at once that the map $\varphi \mapsto \varphi^{\Gamma}$ on sentences is Prim $(\varkappa, \Gamma, \mathscr{P}_{<\varkappa})$. We turn to the map $A \mapsto \Gamma(A)$.

Let A be a Σ-structure. Then for each formula $\Gamma^s_\imath$ of Γ, we have

$$\{\vec{a} \in |A| : A \models \Gamma^s_\tau[\vec{a}]\} =$$

$$\{\vec{a} \in \mathcal{P}_{<\varkappa}(FV(\Gamma^s_\tau) \times |A|) : R_0(A, \Gamma^s_\tau, \vec{a})\},$$

where R_0 is as in Lemma 12. Hence X^s in (9) is a Prim ($\varkappa$, Γ, $\mathcal{P}_{<\varkappa}$) function of A. Ω'' in (9) is a Prim (Γ) function of A. To construct $\Gamma(A)$ from X^s and Ω'', it suffices to determine the set

$$\Psi = \{<\psi, \vec{\tau}> : \psi(\vec{v}) \text{ atomic}, \ \vec{\tau} \text{ from } X^s, \ \Gamma(A) \models \psi[\vec{\tau}]\}.$$

(Cf. Lemma 1.) Given $\psi, \vec{\tau}$, we can define the set of functions $w: FV(\psi) \to Tm$ and $\vec{a} \in |A|$ (for each $v_\alpha \in FV(\psi)$) such that for each α, τ_α is the closed term $(w(v_\alpha))(\vec{a}_\alpha)$. Then for each such w,

$$\Gamma(A) \models \psi[\vec{\tau}^{\sim}] \text{ iff } A \models \psi^\Gamma_w[(\vec{a}_\alpha)_\alpha] \tag{11}$$

by the proof of Theorem 11. Now the set of such ψ^Γ_w is a Prim ($\varkappa$, Γ, $\mathcal{P}_{<\varkappa}$) function of $\psi, \vec{\tau}$, A, and the right-hand side of (11) is independent of the choice of w, $\vec{a}_\alpha$. Hence the right-hand side of (11) is a Prim ($\varkappa$, Γ, $\mathcal{P}_{<\varkappa}$) relation in ψ, $\vec{\tau}$, A. It follows that Ψ is a Prim ($\varkappa$, Γ, $\mathcal{P}_{<\varkappa}$) function of A as required.

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Theorems 11 and 13 combine to yield the Theorem of [8].

Correction to [8]. In the Theorem of [8] I had "Prim $(\omega, \Gamma, \mathcal{P}_{<\varkappa})$" in place of "Prim $(\varkappa, \Gamma, \mathcal{P}_{<\varkappa})$". Since I have been unable to reconstruct the device which led to the stronger result, I withdraw to the weaker one. No applications mentioned in [8] are disrupted.

2.3 *Preservation of equivalence*

Theorems 11 and 13 and their proofs yield a preservation theorem for word-constructions. To aid comparison with Feferman's local functors [4], we cast this in a strong form which mentions quantifier-rank.

The quantifier-rank $qr(\varphi)$ of a formula φ of $L_{\infty\varkappa}$ is defined as follows, by induction on the complexity of φ:

$$qr(\varphi) = 0 \quad \text{when } \varphi \text{ is atomic}$$
$$qr(\neg\varphi) = qr(\varphi)$$
$$qr(\bigwedge\Phi) = \sup\{qr(\varphi) : \varphi \in \Phi\}$$
(and similarly for disjunction)
$$qr(\exists\{v : v \in I\}\varphi) = qr(\varphi) + 1 \quad \text{(and similarly for } \forall\text{)}.$$

We write $L^{\alpha}_{\lambda\varkappa}$ for the fragment of $L_{\lambda\varkappa}$ got by restricting to formulae of quantifier-rank $\leqq \alpha$.

The quantifier-rank $qr(\Gamma)$ of the word-construction Γ is defined to be

$$\sup\{qr(\Gamma^{s}_{\tau}) : \Gamma^{s}_{\tau} \text{ in } \Gamma\} \cup (\sup\{qr(\Gamma^{At}_{\varphi}) : \Gamma^{At}_{\varphi} \text{ in } \Gamma\} + 1).$$

THEOREM 14 (Preservation theorem). *Let $\varkappa$ be a regular cardinal, Γ a $\varkappa$-word-construction from Σ to Ω, M a transitive* Prim $(\varkappa, \Gamma, \mathcal{P}_{<\varkappa})$*-closed set, and α an ordinal such that* $qr(\Gamma) + \alpha = \alpha$. *Put* $L(\Sigma) = L^{\alpha}_{\infty\varkappa}(\Sigma) \cap M$ *and*

$L(\Omega) = L^{\alpha}_{\infty\varkappa}(\Omega) \cap M$. *Then for all* Σ*-structures* A, A',

a. *if* $A \equiv_{L(\Sigma)} A'$ *then* $\Gamma(A) \equiv_{L(\Omega)} \Gamma(A')$;

b. *if* $e: A \to A'$ *is an* $L(\Sigma)$*-elementary embedding, then* $\Gamma(e) : \Gamma(A) \to \Gamma(A')$ *is* $L(\Omega)$*-elementary.*

Proof. We refer to the proof of Theorem 11. Let φ be a formula of $L_{\infty\varkappa}(\Omega)$; then by induction on the complexity of φ, $qr(\varphi^{\Gamma}_{w}) \leqq qr(\Gamma) + qr(\varphi)$. It follows that if φ is a formula of $L^{\alpha}_{\infty\varkappa}(\Omega)$, then φ^{Γ}_{w} is a formula of $L^{\alpha}_{\infty\varkappa}(\Sigma)$. By Theorem 13, M is closed under the map $\varphi \mapsto (\varphi_{w})^{\Gamma}_{w}$; hence if φ is a formula of $L(\Omega)$, then φ^{Γ}_{w} is a formula of $L(\Sigma)$. The theorem follows at once.

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For example, if $\varkappa = \omega$ then $\mathcal{P}_{<\varkappa}$ is a Prim $(\varkappa)$ function; if furthermore Γ is recursively definable, then M can be any Prim-closed set containing ω. We then get the languages $L_{\lambda\omega}$ $(\lambda > \omega)$ by taking M to be the set $H(\lambda)$ of sets hereditarily of cardinality $< \lambda$; likewise by taking M to be any countable admissible set containing ω, we get the countable admissible languages of Barwise.

The connection between the effectivity of $A \mapsto \Gamma(A)$ and the preservation result seems to be more than accidental: Nadel ([14] Theorem 2) shows that a construction which is Σ_1-definable from a hereditarily countable parameter and preserves isomorphism also preserves $\equiv_{\infty\omega}$.

EXAMPLE 15. The following word-construction consists of atomic formulae, and fails to preserve equivalence for one-quantifier formulae:

$$\Gamma^0_{v_0} \equiv v_0 = v_0$$

$$\Gamma^{At}_{Sv_0 = v_1} \equiv Sv_0 = v_1$$

$$\Gamma^{At}_{Sv_0 = Sv_1} \equiv v_0 = v_0$$

Let A, A' be respectively the natural numbers and the integers, with S interpreted as the successor function. By the back-and-forth criterion, A is $L^1_{\infty\omega}$-equivalent to A'. But $\Gamma(A)$ has two elements, while $\Gamma(A')$ has only one, so that only $\Gamma(A')$ satisfies $\forall v_0 v_1\ v_0 = v_1$.

2.4 *Closure under composition*

THEOREM 16. *Let $\varkappa$, be a regular cardinal, Γ a $\varkappa$-word-construction from Σ to Ω, and Δ a $\varkappa$-word-construction from Ω to Ξ. Then there is a composite $\varkappa$-word-construction $\Delta . \Gamma$ from Σ to Ξ, such that for each Σ-structure A, $(\Delta . \Gamma)(A) \cong \Delta(\Gamma(A))$ naturally. The composition map $(\Delta, \Gamma) \mapsto \Delta . \Gamma$ is* Prim $(\varkappa, \mathcal{P}_{<\varkappa})$.

Proof. Let Ω' (resp. Ξ') be the similarity type got from Ω (resp. Ξ) by adding the extra symbols of Γ (resp. Δ). We assume for simplicity that all these similarity types are one-sorted, and that Ω' and Ξ' have no symbols in common. Let Ξ^* be the union of Ω' and Ξ'. Let Tm_Γ be the closure of dom Γ^0 under the function symbols of Ω and under change of variables.

For each term τ of $L(\Xi^*)$ and each atomic formula φ of $L(\Xi^*)$, we put

$$(\Delta\,.\,\Gamma)^{O}_{\tau} \equiv_{df} \bigvee_{\substack{\sigma \in \operatorname{dom} \Delta^{O}, \\ w:\, FV(\sigma) \to Tm_{\Gamma}, \\ \tau \text{ is } \sigma(w(v_\alpha))_\alpha}} (\Delta^{O}_{\sigma})_{w}(\overset{\Gamma\,\to}{u}),$$

$$(\Delta\,.\,\Gamma)^{At}_{\varphi} \equiv_{df} \bigvee_{\substack{\psi \in \operatorname{dom} \Delta^{At}, \\ w:\, FV(\psi) \to Tm_{\Gamma}, \\ \varphi \text{ is } \psi(w(v_\alpha))_\alpha}} (\Delta^{At}_{\psi})_{w}(\overset{\Gamma\,\to}{u}),$$

where $\vec{u}$ is the sequence of variables whose term u^{α}_{β} is v_{β}.

(We may omit those formulae where the disjunction is empty).

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2.5 *Limits as word-constructions*

Let D be a small category and Ω a similarity type; for simplicity we assume Ω is one-sorted. We define Ω^D to be the following many-sorted similarity type. Ω^D has a sort Ω^d for each object d of D; Ω^d is a copy of Ω. For each morphism x: $d \to e$ of D, Ω^D has a 1-ary function symbol M_x from sort Ω^d to sort Ω^e.

Let F: $D \to \Omega\text{-}Str$ be a functor. We can represent F as a single structure A, as follows. For each object d of D, the Ω^d-th sort of A is Fd; for each morphism x: $d \to e$ of D, M_x is interpreted in A as Fx: $|Fd| \to |Fe|$. Then A is an Ω^D-structure which encodes F. Sometimes (as in Theorem 17 below) it is technically necessary to make the sorts of a many-sorted structure pairwise disjoint. In this case Fd in A must be replaced by an isomorphic copy of Fd, and A represents F only up to natural isomorphism of functors; we call such a structure A a *disjoint* representation of F.

THEOREM 17. *Let D be a small category, $\varkappa$ a regular cardinal and Ω a similarity type of length $\leqq \varkappa$. Then*

a. *there is a $\varkappa$-word-construction $\varinjlim_D$ from Ω^D to Ω such that if the Ω^D-structure A is a disjoint representation of of F: $D \to \Omega$-Str, then $\varinjlim_D(A) = \varinjlim(F)$;*

b. *the same with $\varprojlim$ for $\varinjlim$, assuming D has $< \varkappa$ objects.*

Proof. a. Invoking Example 3 above, we take $\Gamma = \varinjlim_D$ to be as follows:

$\Gamma^0_v \equiv_{df} v = v$ (v a variable of sort Ω^d, d an object of D)

$\Gamma^{At}_\varphi \equiv_{df} \varphi$ (φ an atomic formula in sort Ω^d, d an object of D)

$\Gamma^{At}_{v = w} \equiv_{df} M_x(v) = w$ ($x : d \to e$ a morphism of D, v a variable of sort Ω^d, w a variable of sort Ω^e)

Then $(\Gamma(A), \sim)$ is the required right limit cone.

b. We use the product-equaliser construction of left limits. Enumerate the objects of D as d_α $(\alpha < \mu)$ for some $\mu < \varkappa$, and introduce the μ-ary function symbol ζ, whose α-th slot is of sort Ω^{d_α}. Then define $\Gamma = \varprojlim_D$ as follows:

$$\Gamma^0_{\zeta(\vec{v})} \equiv_{df} \bigwedge_{\substack{x : d_\alpha \to d_\beta \text{ a} \\ \text{morphism of } D}} M_x(v_\alpha) = v_\beta$$

$$\Gamma^{At}_{\varphi(\zeta_0(\vec{v}_0), \zeta_1(\vec{v}_1), \ldots)} \equiv_{df} \bigwedge_{\alpha < \mu} \varphi(v_0^\alpha, v_1^\alpha, \ldots)$$

for every atomic formula φ of $L(\Omega)$, where $\vec{v}_i$ is $(v_i^0, v_i^1, \ldots)$. The required left limit cone is $(\Gamma(A),\eta)$ where $\eta_{d_\alpha} : \Gamma(A) \to$ (d_α-th sort of A) is the map $\zeta(\vec{a})^\sim \mapsto a_\alpha$.

[]

If F, F' are functors from D to Ω-*Str*, with disjoint representations A, A' respectively, and $\xi: F \to F'$ is a natural transformation, then ξ induces a homomorphism $\overline{\xi}: A \to A'$. The word-constructions $\underset{\rightarrow}{\lim}_D$ and $\underset{\leftarrow}{\lim}_D$ defined in the proof above are both $\exists_1^+$, so by Theorem 10, $\underset{\rightarrow}{\lim}_D(\overline{\xi}): \underset{\rightarrow}{\lim}_D(A) \to \underset{\rightarrow}{\lim}_D(A')$ and $\underset{\leftarrow}{\lim}_D(\overline{\xi}): \underset{\leftarrow}{\lim}_D(A) \to \underset{\leftarrow}{\lim}_D(A')$ are both defined. They are of course the morphisms induced by the limit property.

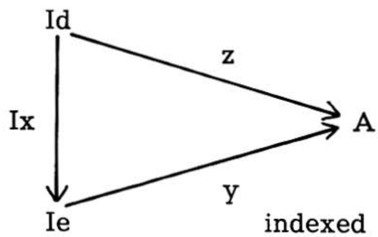
3: *Functors and word-constructions*

3.1 *Left Kan extensions*

We aim to characterise the associated functors of $\exists_1^+$ word-constructions as those functors which preserve filtered right limits. This is possible because Kan extensions (which, to quote Mac Lane [11] p. 244, 'subsume all the other fundamental concepts of category theory') can under certain conditions be encoded as word-constructions.

We begin by reviewing comma categories and left Kan extensions; for more details see Chapter X of Mac Lane [11] or Chapter 17 of Schubert [16].

Let A be any category, D a small category and I a functor from D to A. For each object A of A, the comma category I/A is defined to have as objects the morphisms z of A of form z: $Id \to A$ (d an object of D) indexed by d. The morphisms of I/A, say from z: $Id \to A$ to y: $Ie \to A$, are the commutative diagrams of form



(x: $d \to e$ in D) (12)

indexed by x

There is a projection functor Q^I_A: $I/A \to D$, which takes z: $Id \to A$ to d and (12) to x. If D is a subcategory of A and I is the inclusion, we write D/A for I/A and Q^D_A for Q^I_A.

If I: $D \to A$ and G: $D \to B$ are functors, a *left Kan extension* of G along I is a functor F: $A \to B$ together with a natural transformation η: $G \to FI$ such that for every functor H: $A \to B$, the map

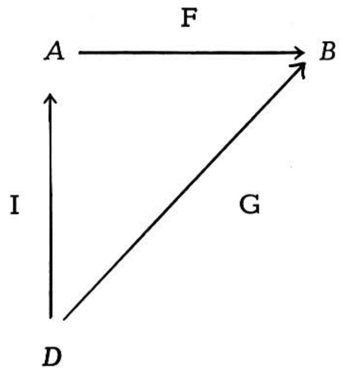
$$B^A(F, H) \to B^D(G, HI), \ \zeta \mapsto \zeta I.\eta$$

is a bijection.

From the definition it follows at once that if F_0, F_1 are left Kan extensions of G along I, then F_0 is naturally isomorphic to F_1. We shall often consider the case when D is a subcategory of A and I is the inclusion; in this case we talk of the left Kan extension of G along D, and we write $F|D$ for FI.

A sufficient condition for the left Kan extension to exist is as follows. For each object A of A, GQ^I_A is a functor from I/A to B. Suppose $\varinjlim GQ^I_A$ exists for each A. Then G has a left Kan extension F along I, such that $FA = \varinjlim GQ^I_A$ for all A. If x: $A \to B$ is a morphism of A, then x induces Fx: $FA \to FB$ functorially through the right limit cones. Mac Lane [11] refers to left Kan extensions constructed in this way as *pointwise* left Kan extensions. Clearly they exist when B is right complete, for instance if B is a quasivariety.

THEOREM 18. a. *In the diagram of functors*



assume Σ, Ω *are of length* $\leqq \varkappa$ ($\varkappa$ regular), A *is a full subcategory of* Σ-*Str*, B *is a* $\varkappa$-*quasivariety* $=$ (Ω, T)-*Str*, D *is small, every* Σ-*structure* Id (d in D) *has a set of* $< \varkappa$ *generators, and* F *is a left Kan extension of* G *along* I. *Then* F *is naturally isomorphic to the restriction to* A *of the associated functor of an* $\exists_1^+$ $\varkappa$-*word-construction* Γ *from* Σ *to* Ω; Γ *is a* Prim $(\varkappa, \mathscr{P}_{<\varkappa})$ *function of* I, G.

b. *The same, with* Σ-Str_e *for* Σ-*Str and* $\exists_1$ *for* $\exists_1^+$.

Proof. a. It suffices to define a $\varkappa$-word-construction Γ whose associated functor, restricted to A, is a left Kan extension of G along I, since the universal property of Kan extensions then guarantees that Γ on A is naturally isomorphic to F. We follow Dyck's Theorem, constructing the pointwise Kan extension.

Assume for simplicity that Σ and Ω are one-sorted, with the single sort O. For each object e of D and each map f: $\alpha \rightarrow |\mathrm{Ie}|$ with $\alpha < \varkappa$ and im f a set of generators of Ie as Σ-structure, and each $\mathrm{b} \in |\mathrm{Ge}|$, we introduce the new function symbol $\zeta_{f,b}$ of arity α; Ω' is Ω with the $\zeta^e_{f,b}$ added. The intention of the definitions below is that $\zeta^e_{f,b}(\vec{a})$ will be a generator of ΓA iff there is a homomorphism h: $Ie \to A$ such that $hf_\beta = a_\beta$ for each $\beta < \alpha$, and in this case $\zeta^e_{f,b}(\vec{a})$ encodes h: $Ie \to A$ as object of I/A, together with the element $b \in |Ge|$.

We define Γ as follows:

$$\Gamma^0_{\zeta^e_{f,b}}(\vec{v}) \equiv_{df} \bigwedge_{\substack{\varphi(v_\beta)_{\beta<\alpha} \text{ an atomic formula of} \\ L(\Sigma) \text{ such that } Ie \models \varphi[f_\beta]_{\beta<\alpha}}} \varphi(\vec{v})$$

$$\Gamma^{At}_\psi \equiv_{df} \bigvee_{\substack{\{\varphi_\alpha : \alpha<\delta\},\ T \models \psi, \\ \delta < \varkappa, \\ \varphi_\alpha \text{ atomic formulae of } L(\Omega')}} \exists \vec{u} \bigwedge_{\alpha<\delta}$$

$$[\bigvee_{\substack{\varphi' \text{ atomic formula of } L(\Omega), \\ \varphi_\alpha \text{ is } \varphi'(\zeta^e_{f_\beta,b}(\vec{v}))_{\beta<\gamma}, \\ Ge \models \varphi'[b_\beta]_{\beta<\gamma}}} \bigwedge_{\beta<\gamma} \Gamma^0_{\zeta^e_{f,b_\beta}}(\vec{v})$$

$$\vee \bigvee_{\substack{\varphi_\alpha \text{ is } \zeta^d_{f,b}(\vec{v}) = \zeta^e_{g,c}(\vec{w}) \\ x : d \to e \text{ a morphism of } D, \\ Gx(b) = c}} \{\Gamma^0_{\zeta^d_{f,b}}(\vec{v}) \wedge \Gamma^0_{\zeta^e_{g,c}}(\vec{w})$$

$$\bigwedge\!\!\!\!\bigwedge \qquad v_\beta = \sigma(\vec{w}_\gamma)\}],$$

σ a term of $L(\Sigma)$,

$$Ix(f_\beta) = \sigma(\vec{g}_\gamma)$$

for each atomic formula ψ of $L(\Omega')$, where $\vec{u}$ is the sequence of all variables occurring free in the φ_α but not in ψ. Note that we rely on Corollary 5 for the bound on δ.

b. The proof is the same, except that now the comma category objects are the embeddings h: Ie $\to$ A. Hence we simply replace 'atomic formula' in the definition of $\Gamma^{0_e}_{\zeta_{f,b}(\vec{v})}$ by 'atomic or negated atomic formula'. This converts Γ from an $\exists^+_1$ $\varkappa$-word-construction to an $\exists_1$ $\varkappa$-word-construction.

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3.2 *Filtered limits*

We assemble some technicalities needed for the normal form theorem.

Let $\varkappa$ be a regular cardinal, C a small category. We say that C is *$\varkappa$-filtered* when

(1) for every family $\{c_\alpha : \alpha < \mu\}$ of objects of C, with $\mu < \varkappa$, there is an object d of C together with morphisms $x_\alpha : c_\alpha \to d$;

(2) for every pair of objects c, d of C, and family $\{(x_\alpha : c \to d) : \alpha < \mu\}$ of morphisms of C, with $\mu < \varkappa$, there is a morphism y: d $\to$ e of C such that $yx_\alpha = yx_\beta$ for all $\alpha, \beta < \mu$.

If C is $\varkappa$-filtered, then a functor F: $C \to B$ is called a *$\varkappa$-filtered diagram,* and its right limit is called a *$\varkappa$-filtered limit.*

Examples are found as follows.

Let Σ be a similarity type of length $\leqq \varkappa$. We say that a Σ-structure A is *$\varkappa$-presented* if A has form df $< \Sigma' X, \Phi >$, where X is a set of individual constants not in Σ, Σ' comes from Σ by adding these constants, and $|X|$, $|\Phi| < \varkappa$. We say A is *$\varkappa$-generated* under the same conditions but without the restriction on $|\Phi|$. A subcategory A of Σ-*Str* will be called *$\varkappa$-presented* (resp. *$\varkappa$-generated*) when every object of A is $\varkappa$-presented (resp. $\varkappa$-generated). A category D is said to be a *representative $\varkappa$-presented* (resp. *representative $\varkappa$-generated)* subcategory of A if D is a small, full, $\varkappa$-presented (resp. $\varkappa$-generated) subcategory of A containing at least one representative of each isomorphism type of $\varkappa$-presented (resp. $\varkappa$-generated) object of A.

There is a Prim $(\varkappa, \mathcal{P}_{<\varkappa})$ map which takes each similarity type Σ of length $\leqq \varkappa$ to a representative $\varkappa$-presented subcategory of Σ-*Str*, and another Prim $(\varkappa, \mathcal{P}_{<\varkappa})$ map which finds a representative $\varkappa$-generated subcategory of Σ-Str_e.

LEMMA 19. a. *Let $\varkappa$ be a regular cardinal and Σ a similarity type of length $\leqq \varkappa$. If D is a representative $\varkappa$-presented subcategory of Σ-Str, then for each object* A *of Σ-Str, the comma category D/*A *is $\varkappa$-filtered and* $A = \varinjlim Q^D_A$.

b. *The same, with '$\varkappa$-presented' and 'Σ-Str' replaced by '$\varkappa$-generated' and 'Σ-Str_e'.*

Proof. a. To show property (1) of $\varkappa$-filtered categories, it suffices to check that D has coproducts of families of cardinality $< \varkappa$. Since $\varkappa$ is regular, this follows easily from the definition of D and the construction of coproducts by Dyck's Theorem (Example 3). To show property (2) it suffices to check that every family $\{(y_\alpha : D_1 \to D_2) : \alpha < \mu\}$ of fewer than $\varkappa$ morphisms in D has a coequaliser in D. Such a coequaliser can be found as follows: take a set of $<\varkappa$ generators of D_1, and add to the relations of some presentation of D_2 the equations needed to identify $y_\alpha(x)$ with $y_\beta(x)$, for each generator x of D_1 and each $\alpha, \beta < \mu$.

We form a right cone (A, η) for Q^D_A by putting $\eta_x = x$ for each object $x\colon D \rightarrow A$ of D/A. It is well-known and readily verified that (A, η) is a right limit cone.

b. Up to isomorphism, the objects of D/A are the inclusions $y\colon D \subseteq A$ of $\varkappa$-generated substructures of A. Since $\varkappa$ is regular, the union of fewer than $\varkappa$ $\varkappa$-generated substructures of A is a $\varkappa$-generated substructure of A, so that (1) holds. (2) is trivial since embeddings are monomorphisms. The right limit of Q^D_A is built as in part a.

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The limit formula in Lemma 19 will now help us to show that our definition of '$\varkappa$-presented' agrees with the categorical definition on p. 63 of Gabriel and Ulmer [6]; our definition of '$\varkappa$-generated' also agrees with theirs if the category in question is Σ-Str_e.

LEMMA 20. a. *Let $\varkappa$ be a regular cardinal and Σ a similarity type of length $\leqq \varkappa$. Then a Σ-structure A is $\varkappa$-presented iff the functor* $B \mapsto \Sigma\text{-}Str\,(A, B)$ *from Σ-Str to Set preserves $\varkappa$-filtered limits.*

b. *The same, with '$\varkappa$-presented' and 'Σ-Str' replaced by '$\varkappa$-generated' and 'Σ-Str_e'.*

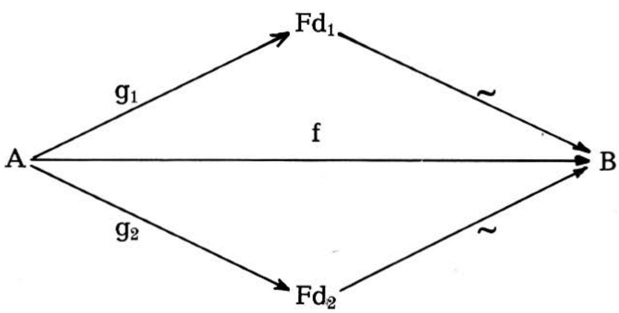
Proof. a. First suppose that A is $\varkappa$-presented, and let $F\colon C \rightarrow \Sigma\text{-}Str$ be a $\varkappa$-filtered diagram. Use Dyck's Theorem as in Example 3 to construct a right limit of F as $(B, \sim) = (\mathrm{df} < \Sigma', X, \Phi >, \sim)$. We must show that every homomorphism $f\colon A \rightarrow B$ factors through some Fd, d an object of C. For this, choose a set $\{a_\alpha : \alpha < \mu\}$ of $<\varkappa$ generators of A, and for each $\alpha < \mu$ pick a closed term $\tau_\alpha \in \overline{X}$ such that $f(a_\alpha) = \tau_\alpha^{\sim}$. Then the map $a_\alpha \mapsto \tau_\alpha$ carries the positive diagram of A to a set of $<\varkappa$

atomic sentences true in B. By Corollary 5 and Example 3, these atomic sentences are entailed by a set Ψ of $< \varkappa$ atomic sentences of the following two forms:

i. sentences from the positive diagrams of the structures Fd, where d are objects of *C*;
ii. sentences "$c_{d,a} = c_{e,Fy(a)}$" where $y\colon d \to e$ is a morphism of *C*.

Using the fact that *C* is $\varkappa$-filtered, we may assume without loss that Ψ is part of the positive diagram of some one structure Fd; property (2) of $\varkappa$-filtered categories is needed to make the sentences ii. true in Fd. There is then a homomorphism $g\colon A \to Fd$ which takes each a_α to the element of Fd named by τ_α, and we have $f = \sim . g$.

Next we must show that if $f\colon A \to B$ factors in two ways as



then there are morphisms $y_1\colon d_1 \to e$ and $y_2\colon d_2 \to e$ of *C* such $Fy_1 . g_1 = Fy_2 . g_2$. Using the same generators of A as before, we consider the equations

$$c_{d_1, g_1 a_\alpha} = c_{d_2, g_2 a_\alpha} \qquad (\alpha < \mu).$$

These equations come out true in B, whence an argument like that above finds suitable morphisms y_1, y_2.

Bearing in mind the form of $\varkappa$-filtered limits in *Set*, we have proved that Σ-*Str*(A,-) preserves $\varkappa$-filtered limits when A is $\varkappa$-presented. Conversely assume that Σ-*Str*(A,-) preserves $\varkappa$-filtered limits, where A is some Σ-structure. Let D be a representative $\varkappa$-presented subcategory of Σ-*Str*. By Lemma 19, $A = \varinjlim Q^D_A$ is a $\varkappa$-filtered limit. Since Σ-Str(A,-) preserves this limit, the identity $1_A \colon A \to A$ factors through some object of D. Hence A is a retract of a $\varkappa$-presented structure, and so A is itself $\varkappa$-presented.

b. The proof is similar but a little easier.

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We shall need to know how formulae are preserved in $\varkappa$-filtered limits; this prompts the following definitions. A formula φ of $L_{\infty\varkappa}(\Sigma)$ will be described as *$\varkappa$-presenting* if φ has the form

$$\bigvee_{\alpha<\mu} \exists \vec{u}_\alpha \psi_\alpha \tag{13}$$

where each ψ_α is a conjunction of $<\varkappa$ atomic formulae; φ will be described as *$\varkappa$-generating* if φ has the form (13) where each ψ_α is a conjunction of any number of atomic or negated atomic formulae.

LEMMA 21. a. *Let $\varkappa$ be a regular cardinal and Σ a similarity type of length $\leq \varkappa$, and let* $F \colon C \to \Sigma$-*Str be a $\varkappa$-filtered diagram with right limit* (A, η). *Then for every $\varkappa$-presenting formula φ of* $L_{\infty\varkappa}(\Sigma)$ *and every* $\vec{a}$ *in* $|A|$,

$A \models \varphi[\vec{a}]$ *iff for some object c of C and some* $\vec{b}$ *in* $|Fc|$,

$$\eta_c(\vec{b}) = \vec{a} \text{ and } Fc \models \varphi[\vec{b}].$$

b. *The same, with "ϰ-generating" and "Σ-Str$_e$" in place of "ϰ-presenting" and "Σ-Str".*

Proof. a. Right to left follows at once from the fact that φ is $\exists_1^+$ and hence is preserved by homomorphisms. It suffices to prove left to right in the case where φ is a conjunction of fewer than $\varkappa$ atomic formulae. The left-hand side then says in effect that there is a certain homomorphism f: B $\to$ A with B $\varkappa$-presented. By Lemma 20 the functor Σ-*Str*(B,-) preserves $\varkappa$-filtered limits, so that f factors through some η_c; this is the right-hand side.

b. Similar.

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3.3 *The normal form theorem*

The following theorem characterises those functors which can be expressed as associated functors of $\exists_1^+$ or $\exists_1$ word-constructions; word-constructions thus provide a syntactic normal form for the functors in question.

We say a $\varkappa$-word-construction Γ from Σ to Ω is *presenting* (resp. *generating*) if every formula Γ^s_τ, Γ^{At}_φ of Γ is $\varkappa$-presenting (resp. $\varkappa$-generating). Note that a word-construction is $\exists_1^+$ (resp. $\exists_1$) iff up to logical equivalence it is $\varkappa$-presenting (resp. $\varkappa$-generating) for all large enough $\varkappa$. On the other hand Example 23 will describe an $\exists_1^+$ ω-word-construction which is not a presenting ω-word-construction.

THEOREM 22 (Normal form theorem). a. *Let $\varkappa$ be a regular cardinal, Σ and Ω similarity types of length $\leqq \varkappa$, and* F *a functor from Σ-Str to Ω-Str. Then the following are equivalent:*

(1) *For every representative $\varkappa$-presented subcategory D of Σ-Str, F is a left Kan extension of $F|D$ along D.*
(2) *For some $\varkappa$-presented small subcategory D of Σ-Str, F is a left Kan extension of $F|D$ along D.*
(3) F *is naturally isomorphic to the associated functor of a presenting $\varkappa$-word-construction from Σ to Ω.*
(4) F *preserves $\varkappa$-filtered limits.*

Proof of a. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3). Let D be as in (2). We apply Theorem 18(a) with $A = \Sigma\text{-}Str$, $B = \Omega\text{-}Str$, $I\colon D \subseteq A$, $G = F|D$. Consider the formulae in the proof of Theorem 18(a). The formulae $\Gamma\overset{0}{\zeta}{}^{e}_{f,b}(\vec{v})$ are already of form (13), and since the objects in D are $\varkappa$-presented, we may take the conjunctions to consist of fewer than $\varkappa$ atomic formulae. The formulae Γ^{At}_{ψ} are logically equivalent to $\varkappa$-presenting formulae, since any formula

$$\exists \vec{u} \bigwedge_{\alpha<\delta} \bigvee_{\beta<\mu_\alpha} \chi^{\alpha}_{\beta}$$

can be paraphrased as

$$\bigvee_{\substack{f\colon \delta \to \bigcup_{\alpha<\delta}\mu_\alpha, \\ f(\alpha)<\mu_\alpha \text{ for} \\ \text{each } \alpha}} \exists \vec{u} \bigwedge_{\alpha<\delta} \chi_{f(\alpha)}.$$

(3) $\Rightarrow$ (4). Let Γ be a presenting $\varkappa$-word-construction from Σ to Ω with associated functor $\Gamma\colon \Sigma\text{-}Str \to \Omega\text{-}Str$. Let C be a $\varkappa$-filtered small category, $G\colon C \to \Sigma\text{-}Str$ a functor, and (A, η) a right limit of G in Σ-Str. We must show that $(\Gamma A, \Gamma\eta)$ is a right limit of ΓG in Ω-Str. Let (B, ζ) be a right cone for ΓG.

We define f: $|\Gamma A| \to |B|$ as follows, using notation from section 1.3.

ΓA is generated as Ω-structure by the elements $\tau(\vec{a})^{\sim}$ where $\tau \in \mathrm{dom}\,\Gamma^s$, $\vec{a}$ is in $|A|$, and $A \models \Gamma^s_\tau[\vec{a}]$. Consider one such $\tau(\vec{a})^{\sim}$. Since Γ is presenting, Lemma 21(a) implies that there is an object c of C with $\vec{b}$ in $|Gc|$ such that $\eta_c(\vec{b}) = \vec{a}$ and $Gc \models \Gamma^s_\tau[\vec{b}]$, so $\tau(\vec{b}) \in X^s_{Gc}$. We put

$$f(\tau(\vec{a})^{\sim}) = \zeta_c(\tau(\vec{b})^{\sim}).$$

To justify this definition, suppose $\tau(\vec{a})^{\sim} = \sigma(\vec{a}')^{\sim}$; then $\Phi_A \models \tau(\vec{a}) = \sigma(\vec{a}')$ by Lemma 1. Using Corollary 5 and Lemma 21 again, we find an object e of C with $\vec{d}$, $\vec{d}'$ in $|Ge|$ such that $\eta_e(\vec{d}) = \vec{a}$, $\eta_e(\vec{d}') = \vec{a}'$ and $\Phi_{Ge} \models \tau(\vec{d}) = \sigma(\vec{d}')$. Since C is $\varkappa$-filtered we may choose e so that there is x: $c \to e$ in C such that $Gx(\vec{b}) = \vec{d}$ and $\sigma(\vec{d}') \in X^s_{Ge}$. By assumption on ζ, $\zeta_c = \zeta_e . Gx$, so

$$\zeta_c(\tau(\vec{b})^{\sim}) = \zeta_e(\Gamma Gx . \tau(\vec{b})^{\sim}) = \zeta_e(\overline{Gx}\tau(\vec{b}))^{\sim}$$

$$= \zeta_e(\tau(Gx . \vec{b})^{\sim}) = \zeta_e(\tau(\vec{d})^{\sim}) = \zeta_e(\sigma(\vec{d}')^{\sim}).$$

This shows that the definition of f is sound. Similar arguments show that f may be extended from the generators $\tau(\vec{a})^{\sim}$ to the whole of $|\Gamma A|$, so as to form a homomorphism f: $\Gamma A \to B$.

By the definition of f, if c is an object of C and $y \in |\Gamma G c|$, then

$$\zeta_c(y) = f\Gamma(\eta_c)(y). \qquad (14)$$

(14) says that f is a morphism of cones, f: $(\Gamma A, \Gamma\eta) \rightarrow (B, \zeta)$. Also (14) determines f uniquely, because (by Lemma 20(a)) $|\Gamma A|$ is the union of the images of the $\Gamma(\eta_c)$. Thus there is a unique morphism of cones from $(\Gamma A, \Gamma\eta)$ to (B, ζ).

(4) $\Rightarrow$ (1). Assume (4), and let D be a representative $\varkappa$-presented subcategory of Σ-*Str*. By Lemma 19(a), if A is any Σ-structure, then $A = \varinjlim Q^D_A$ and this is a $\varkappa$-filtered limit. Hence $FA = \varinjlim FQ^D_A$ by the assumption on F, and $\varinjlim FQ^D_A = \varinjlim (F|D)Q^D_A$. This tells us that F is the pointwise left Kan extension of $F|D$ along D.

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THEOREM 22. b. *Let $\varkappa$ be a regular cardinal, Σ and Ω similarity types of length $\leqq \varkappa$, and F a functor from Σ-Str_e to Ω-Str. Then the following are equivalent:*

(1) *For every representative $\varkappa$-generated subcategory D of Σ-Str_e, F is a left Kan extension of $F|D$ along D.*
(2) *For some $\varkappa$-generated small subcategory D of Σ-Str_e, F is a left Kan extension of $F|D$ along D.*
(3) *F is naturally isomorphic to the associated functor of a generating $\varkappa$-word-construction from Σ to Ω, restricted to Σ-Str_e.*
(4) *F preserves $\varkappa$-filtered limits.*

Proof of b. Just as a.

[ ]

Of course the entailments (2) $\Rightarrow$ (4) in Theorem 22 can be proved more directly; see § 14 of Gabriel and Ulmer [6] for the 'part a.' version.

EXAMPLE 23: a functor $F\colon \Sigma\text{-}Str \rightarrow \Omega\text{-}Str$ such that $F|\Sigma\text{-}Str_e$ preserves ω-filtered limits but F fails to preserve ω-filtered limits. Take Σ and Ω to be the same one-sorted similarity type with 1-ary functions f, g and individual constant O. *1* is the terminal Ω-structure whose sole element is O. F is defined on Σ-structures A by:

$$F(A) = \begin{cases} A & \text{if for some } n \geqq O,\ g(f^n(O)) \neq f^n(O) \text{ in A} \\ 1 & \text{otherwise.} \end{cases}$$

If $F(A) = A$, $F(B) = B$ and $x\colon A \rightarrow B$, then put $Fx = x$. Fx elsewhere is defined to be the unique morphism to the terminal object. For each $\alpha \leqq \omega$, let A_α be the object df $< \Sigma, O, \{gf^nO = f^nO : n < \alpha\} >$, and $x_\alpha : A_\alpha \rightarrow A_{\alpha+1}$ the unique homomorphism when $\alpha < \omega$. Then $A_\omega = \varinjlim_{\alpha < \omega} A_\alpha$, an ω-filtered limit, but $FA_\omega = 1 \neq A_\omega = \varinjlim FA_\alpha$. F is associated to an $\exists^+_1$ generating ω-word-construction.

3.4 Extensions

Our normal form theorem was stated only for functors defined on Σ-*Str* or $\Sigma\text{-}Str_e$. But clearly it also applies to functors defined on some smaller category, provided they can be extended in some appropriate way. There is a very natural sufficient condition for this to hold, as follows.

Let *A* be a full subcategory for Σ-*Str* (resp. of $\Sigma\text{-}Str_e$), and I the inclusion functor. A small subcategory *D* of *A* will be said to be *pointwise dense* in *A* if for every object A of *A*, $IA = \varinjlim IQ^D_A$.

THEOREM 24. a. *Let* ϰ *be a regular cardinal,* Σ *and* Ω *similarity types of length* $\leqq$ ϰ, *D a pointwise dense small* ϰ*-pre-*

sented subcategory of the full subcategory A of Σ-Str, and F: $A \to \Omega$*-Str a functor which preserves all the right limits* $\varinjlim Q^D_A$.

Then F *is the restriction to* A *of a functor* G: Σ-*Str* $\to \Omega$-*Str which preserves* $\varkappa$*-filtered limits.*

b. *The same, with '*$\varkappa$*-generated' and '*Σ-Str_e*' in place of '*$\varkappa$*-presented' and '*Σ-*Str'.*

Proof. a. Since Ω-*Str* is right complete, F$|D$ has a pointwise left Kan extension G: Σ-*Str* $\to \Omega$-*Str* along the inclusion of D in Σ-*Str*. By definition of G, GA $= \varinjlim FQ^D_A$ for each object A of A, while FA $= \varinjlim FQ^D_A$ by assumption. Hence G can be taken to be an extension of F. The rest is by (2) $\Rightarrow$ (4) in Theorem 22.

b. Similar.

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Theorem 24 applies in the following situations — we omit the proof, which is an encore of Lemma 19.

THEOREM 25. *Let* $\varkappa$ *be a regular cardinal,* Σ *a similarity type of length* $\leqq \varkappa$.

a. *If* A *is a* $\varkappa$*-quasivariety* (Σ, T)-*Str and* D *is a representative* $\varkappa$*-presented subcategory of* A*, then for each object* A *of* A*, the comma category* $D/$A *is* $\varkappa$*-filtered, and* D *is pointwise dense in* A.

b. *If* A *is a full subcategory of* Σ-Str_e *which is closed under substructures, and* D *is a representative* $\varkappa$*-generated subcategory of* A*, then for each object* A *of* A*, the comma category* D/A *is* $\varkappa$*-filtered, and* D *is pointwise dense in* A

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Thus every functor between quasivarieties which preserves ω-filtered limits is (up to natural isomorphism) the associated functor of an $\exists^+_1$ ω-word-constructon. This includes all left adjoints between quasivarieties, since(by the Eckmann-Hilton theorem) they preserve all right limits. We have here taken on trust the well-known fact that ω-filtered limits are created in the quasivariety (Ω, T)-*Str* from Ω-*Str*.

We turn to some closure conditions.

THEOREM 26. *Let $\varkappa$ be a regular cardinal, Ω a similarity type of length $\leqq \varkappa$, and A a category. Then the class of functors* F: $A \to \Omega$-Str *which preserve $\varkappa$-filtered limits is closed under right limits, and under left limits of diagrams with $< \varkappa$ objects.*

Proof. Let G: $C \to \Omega\text{-}Str^A$ be a diagram of functors which preserve $\varkappa$-filtered limits, and let $\underset{\to}{\lim}$ G: $A \to \Omega$-*Str* be given. Let F: $D \to A$ be a $\varkappa$-filtered diagram. By assumption, $G_c\,(\underset{\to}{\lim}\ F) = \underset{\to}{\lim}\ G_cF$ for each object c of C and for any right limit $\underset{\mapsto}{\lim}$ F of F. We have a bifunctor from $C \times D$ to Ω-*Str*, namely $<c, d> \ \mapsto G_cF(d)$. Then

$$\underset{\to}{\lim}\ G\,.\,\underset{\to}{\lim}\ F$$

$= \underset{\to}{\lim}_c(G_c\underset{\to}{\lim}\ F)$ by pointwise construction of limits

$= \underset{\to}{\lim}_c\,.\,\underset{\to}{\lim}_d\ G_cF(d)$ by assumption on G_c

$= \underset{\to}{\lim}_d\,.\,\underset{\to}{\lim}_c\ G_cF(d)$ by commutativity of right limits

$= \underset{\to}{\lim}_d\ (\underset{\to}{\lim}\ G)F(d)$ by pointwise construction

$= \underset{\to}{\lim}\ (\underset{\to}{\lim}G\,.\,F)$.

Hence $\underset{\to}{\lim}$ G preserves the right limit of F when this exists.

For left limits of diagrams with $< \varkappa$ objects, the proof is the same, except that the commutativity of right limits is replaced by

$$\underleftarrow{\lim}_c \cdot \underrightarrow{\lim}_d\, G_cF(d) = \underrightarrow{\lim}_d \cdot \underleftarrow{\lim}_c\, G_cF(d).$$

The argument for this is just like the familiar case $\varkappa = \omega$, Ω-*Str* = *Set*.

EXAMPLE 27. (Cf. Eklof [3] Corollary 4.5) Let $\varkappa$ be a regular cardinal, *A* a $\varkappa$-quasivariety, and F: $A \to A$ a subfunctor of the identity which preserves $\varkappa$-filtered limits. We may define inductively

$$F^0 = \text{identity}, \quad F^{\alpha} = \bigcap_{\beta < \alpha} F(F^{\beta}).$$

Using Theorem 26, every F^{α} $(\alpha < \varkappa)$ preserves $\varkappa$-filtered limits, and so by Theorems 25 and 24 extends to some functor G: Σ-*Str* $\to$ Σ-*Str* which preserves $\varkappa$-filtered limits. The normal form theorem associates G to a presenting $\varkappa$-word-construction. Examples ready to hand are torsion radicals on noetherian rings.

Clearly the class of functors which preserve $\varkappa$-filtered limits is closed under composition. Composition is in general not defined for functors of form F: Σ-$Str_e \to \Omega$-*Str*; even where there is a natural composition, it need not preserve $\varkappa$-filtered limits in Σ-Str_e.

EXAMPLE 28. Let F: Σ-*Str* $\to$ Ω-*Str* be as in Example 23, so that $F|\Sigma$-Str_e preserves ω-filtered limits. Let Ξ be Σ with the added 2-ary function symbol h. Define G: Ξ-*Str* $\to$ Σ-*Str* on objects by

GA = reduct to Σ of A/C, where C is the congruence on A generated by the pairs $\langle a, b\rangle$ such that $\exists c\ h(a, c) = b$.

This definition induces a map Gx: GA $\to$ GB for each morphism x: A $\to$ B of Ξ-*Str*, and hence a functor G: Ξ-*Str* $\to$ Σ-*Str* which preserves ω-filtered limits. In particular $G|\Xi$-Str_e preserves ω-filtered limits. Let Ξ' come from Ξ by adding the countably many new individual constants $x_0, x_1, \ldots$; for each $\alpha \leqq \omega$ define a Ξ-structure B_α by the Ξ-presentation

$$B_\alpha = df < \Xi', \{x_n : n < \alpha\}, \{h(gf^nO, x_n) = f^nO : n < \alpha\} \cup \{h(x_n, x_n) = O : n < \alpha\}>.$$

Define $y_\alpha : B_\alpha \to B_{\alpha+1}$ to be the unique homomorphism taking each $\tilde{x}_n$ in B_α to $\tilde{x}_n$ in $B_{\alpha+1}$, for each $\alpha < \omega$. Then $\lim_{\to \alpha < \omega} B_\alpha = B_\omega$ is an ω-filtered limit in Ξ-Str_e, and $\lim_{\to} GB_\alpha = \lim_{\to} A_\alpha = = A_\omega = GB_\omega$. But then $\lim_{\to} FGB_\alpha \neq FGB_\omega$ by Example 23.

(I believe the question answered here was raised by Eklof in conversation with Sabbagh.)

3.5 *Functions of several structures*

One can define an *α-ary construction* from $(\Sigma_\beta)_{\beta < \alpha}$ to Ω to be a function which takes some or all α-tuples $<A_\beta>_{\beta < \alpha}$ (where each A_β is a Σ_β-structure) to an Ω-structure $F(A_\beta)_{\beta < \alpha}$. An example is the construction $\amalg$ which takes $<A_\beta>_{\beta < \alpha}$ to the disjoint sum $\amalg_{\beta < \alpha} A_\beta$, a many-sorted structure; we we write $\amalg_{\beta < \alpha} \Sigma_\beta$ for the similarity type of the disjoint sum. Natural homomorphisms can be defined which make $\amalg$ a functor from the product category $\amalg_{\beta < \alpha} (\Sigma_\beta$-$Str)$ to $\amalg_{\beta < \alpha} \Sigma_\beta$-$Str$.

The world is not improved by attempts to define α-ary word-constructions; they are ugly, and everything they do is done better by composing 1-ary word-constructions with $\amalg$. For this reason we confine ourselves to discussing $\amalg$.

THEOREM 29. *Let* $\varkappa$ *be a regular cardinal,* α *an ordinal, and for each* $\beta < \alpha$ *let* Σ_β *be a similarity type of length* $\leqq \varkappa$*. Then the functor* $\amalg$ *preserves all right and left limits, and its restriction to objects is a Prim function. For each formula* φ *of* $L_{\infty\varkappa}(\amalg_{\beta<\alpha}\Sigma_\beta)$ *there are, for each* $\beta < \alpha$*, sequences* $\langle \varphi^\beta_\gamma \rangle_{\gamma<\delta}$ *of formulae of* $L_{\infty\varkappa}(\Sigma_\beta)$ *such that for every sequence* $\langle A_\beta \rangle_{\beta<\alpha}$ *of structures and sequences* $\vec{a}_\alpha$ *from* $|A_\alpha|$*,*

$$\amalg_{\beta<\alpha} A_\beta \models \varphi[\vec{a}_\beta]_{\beta<\alpha} \quad \textit{iff for every } \gamma < \delta \textit{ there is } \beta < \alpha$$

$$\textit{such that } A_\beta \models \psi^\beta_\gamma[\vec{a}_\beta].$$

The ordinal δ *is* $< \beth_{|TC(\varphi)|^+}$.

Proof. The statement about φ is proved by induction on the complexity of φ. Cf. the similar theorem of Malitz on products, Theorem 2.1 of [13].

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It follows, as in Theorem 14, that $\amalg$ preserves $L_{\lambda\varkappa}$-equivalence when λ is a strongly inaccessible cardinal $\geqq \varkappa$. This is a theorem of Malitz [13], who shows that it cannot be improved.

4: Related work

4.1 *Feferman's* $\varkappa$*-local functors*

Solomon Feferman ([4] section 3.2) defines local functors as follows. Let $\varkappa$ be any cardinal > 1, and let Σ and Ω be similarity types (of any length), and A a full subcategory of Σ-Str_e. Then a functor $F: A \to \Omega$-Str is said to be $\varkappa$*-local* if

i. A is closed under substructures;
ii. F preserves $\subseteq$;
iii. if A is an object of A and Z a set of $< \varkappa$ elements of FA, then $Z \subseteq |FB|$ for some $\varkappa$-generated substructure B of A.

Feferman's 'main preservation theorem' (his Theorem 6) states that every $\varkappa$-local functor preserves $\equiv_L$ and $\preccurlyeq_L$ for every language $L = L^{\alpha}_{\infty\varkappa}$.

In two ways Feferman's definitions are broader than ours: he allows $\varkappa$ to be singular, and he puts no restriction on the number of variables occurring in a term or free in a formula. It looks as if our methods should generalise in these two ways, but messily.

Suppose F: $A \rightarrow \Omega$-*Str* is $\varkappa$-local and that Σ and Ω have length $\leqq \varkappa$, and that $\varkappa$ is regular. By ii. and iii., F preserves $\varkappa$-filtered limits. Hence by Theorems 25(b) and 24(b), F extends to a functor G: Σ-$Str_e \rightarrow \Omega$-*Str* which preserves $\varkappa$-filtered limits. The normal form theorem, Theorem 22(b), says that G is (up to natural isomorphism) associated to a generating $\varkappa$-word-construction. Feferman's 'main preservation theorem' (for infinite α) then follows from our preservation theorem, Theorem 14.

Feferman's proof — by a back-and-forth argument — is infinitely more pleasant than ours. So there is some point in remarking that our uniform reduction and effectivity theorems give a lot more information about Feferman's functors, and this further information seems at the moment to be beyond the range of back-and-forth methods.

Feferman also defines $\varkappa$-local functors of many structures, and proves the same preservation theorem for these as for 1-ary functors. Using section 3.5 above, we can capture this generalisation by our methods too. But there seems little point in doing this. I know no interesting facts about $\varkappa$-local functors of many structures which are not best proved by Feferman's method in company with Theorem 1 of Benda [1]. Olin [15] has examples to show how products of modules may fail to preserve infinitary equivalence.

Gabriel Sabbagh (unpublished, but reported in Eklof [3]) observed that if we are not interested in small quantifier-rank α, we can replace ii. and iii. in the definition of '$\varkappa$-local' by the assumption that F preserves $\varkappa$-filtered limits (of embeddings). The functors which Sabbagh thus defined are essentially those of Theorem 22(b). Example 15 shows that these functors may fail to satisfy Feferman's theorem for small α.

Feferman remarks that «The general notion of $\varkappa$-local functor seems not to have been considered in category theory» ([4] p. 74). In the light of our treatment above, this might be a little unfair to Kan, Isbell, Lambek, Gabriel, Ulmer. Nevertheless Feferman must take full credit for establishing the connection with infinitary logic.

4.2 Classes of functors defined by Eklof

In two papers [2] and [3], Paul Eklof has extended Feferman's ideas in two ways.

In [2], Eklof makes the following definitions. $\mathcal{U}_p$ is the category whose objects are the algebraically closed fields of characteristic p and infinite transcendence degree, with embeddings for morphisms; p is O or a prime. A functor F: $\mathcal{U}_p \rightarrow \Omega\text{-}Str$ is called ω-*local* if F preserves embeddings and ω-filtered limits. Eklof proves by a neat back-and-forth argument that if F is ω-local then $FA \equiv_{\infty\omega} FB$ for all objects A, B of $\mathcal{U}_p$.

Eklof says twice that his theorem is a special case of Feferman's preservation theorem in [4]. I think Eklof is too modest; the resemblance between his result and Feferman's is only skin-deep. In support of this remark we prove:

THEOREM 30. *Let Σ be such that $\mathcal{U}_p$ is a subcategory of $\Sigma\text{-}Str_e$. Then there is an ω-local functor $F: \mathcal{U}_p \rightarrow \Omega\text{-}Str$ (in Eklof's sense) which is not naturally isomorphic to $\Gamma|\mathcal{U}_p$ for the associated functor Γ of any ω-word-construction from Σ to Ω.*

Proof. Let k be the prime field of characteristic p, $\overline{k}$ the algebraic closure of k, and G the Galois group $Gal(\overline{k}/k)$. For each element g of G, let $L_g \in \mathrm{Aut}(G)$ be left multiplication by g. By global Choice, pick for each object A of $\mathcal{U}_p$ an embedding $e_A \colon \bar{k} \to A$. For each embedding $x \colon A \to B$ which is in $\mathcal{U}_p$, define $g(x) \in G$ by commutativity of the diagram

$$\begin{array}{ccc} \bar{k} & \xrightarrow{\;e_A\;} & A \\ {\scriptstyle g(x)}\downarrow & & \downarrow{\scriptstyle x} \\ \bar{k} & \xrightarrow[\;e_B\;]{} & B \end{array}$$

Define $F \colon \mathcal{U}_p \to \mathit{Set}$ by putting $FA = |G|$ for each object A, and $Fx = L_{g(x)}$ for each morphism x. Then F is ω-local.

Let Γ be any ω-word-construction with associated functor $\Gamma \colon \Sigma\text{-}\mathit{Str}_\Gamma \to \mathit{Set}$. We assert that F is not naturally isomorphic to $\Gamma | \mathcal{U}_p$. For suppose $\eta \colon F \to \Gamma | \mathcal{U}_p$ is a natural isomorphism. Take any object A of $\mathcal{U}_p$ and pick any element $\tau(\vec{a})^{\sim}$ of $|\Gamma A|$ such that $\tau \in \mathrm{dom}\,\Gamma^0$ and $A \models \Gamma^0_\tau[\vec{a}]$. Since Γ is an ω-word-construction, $\vec{a}$ can be assumed finite. If x is an automorphism of A which pointwise fixes $\vec{a}$, then $\Gamma x(\tau(\vec{a})^{\sim}) = \tau(\bar{x}\vec{a})^{\sim} = \tau(\vec{a})^{\sim}$, so that (via η) $L_{g(x)}$ and $g(x)$ must be identities. But clearly A has automorphisms which pointwise fix the finite sequence $\vec{a}$ and yet move some algebraic elements. This contradiction proves our assertion.

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Thus it seems that Eklof's theorem in [2] is strictly incomparable both with our work and with Feferman's.

We turn to Eklof's paper [3]. In this note Eklof defines a further class of functors, the $(\varkappa, \infty)$-*local functors*. This class includes those of Theorem 22(b) (with the same $\varkappa$), and is closed under right limits, under left limits of diagrams with $<\varkappa$ objects, and under composition. He shows that all functors in this class preserve $L_{\infty\varkappa}$-equivalence. Note that the functors of Theorem 22(b) have all these properties except closure under composition (cf. Example 28).

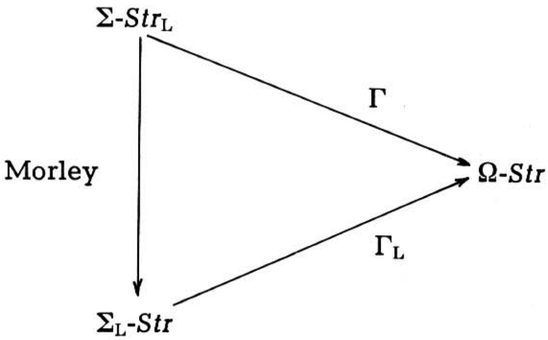
Using Theorems 22(b), 17 and 16, the class of associated functors of $\exists_1$ $\varkappa$-word-constructions also has the properties of Eklof's class (and more besides). It is natural to ask what the relationship is between this class and Eklof's. I confine myself to brief and oversimplified remarks because the situation is not yet entirely clear.

$\exists_1$ $\varkappa$-word-constructions do not in general preserve $\varkappa$-filtered limits (Example 23). But they do if we restrict the morphisms of the domain category. More precisely, call the fragment L of $L_{\infty\varkappa}$ *transitive* if L contains all subformulae of formulae in L. If A is a subcategory of Σ-*Str* and L a transitive fragment of $L_{\infty\varkappa}(\Sigma)$, write A_L for the subcategory of A whose objects are those of A, and whose morphisms are those of A which preserve all formulae of L. For example, A_e is precisely $A_{L^0_{\infty\varkappa}(\Sigma)}$.

REMARK 31. If Γ is a $\varkappa$-word-construction from Σ to Ω, and L a transitive fragment of $L_{\infty\varkappa}(\Sigma)$ which contains all formulae Γ^s_τ, Γ^{At}_φ and their negations, then Γ: $\Sigma\text{-}Str_L \to \Omega\text{-}Str$ preserves $\varkappa$-filtered limits.

The essential point here is that the Tarski-Vaught theorem on elementary chains extends to $\varkappa$-filtered limits in $\Sigma\text{-}Str_L$.

We may now add to Σ a new relation symbol R_φ for each formula φ of L, to get an enlarged similarity type Σ_L. If we interpret R_φ as φ in every object of $\Sigma\text{-}Str_L$, the result is a functor Morley: $\Sigma\text{-}Str_{\mathrm{L}} \to \Sigma_{\mathrm{L}}\text{-}Str$, known as *Morleyisation.* Morley is an embedding; it preserves $\varkappa$-filtered limits, and its image is a full subcategory of $\Sigma_{\mathrm{L}}\text{-}Str$. Translating φ into R_{φ}, the $\varkappa$-word-construction Γ in Remark 31 can be rendered into a presenting $\varkappa$-word-construction Γ_{L} from Σ_{L} to Ω with all formulae atomic, and we have a commutative diagram of functors



(15)

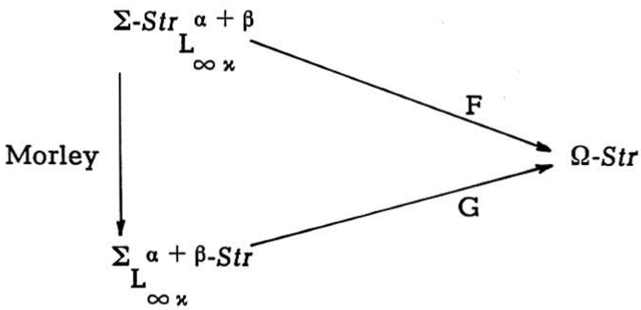
Now start again at the other end. Let L be $L^{\alpha}_{\infty\varkappa}(\Sigma)$, and define $F\colon \Sigma\text{-}Str_e \to \Omega\text{-}Str$ to be *$(\varkappa, \alpha)$-local* if $F|\Sigma\text{-}Str_L$ is equal to G.Morley for some functor $G\colon \Sigma_{\mathrm{L}}\text{-}Str \to \Omega\text{-}Str$ which preserves embeddings and $\varkappa$-filtered limits. We make the following observations.

(1) The associated functor of any $\exists_1$ $\varkappa$-word-construction from Σ to Ω is $(\varkappa, \alpha)$-local whenever all formulae of Γ are in $L^{\alpha}_{\infty\varkappa}(\Sigma)$. This follows from (15).

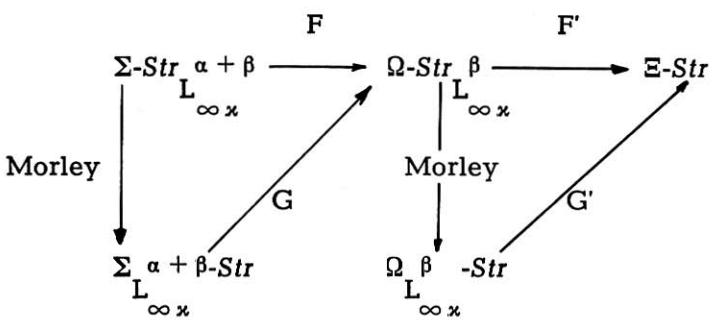
(2) If F is $(\varkappa, \alpha)$-local, say $F|\Sigma\text{-}Str_{\mathrm{L}} =$ G.Morley; then G is associated to a presenting $\varkappa$-word-construction Γ by the normal form theorem, so that formulae ψ of $L(\Omega)$ are reduced to formulae ψ^{Γ} of $L(\Sigma_{\mathrm{L}})$ by the uniform reduction theorem (Theorem 11). Replacing R_{φ} in ψ^{Γ} by φ, we find formulae ψ^{F} of

L(Σ) which are equivalent to ψ^{Γ} on all objects in the image of Morley. Thus F preserves L'-equivalence for every 'reasonable' infinitary language L' ⊇ L, just as in the preservation theorem (Theorem 14). In particular, $(\varkappa, \alpha)$-local functors preserve $\equiv_{\infty\varkappa}$.

(3) Suppose F: $\Sigma\text{-}Str_e \to \Omega\text{-}Str$ is $(\varkappa, \alpha)$-local and F': $\Omega\text{-}Str_e \to$ $\Xi\text{-}Str$ is $(\varkappa, \beta)$-local. Then a fortiori F is $(\varkappa, \alpha + \beta)$-local ,and so we have a commutative diagram



where G is associated to a presenting $\varkappa$-word-construction with all formulae atomic. Using the uniform reduction theorem (sharpened a little if β is small), we may show that the image of G lies in $\Omega\text{-}Str^{\beta}_{L_{\infty\varkappa}}$. Incorporating the assumption on F', we thus have a commutative diagram



where F'F = G'.Morley.G.Morley, and G'.Morley.G preserves

$\varkappa$-filtered limits. Thus F'F is $(\varkappa, \alpha + \beta)$-local. We infer that the class of functors which are $(\varkappa, \alpha)$-local for some α is closed under composition.

(4) The class of $(\varkappa, \alpha)$-local functors from Σ to Ω is closed under all the same limits as the class of functors $G: \Sigma_{L_{\infty\varkappa}}^{\alpha}\text{-}Str \to \Omega\text{-}Str$ which preserve embeddings and $\varkappa$-filtered limits; cf. Theorem 26.

(5) Without loss we could assume that a $(\varkappa, \alpha)$-local functor is only defined on the category $\Sigma\text{-}Str_L$ mentioned in the definition. Note that this restricted functor may be extendable in more than one way to a functor defined on the whole of Σ-*Str*.

4.3 Gaifman's single-valued operations

Gaifman [7] defines a *single-valued operation* from Σ to Ω to be a pair of theories (T_1, T_2) — possibly higher-order — such that if A is a Σ-structure and $A \models T_1$ then

(1) there is a composite structure (A, B, r) such that B is an Ω-structure and r is a collection of functions and relations, and $(A, B, r) \models T_2$;
(2) if (A, B, r) and (A, B', r') are as in (1), then $(A, B, r) \cong_A (A, B', r')$.

We shall call a single-valued operation (T_1, T_2) a *Gaifman operation;* it is a *first-order* operation when T_1 and T_2 are first-order theories, an $L_{\omega_1\omega}$ operation when T_1 and T_2 are theories in $L_{\omega_1\omega}$, etc.

Feferman remarked (in correspondence) that word-constructions are a special case of Gaifman operations. A precise statement follows.

THEOREM 32. *Let $\varkappa$ be a regular cardinal and Γ a $\varkappa$-word-construction from Σ to Ω. Then there is an $L_{\infty\varkappa}$ Gaifman operation (O, T_Γ) such that every Σ-structure A can be completed to a model $(A, \Gamma(A), r_A)$ of T_Γ. The map $\Gamma \mapsto T_\Gamma$ is* Prim $(\varkappa, \mathcal{P}_{<\varkappa})$.

Proof. Assume Σ and Ω are one-sorted. Then $(A, \Gamma(A), r_A)$ will be a two-sorted structure with A in the first sort and $\Gamma(A)$ in the second; call these sorts Σ and Ω respectively. We shall use v for variables of sort Σ, and x, y for variables of sort Ω. For each term $\tau \in \mathrm{dom}\,\Gamma^0$ we introduce a relation symbol R_τ.

T_Γ shall consist of the sentences

1) $\forall \vec{v}\, \exists_{\leqslant 1} x\, R_\tau(\vec{v}, x)$ $\qquad (\tau \in \mathrm{dom}\,\Gamma^0)$

2) $\forall x \bigvee_{\substack{\sigma \text{ an } \alpha\text{-ary term of } L(\Omega), \\ (\tau_\beta)_{\beta<\alpha} \text{ from dom } \Gamma^0}} \exists \{y_\beta : \beta < \alpha\} [x = \sigma(y) \wedge \bigwedge_{\beta<\alpha} \exists \vec{v}\, R_{\tau_\beta}(\vec{v}, y_\beta)]$

3) $\forall \vec{v}\, [\exists x R_\tau(\vec{v}, x) \leftrightarrow \Gamma^0_\tau(\vec{v})]$ $\qquad (\tau \in \mathrm{dom}\,\Gamma^0)$

4) $\forall \{\vec{v}_\beta : \beta < \alpha\} \forall \{x_\beta : \beta < \alpha\} [\bigwedge_{\beta<\alpha} R_{\tau_\beta}(\vec{v}_\beta, x_\beta) \rightarrow$

$[\varphi(\vec{x}) \leftrightarrow \varphi_w^\Gamma(\vec{v}_\beta)_{\beta<\alpha}]]$ $\qquad$ (φ an α-ary atomic formula of $L(\Omega)$, $\tau_\beta \in \mathrm{dom}\,\Gamma^0$ for each $\beta < \alpha$)

where in 4), w: $FV(\varphi) \to \mathrm{dom}\,\Gamma^0$ is such that $w(x_\beta) = \tau_\beta$ for each $\beta < \alpha$, and in $\varphi_w^\Gamma(\vec{v}_\beta)_{\beta<\alpha}$ the variable x_γ^β of φ_w^Γ is replaced by the γ-th variable in the sequence $\vec{v}_\beta$. These sentences suffice to define $\Gamma(A)$ up to isomorphism over A.

[]

In particular if Σ and Ω are countable, Γ is recursively defined, and L is $L_{\omega_1\omega}$ or a countable admissible language (not $L_{\omega\omega}$), then Γ can be expressed as an L Gaifman operation, and Theorem 11 (uniform reduction) follows from Feferman's many-sorted interpolation theorem as in [7]; see also Feferman [5]. No similar argument is known for languages stronger than $L_{\omega_1\omega}$. (There are grounds for hope: Isbell [19] extends an algebraic consequence of Beth's theorem to varieties in arbitrary $L_{\infty\varkappa}$, by an argument with free structures.)

In a recent preprint [18], Gaifman states some conjectures and results about the relationship between first-order Gaifman operations and ω-word-constructions. In place of many-sorted structures he now uses reducts of one-sorted structures; the reader will have no difficulty in translating. We quote:

"Let L_0, L_1 be countable first-order languages such that $L_0 \subset L_1$. Let P(v) be one-place predicate in L_1 and not in L_0. If M is a model for L_1 let M^P be the submodel whose universe is $\{x : M \models P(x)\}$ and let $M^P|L_0$ be its reduct to L_0. Finally, let T be a theory in L_1. We are interested in characterizing the following property of T:

(I) If $M_i \models T$, $i = 1, 2$, and $M_1^P|L_0 = M_2^P|L_0 = M_0$, then M_1 is isomorphic to M_2 over M_0."

Next Gaifman introduces the notion of a *defining schema.*

"This schema, say D, consists of (i) a function associating with every k-place predicate $R(v_1, \ldots, v_k)$ of L_1 a formula $\varphi_R(\vec{x}_1, \ldots, \vec{x}_k)$ of L_0, where $\vec{x}_i = x_{i,1}, \ldots, x_{i,n}$, and $n > 0$ is fixed (including the case where $R(v_1, v_2)$ is $v_1 = v_2$) and (ii) a formula $\psi(\vec{u}, v)$ of L_0 where $\vec{u} = u_1, \ldots, u_n$."

Now let $M_0 = M^P | L^0$. We say that M is *defined* in M_0 by D "if there is a function I, defined for n-tuples of members of M_0 such that $\{I(x_1, \ldots, x_n) : x_1, \ldots, x_n \in M_0\}$ is the universe of M and

(a) $M \models R(I(\vec{x}_1), \ldots, I(\vec{x}_k)) \Leftrightarrow M_0 \models \varphi_R(\vec{x}_1, \ldots, \vec{x}_k)$ for all predicates of L_1 (including the equality).

(b) For all $y \in M_0$, $I(\vec{x}) = y \Leftrightarrow M_0 \models \psi(\vec{x}, y)$."

Gaifman next introduces the following property of T:

(1) There is a defining schema D such that, for every M, $M \models T \Rightarrow M$ is defined in $M^P | L_0$ by D.

He conjectures:

$$\text{(A)} \qquad (1) \Leftrightarrow (\mathrm{I}).$$

We remark that (1) immediately implies that the map $M_0 \to M$ (for models M of T) can be expressed as an ω-word-construction with first-order formulae. Gaifman can show that certain strengthenings of (I) (e.g. that M is always rigid over M^P in addition to (I)) do imply corresponding strengthenings of (1). Shelah has a similar result (unpublished) for $L_{\omega_1 \omega}$. Also (1) implies (I) easily.

However, Gaifman's conjecture (A) is false in general. We prove this by following the analogy with Theorem 7 above.

Let M be a model of T, and suppose M is defined in M_0 by the defining schema D. Let f be an automorphism of M_0. We define a map $f^D : |M| \to |M|$ by

$$f^D(I(x_1, \ldots, x_n)) = I(fx_1, \ldots, fx_n).$$

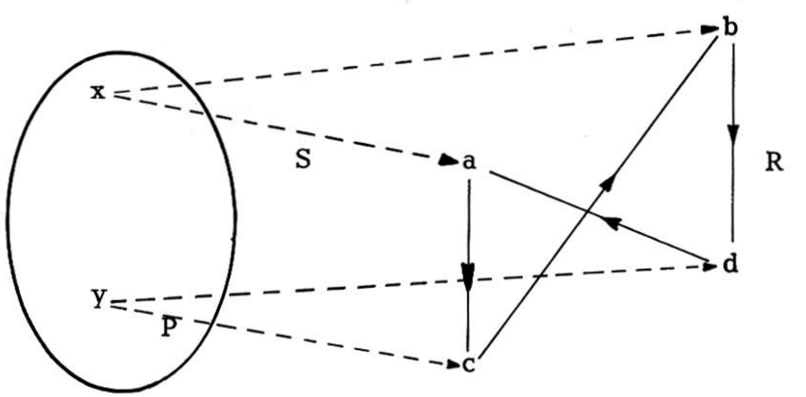
Since equality is included in clause (a), f^D is a well-defined map; also by clause (a), f^D is an automorphism of M. By clause (b), if $\vec{x}$, y are in M_0 then

$$I(\vec{x}) = y \quad \text{iff} \quad M_0 \models \psi(\vec{x}, y) \quad \text{iff} \quad M_0 \models \psi(f\vec{x}, fy) \quad \text{iff} \quad f^D(I\vec{x}) = fy.$$

Since I is surjective, it follows that $f^D y = fy$ for all $y \in |M_0|$. Thus f^D extends f. Also we see easily that the map $f \mapsto f^D$ is a homomorphism from the automorphism group of M_0 to that of M. Hence every automorphism of M_0 extends to an automorphism of M of the same order.

EXAMPLE 33: a counterexample to conjecture (A). We define T as follows. L_0 has no non-logical symbols. Besides P(v), L_1 has binary relation symbols R, S. T says:

> There are exactly six elements, of which exactly two satisfy P.
> If the elements satisfying P are x, y, and the others are a, b, c, d, then the positive diagram of any model of T has form "Px, Py, Sxa, Sxb, Syc, Syd, Rac, Rcb, Rbd, Rda".



Obviously T satisfies (I). To show that (1) fails in T, it suffices to find a model M of T and an automorphism f of M_0 which cannot be extended to an automorphism of M of the same order. Let M be as pictured above, and let f be the automorphism of M_0 which transposes x and y. Then f has order 2, whereas any extension of f to an automorphism of the whole of M has order 4.

In fact this argument shows that the Gaifman operation expressed by T cannot be expressed as a $\varkappa$-word-construction for any cardinal $\varkappa$.

A slight extension of Example 33 gives us a counterexample to another conjecture of Gaifman in the same preprint [18]. Gaifman says that M is defined in M_0 from *parameters* by D if the formulae φ_R of D are allowed to have additional free variables, to be taken over by parameters in the model. His conjecture (A*) is that (1*) and (I*) are equivalent, where (1*) and (I*) are:

(1*) There are finitely many defining schemas $D_1, \ldots, D_t$ such that whenever $M \models T$ then M is definable in $M^P | L_0$ from parameters by some D_i.

(I*) For every model M_0 for L_0 there are at most $\aleph_0$ + card(M_0) models M which are non-isomorphic over M such that $M^P | L_0 = M_0$ and $M \models T$.

EXAMPLE 34: a counterexample to conjecture (A*). We define T as in example 33, with the following alterations. L_0 has an extra binary relation symbol E; T says that E is an equivalence relation whose equivalence classes each have just six elements, and each equivalence class is exactly like a model of the theory of Example 33. T thus defined satisfies (I), so clearly it satisfies the weaker condition (I*). If M is a model of T which is defined in M_0 from parameters $\vec{p}$ by a defining schema D, then (by an argument like that above) any automorphism of M_0 which pointwise fixes $\vec{p}$ must extend to an automorphism of M of the same order. We thus find a counterexample by taking an M with infinitely many equivalence classes, and applying the argument of Example 33 to an equivalence class not containing any of the finitely many parameters $\vec{p}$.

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April 1975

ADDED IN PROOF:

1. Word-constructions can be seen as a generalisation to infinitary languages of the «defining schemas» of Gaifman (cf. para. 4.3 above). Gaifman's notion is the earlier by two years.

2. Since seeing Example 33 above, Gaifman has suggested a revised and slightly more complicated version of his conjecture (A) which allows the formulae φ_R to refer to a linear ordering of M_0.

3. As hoped on p. 129 above, we now have a many-sorted interpolation theorem for larger infinitary languages; it is restricted to Horn formulae. Details will appear elsewhere.