# ON $\aleph_0$-CATEGORICAL EXTRA-SPECIAL p-GROUPS

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*Abstract:* (°) Let p be an odd prime. We shall prove in this note, that the first-order theory $T_p$ of extra-special p-groups G of exponent p is $\aleph_0$-categorical but not $\aleph_1$-categorical. We also show that these groups G have a finite cyclic center of order p and that the first-order theory of G/Z(G) is both $\aleph_0$-categorical and $\aleph_1$-categorical ! These groups G have several further interesting properties: they are nilpotent of class 2, they have many maximal normal subgroups, $Z(G) = G' = \Phi(G)$ and they are FC-groups. We use these groups in order to disprove some conjectures concerning categoricity of non-abelian groups. We also state without proof some further results concerning $\aleph_0$-categoricity of non-abelian groups.

## § 1. *Introduction*

Let T be a theory formulated in some countable first-order language L, and let $\mathfrak{m}$ be an infinite cardinal number. Then T is called categorical in power $\mathfrak{m}$ (or: $\mathfrak{m}$-categorical) if any two models $\mathfrak{A}$ and $\mathfrak{B}$ of T which are both of power $\mathfrak{m}$ are isomorphic. If $\mathfrak{C}$ is an arbitrary structure, then let Th($\mathfrak{C}$) denote the first-order theory of $\mathfrak{C}$. Thus Th($\mathfrak{C}$) is the set of sentences which are true in $\mathfrak{C}$. It would be desirable if some of the most important mathematical structures $\mathfrak{C}$ would have first-order theories which are categorical in power $\mathfrak{m}$, where $\mathfrak{m}$ is the cardinality of $\mathfrak{C}$. As it is well known, the first-order theory of $\mathcal{N} = \langle \mathbb{N}, +, \cdot, 0, 1 \rangle$ is not $\aleph_0$-categorical, and the first-order theory of the field $\mathbb{R}$ of real numbers is not categorical in power $2^{\aleph_0}$ (here $\mathbb{N} = \{0, 1, 2, ..\}$ is the set of positive integers). But on the contrary the first-order theory of the field of complex numbers $\mathbb{C}$ is categorical in power $2^{\aleph_0}$. (This fact follows from results of E. Steinitz and A. Tarski).

There is a great difference between the techniques used to investigate (i) $\aleph_0$-categoricity and (ii) categoricity in uncountable powers. In the study of $\aleph_0$-categorical theories the theorem of Engeler, Ryll-Nardzewski and Svenonius plays a dominant rôle. In the study of theories categorical in uncountable powers results and methods due to Baldwin, Lachlan, Morley and Shelah are fundamental.

In this paper we shall confine our attention to $\aleph_0$-categoricity, and in particular to non-abelian groups, whose theories are $\aleph_0$-categorical. It is easily seen that an abelian group G has an $\aleph_0$-categorical theory if and only if G is of bounded order (see e.g. P. Eklof - E. R. Fisher [2] or G. Rosenstein [8]). A classification of non-abelian groups whose theories are $\aleph_0$-categorical is not yet known. In fact very little is known here and the only existing contributions to this field are two papers by J. G. Rosenstein [8], [9]. The aim of this paper is to state some new results concerning non-abelian $\aleph_0$-categorical groups and to disprove some conjectures.

## *§ 2. $\aleph_0$-categorical FC-groups.*

Let G be a multiplicatively written group. If $g \in G$, then $Cl_G(g) = \{x^{-1}gx ; x \in G\}$ is the conjugate-class of g in G. An element g of the Group G is called an FC-element, if $Cl_G(g)$ is finite. Let $FC_1(G)$ denote the set of all FC-elements of G, and let Z(G) be the center of G. Clearly $g \in Z(G)$ if and only if $Cl_G(g)$ consists of only one element: $g \in Z(G) \Leftrightarrow Cl_G(g) = \{g\}$. Since $g \in FC_1(G) \Leftrightarrow$ "$Cl_G(g)$ is finite", one usually calls $FC_1(G)$ the FC-center of G. R. Baer has shown that $FC_1(G)$ is always a characteristic subgroup of G. In analogy with the upper central series one defines (following F. Haimo) the FC-series $FC_\alpha(G)$ (for $\alpha$ an ordinal) as follows: if $\alpha$ is a limit ordinal, then $FC_\alpha(G) = \bigcup_{\beta<\alpha} FC_\beta(G)$, and if $\alpha = \gamma + 1$ then $FC_\alpha(G) = \cup\, FC_1(G/FC_\gamma(G))$, where $\cup X = \{z ; \exists y \in X : z \in y\}$ is the union of X as usual. Let $FCH(G) = \bigcup_\alpha FC_\alpha(G)$, where $\alpha$ runs through all ordinals, then FCH(G) is called the FC-Hypercenter of G. In analogy to the case of the upper central series one defines:

G is an FC-group iff $G = FC_1(G)$,

G is FC-nilpotent iff $G = FC_n(G)$ for some finite number n,

G is FC-hypercentral iff $G = FCH(G)$.

Thus in an FC-group G all elements have *f*inite *c*onjugacy classes. Let us call a group G an IC-group if $Cl_G(g)$ is infinite for every $g \in G$ with $g \neq 1$. Clearly G/FCH(G) is always an IC-group. The letters IC stand for "*i*nfinite *c*onjugacy-classes". Let us mention that G is an FC-group iff G is a normal subgroup in every Ultrapower $G^\omega/J$.

The concept of a BFC-group was introduced by B.H. Neumann. He calls a group G a BFC-group if there is a natural number b such that for every $g \in G$, $Cl_G(g)$ has at most b elements. Thus b is a bound for the cardinalities of all conjugacy-classes in G. Let G' be the commutator-subgroup of G. B. H. Neumann showed that G is a BFC-group if and only if G' is finite (see D. Robinson [7] p. 126).

If the first-order theory of the group G is $\aleph_0$-categorical, then G is uniformly locally finite, i.e. there is a function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that every finite subset $T \subseteq G$ of cardinality n generates a subgroup of cardinality $\leqq f(n)$. We denote by Grp(T) the subgroup of G generated by the subset T. Clearly Grp(T) need not be a normal subgroup of G, and moreover a finite subset T of G need not be contained in a finite normal subgroup of G. But if we assume, that G is an FC-group, then the $\aleph_0$-categoricity of Th(G) implies that every finite subset $T \subseteq G$ of cardinality n is contained in a finite normal subgroup $N \leqq G$ of cardinality at most $f(d \cdot n)$, where d is the cardinality of G' (notice that an FC-group G with $\aleph_0$-categorical theory is necessarily a BFC-group, so that G' is finite). Hence if Th(G) is $\aleph_0$-categorical, then G is locally finite, and if G is moreover FC, then G is locally normal. It is this fact which allows us to say something about the structure of $\aleph_0$-categorical FC-Groups.

We state here without proofs the following results. The proofs will appear elsewhere.

*THEOREM 1:* Let G be an arbitrary group such that Th(G) is $\aleph_0$-categorical. Then the FC-Hypercenter of G is FC-nilpotent: $FCH(G) = FC_n(G)$ for some $n \in \mathbb{N}$, and $FC_1(G)$ is a BFC-group. Moreover all groups $FC_i(G)$ (for $i \in \mathbb{N}$) are definable characteristic subgroups of G.

*THEOREM 2:* Let G be an FC-nilpotent group such that Th(G) is $\aleph_0$-categorical. Then there is a finite normal series $\{1\} = H_0 \leqq H_1 \leqq \ldots \leqq H_m = G$ (i.e. $H_i \trianglelefteq G$ for all $i \leqq m$) such that for all j with $1 \leqq j \leqq m : H_j/H_{j-1}$ is the direct sum of an abelian group $A_j$ and a finite group $F_j$. Moreover all groups $H_j$ are definable characteristic subgroups of G.

Theorem 2 has a number of corrollaries. We mention only the following one:

*Corollary:* Let G be an FC-group such that Th(G) is $\aleph_0$-categorical.

(i) If G is locally nilpotent, then G is nilpotent.
(ii) If G is an Engel-group, then G is nilpotent.
(iii) If G satisfies the normalizer-condition, then G is nilpotent.
(iv) If G is locally solvable, then G is solvable.
(v) If the exponent of G is a prime number, then G is solvable.
(vi) If G has no elements of order 2, then G is solvable.

## § 3. *Description of the groups* $G(p, \leqq)$

Let G be an arbitrary group. If the center Z(G) has finite Index in G, i.e. $[G : Z(G)] < \aleph_0$, then G' is finite (see R. Baer

[1] p. 396, theorem 4). Hence, if G/Z(G) is finite, then G is a BFC-group. We mentioned in section 2 that if Th(G) is $\aleph_0$-categorical and if G is an FC-group, then G is a BFC-group. It had been conjectured, that from the assumption "G is an FC-group such that Th(G) is $\aleph_0$-categorical" it might be possible even to conclude that G/Z(G) is finite ([1a]). If this would be the case, then one could perhaps obtain a complete classification of all FC-groups having an $\aleph_0$-categorical theory by using some results and methods contained in J.G. Rosenstein's paper [8]. However, in the sequel we shall disprove the above mentioned conjecture. We do this by constructing a certain p-group of nilpotency class 2 which is an $\aleph_0$-categorical FC-group with finite center. We shall prove a slightly more general result which says that, if p is an odd prime and G an extra-special p-group of exponent p, then the theory of G is $\aleph_0$-categorical.

*Notation.* Let a, b, c, ... be distinct symbols. A word in the symbols a, b, c, ... is a finite sequence $W = x_1x_2x_3...x_k$ where each of the $x_i$ (for $1 \leqq i \leqslant k$) is one of the symbols a, $a^{-1}$, b, $b^{-1}$, c, $c^{-1}$, ... If a, b, c, ...are distinct symbols and if $W_i$ (for $i \in I$, where I is any index-set) are words in the symbols a, b, c, ..., then $\mathrm{Grp}\{a, b, c, \dots ; \forall i \in I\ (W_i = 1)\}$ denotes the Group H which is generated by the elements a, b, c, ... and has $W_i = 1$ (for $i \in I$) as defining relations (or: relators) (In Magnus-Karras-Solitar [5] p. 4-23, however the notation $H = \langle a, b, c, \dots ; W_i\ (i \in I)\rangle$ is used).

*Definition.* Let $\leqq$ be a linear ordering on the set S and let p be a prime-number. Then $G(p, \leqq)$ denotes the group which has the following presentation:

$$G(p, \leqq) = \mathrm{Grp}\{a, b_i\,;\, i \in S\ \&\ \forall i \in S\ (a^p = b_i^p = 1\ \&\ a \cdot b_i = b_i \cdot a)\ \&\ \forall i \in S\ \forall j \in S\ (i < j \Rightarrow b_j \cdot b_i = a \cdot b_i \cdot b_j)\}.$$

We define ordinal numbers $\alpha$ in the sense of J. von Neumann. We then have $0 = \varnothing$, $1 = \{\varnothing\} = \{0\}, \dots, n+1 = \{0, 1, 2, \dots, n\}, \dots$ and $n+1$ is a linearly ordered set: if $x, y \in n+1$, then $x < y \Leftrightarrow x \in y$. It is convenient to use the following notation: if $\alpha$ is an ordinal number, then

$$G(p, \alpha) = G(p, \leqq), \text{ where } \leqq \text{ is the wellordering on } \alpha.$$

*LEMMA 1:* Let $\leqq$ be a linear ordering on the set S and let p be a prime-number. Then $G(p, \leqq)$ is a group of exponent p.

*Proof.* Since $i < j$ implies $b_j b_i = a b_i b_j$, every element $g \in G(p, \leqq)$ can be written in the following "normal form": (1)

$$g = a^k b_{i_0}^{m(o)} \cdot b_{i_1}^{m(1)} \dots b_{i_n}^{m(n)},$$

such that $i_0 < i_1 < i_2 < \dots < i_n$, $0 \leqq k < p$ and $0 \leqq m(\nu) < p$ for $0 \leqq \nu \leqq n$. Since $x^o = 1$, we may therefore write every element g of $G(p, \leqq)$ in the following form:

$$g = a^k \cdot \prod_{i \in S} b_i^{m(i)},$$

where k and all the m(i) are natural numbers and the formal product is understood to be in the ordering given by $\leqq$. The product of two elements $g_1$ and $g_2$ of $G(p, \leqq)$ can then easily be described as follows: suppose that

$$g_1 = a^u \cdot \prod_{i \in S} b_i^{m(i)}, \quad g_2 = a^v \cdot \prod_{i \in S} b_i^{n(i)},$$

where $u, v, m(i), n(i) < p$, then

$$(*) \quad g_1 : g_2 = a^{u+v+\tau} \cdot \prod_{i \in S} b_i^{m(i)+n(i)},$$

where $\tau = \sum_{i \in S} \delta_i^*$ and $\delta_i = \sum_{i<j} m(j) \cdot n(i)$ and $\delta_i^*$ is the least integer such that $0 \leqq \delta_i^* < p$ and $\delta_i^*$ is congruent to $\delta_i$ modulo p. The formula (*) is nothing else than an iterated application of the rule $i < j \Rightarrow b_j^{m(j)} \cdot b_i^{n(i)} = a^{m(j) \cdot n(i)} \cdot b_i^{n(i)} \cdot b_j^{m(j)}$.

From the formula (*) it follows easily that $g^p = a^o = 1$ for all $g \in G(p, \leqq)$ since all the exponents appearing in the normal form of $g^p$ are congruent 0 modulo p, Q.E.D.

*LEMMA 2:* Let $\leqq$ be a linear ordering on the set S and let p be a prime-number. Put $G = G(p, \leqq)$. Then $G' = \{a^n ; 0 \leqq n < p\} \subseteq Z(G)$. Hence G is nilpotent of class 2. Moreover G is locally finite. (2)

*Proof.* Consider the commutator $[b_i, b_j] = b_i^{-1} b_j^{-1} b_i b_j$. Since $b_i b_j = a^n b_j b_i$ with $n = 1$ if $j < i$, and $n = p - 1$ if $i < j$, it follows that $[b_i, b_j] = a^n$ for some $n \in \mathbb{N}$, $0 \leqq n < p$. From the wellknown commutator identities (see e.g. W.R. Scott [10] p. 56, in particular identity (iv) and (v)) it follows that for all $g_1$ and $g_2$ in $G = G(p, \leqq)$ we have: $[g_1, g_2] = a^k$ for some $k \in \mathbb{N}$, $0 \leqq k < p$. Thus G' is contained in the cyclic group $\{a^n ; 0 \leqq n < p\}$. It is clear, that a belongs to the center of G. Hence $G' \subseteq Z(G)$ and so $G'' = \{1\}$. This means that G is nilpotent of class 2 (i.e. metabelian). Since G' is finite, G is a BFC-group. By lemma 1 both groups, G' and G/G' have exponent p (if they are not trivial). But both groups are abelian and therefore uniformly locally finite. Hence G is uniformly locally finite (compare: D. Robinson [7] p. 35, Theorem 1.45), Q.E.D.

Notice that, if $p = 2$ or $p = 3$, then the locally finiteness of $G = G(p, \leqq)$ follows directly from lemma 1, since then Burnside's problem has a positive solution (see [7] p. 35). In the case $p \geqslant 5$ we need the nilpotency of $G = G(p, \leqq)$ to conclude that G is locally finite. In the next two lemmata we describe the center of the groups $G(p, \leqq)$. We show that only in the case when $\leqq$ is a linear ordering on the finite set S, where S has an odd number of elements, the center of $G(p, \leqq)$ differs from $\{a^n ; 0 \leqq n < p\}$, provided that $p \neq 2$.

*LEMMA 3:* Let $\leqq$ be a linear ordering on the infinite set S and let p be a prime-number. If $p = 2$, then $G(p, \leqq)$ is abelian. If $p \geqslant 3$, then $Z(G(p, \leqq)) = G' = \{a^n ; 0 \leqq n < p\}$.

*Proof.* If $p = 2$, then $g^2 = 1$ for every $g \in G(2, \leqq)$ by lemma 1. Hence $g = g^{-1}$ and therefore $g_1g_2 = (g_1g_2)^{-1} = g_2^{-1}g_1^{-1} = g_2g_1$. Thus $G(2, \leqq)$ is abelian.

Let us assume now that $p \geqslant 3$, and put $G = G(p, \leqq)$. Obviously a is in the center of G and hence $\{a^t ; 0 \leqq t < p\} \subseteq Z(G)$. In order to prove the converse we choose $g \in Z(G)$ arbitrarily. We write g in its normal form:

$$g = a^k \cdot b_{i_1}^{m(1)} \cdot b_{i_2}^{m(2)} \cdot \ldots \cdot b_{i_n}^{m(n)}$$

where $i_1 < i_2 < \ldots < i_n$. We assume that $0 < m(\nu) < p$ for all $1 \leqq \nu \leqq n$. Since S is infinite, there exists $j \in S$ such that $j \notin \{i_1, \ldots, i_n\}$. But $\leqq$ is a linear ordering on S and hence there exists $\mu \in \mathbb{N}$ such that $0 \leqq \mu \leqq n$ and:

$$i_1 < \ldots < i_\mu < j < i_{\mu+1} < \ldots < i_n.$$

Case 1: $\mu = 0$. Then $j < i_1 < \ldots < i_n$. According to the rule $j < i_\nu \Rightarrow b_{i_\nu} b_j = a b_j b_{i_\nu}$ we obtain:

$$gb_j = a^k a^\tau b_j b_{i_1}^{m(1)} \ldots b_{i_n}^{m(n)}, \text{ where } \tau = \sum_{\nu=1}^{n} m(\nu).$$

But $g \in Z(G)$, and hence $gb_j = b_j g = a^k b_j b_{i_1}^{m(1)} \cdot \ldots \cdot b_{i_n}^{m(n)}$. This implies that $\tau \equiv 0$ (mod. p). But also $gb_{i_1} = b_{i_1} g$ and hence $\sum_{\nu=2} m(\nu) \equiv 0$ (mod. p). We conclude that $m_1 \equiv 0$ (mod. p), a contradiction.

Case 2: $\mu \neq 0$. In this case we consider $gb_j = b_j g$ and $gb_\mu = b_\mu g$ in order to conclude $m(\mu) \equiv 0$ (mod. p) in a similar way as above. Thus we get a contradiction also in case 2.

We conclude that necessarily $m(\nu) \equiv 0$ (mod. p) for all $1 \leqq \nu \leqq n$. Hence $g = a^k$. This shows that $Z(G) = \{a^t ; 0 \leqq t < p\}$, Q.E.D.

*LEMMA 4:* Let p be a prime-number and $2 \leqq n \in \mathbb{N}$. If $p \geqq 3$ and if n is even, then $Z(G(p, n)) = \{a^t ; 0 \leqq t < p\}$.

*Proof.* According to our convention $G(p, n) = G(p, \leqq)$, where $\leqq$ is the natural linear ordering on $n = \{0, 1, \ldots, n-1\}$. Suppose there exists an element $z \in Z(G(p, n))$, $z = a^k b_o^{m(o)} \cdot b_1^{m(1)} \cdot \ldots b_{n-1}^{m(n-1)}$, such that $0 \leqq m(\nu) < p$ for all $\nu < n$ and $0 < m(\mu) < p$ for at least one number $\mu$. If there exists a number j with $0 \leqq j < n$ such that $m(j) = 0$ then it follows as in the proof of lemma 3 that $z \notin Z(G(p, n))$, a contradiction to our assumption. Hence if $z = a^k \cdot \prod_{\nu<n} b_\nu^{m(\nu)} \in Z(G(p, n))$ and $m(\mu) \not\equiv 0$ (mod. p) for some $\mu < n$, then $m(\nu) \not\equiv 0$ (mod. p) for all $\nu < n$.

Since $i < j$ implies $b_j b_i = a b_i b_j$ and also $b_i b_j = a^{p-1} \cdot b_j b_i$, we obtain: $z \cdot b_o = a^k \cdot a^{\sigma(o)} \cdot b_o : b_o^{m(o)} \cdot \ldots \cdot b_{n-1}^{m(n-1)}$, where $\sigma(o) = \sum_{1<\nu<n} (m(\nu)$. Further $b_o \cdot z = a^k \cdot b_o \cdot b_o^{m(o)} \cdot \ldots \cdot b_{n-1}^{m(n-1)}$.

Since $z \in Z(G(p, n))$, we obtain $b_o \cdot z = z \cdot b_o$ and hence $\sigma(0) \equiv 0$ modulo p. More generally $z \cdot b_r = b_r \cdot z$ (for $0 \leqq r < n$) yields the following congruences:

$$\sigma(r) = (p-1) \cdot \left(\sum_{\nu<r} m(\nu)\right) + \sum_{\nu=r+1}^{n-1} m(\nu) \equiv 0 \text{ (mod. p)}.$$

Hence $\sigma(r+1) - \sigma(r) = (p - 1 \cdot m(r) - m(r+1) \equiv 0$ (mod. p). This implies $\mu(r+1) + m(r) \equiv 0$ (mod. p) for all $0 \leqq r < n$. But n is odd, and therefore:

$$\sigma(0) = (m(1) + m(2)) + \ldots + (m(n-3) + m(n-2)) + m(n-1)$$
$$= 0 + 0 + \ldots + 0 + m(n-1).$$

Hence $\sigma(0) \equiv 0$ (mod. p) implies $m(n-1) \equiv 0$ (mod. p), a contradiction. Thus all elements of the center have the form $a^t$ for some $t \in \mathbb{N}$, Q.E.D.

The notions of special p-groups and extra-special p-groups are due to P. Hall and G. Higman (see Hall-Higman [4] p. 15). Let G be a finite or infinite group. Then $\Phi(G)$ is the Frattini-subgroup of G (see B.H. Neumann [6]). If H is a finite or infinite p-group, then H is called special, if either H is elementary abelian or H is nilpotent of class 2 and $H' = Z(H) = \Phi(H)$ is elementary abelian. A p-group H is called extra-special if H is special and H' is cyclic of order p. The reader may find information on finite extra-special p-groups in Gorenstein [3] pp. 183-208, Huppert ("Endliche Gruppen I", Berlin 1967) and L. Dornhoff ("Group representation theory, part A", New York 1971).

*LEMMA 5:* Let p be an odd prime and $\leqq$ a linear ordering on the non-empty set S. If the cardinality of S is not a finite odd number, then $G(p, \leqq)$ is an extra-special p-group.

*Proof.* It follows from lemma 1 that $G(p, \leqq)$ is a p-group of exponent p. Lemma 2 shows that the commutator-subgroup of $G(p, \leqq)$ is cyclic of order p. Lemmata 3 and 4 say, that $Z(G) = G'$, where $G = G(p, \leqq)$. Further, by lemma 2 G is nilpotent of class 2. It remains to show that $Z(G) = \Phi(G)$. Since G is nilpotent, we have $G' = \{a^t ; 0 \leqq t < p\} \subseteq \Phi(G)$ (see e.g. W.R. Scott [10] p. 160). For the converse choose $g \in \Phi(G)$, $g = a^k \cdot b_{i_n}^{m(1)} \cdot \ldots \cdot b_{i_n}^{m(n)}$, where $i_1 < i_2 < \ldots < i_n$, $0 < m(\nu) < p$ for all $1 \leqq \nu \leqq n$, and assume that $n \geqq 1$. Define $E = \{b_j ; j \in S \;\&\; j \neq i_1\}$. Since $m(1) \not\equiv 0$ modulo p, clearly $E \cup \{g\}$ generates $G = G(p, \leqq)$, but G is not generated by E alone. Thus $g \notin \Phi(G)$, a contradiction. This shows, that all elements of $\Phi(G)$ are of the form $a^t$ for some $t \in \mathbb{N}$, Q.E.D.

Lemma 5 shows, that if $\leqq$ is a linear ordering on the countably infinite set S, and if p is a prime, $p \neq 2$, then $G(p, \leqq)$ is the union of an ascending sequence of finite extraspecial

S and let $H_k$ be the subgroup of $G(p, \leqq)$ generated by $\{b_{i_t};\ t < 2k\}$. Then by lemma 5 all the groups $H_k$ (for $k \geqq 1$) are extra-special, and $H_1 \subseteq H_2 \subseteq \dots$, $G(p, \leqq) = \bigcup_k H_k$. Finite extraspecial groups are the central product of copies of G(p, 2). Taking all these facts together we are able to show in the next section, that $Th(G(p, \leqq))$ is $\aleph_0$-categorical.

We note that if n is odd and if p is a prime, then G(p, n) is not an extra-special p-group. In fact G(p, 1) is isomorphic to $Z(p) \oplus Z(p)$, where Z(p) is the cyclic group of order p. If $n \geqq 3$ and if n is odd, then $\{a^t;\ 0 \leqq t < p\}$ is a proper subset of the center of G(p, n). In fact, according to our convention (see § 3), G(p, n) is generated by $\{a, b_i;\ i \in n\}$. Let $\sigma(i)$ be 1 if i is odd, and let $\sigma(i)$ be $p-1$ if i is even. Then the following element

$$\sum_{i=0}^{n-1} b_i^{\sigma(i)}$$

belongs to the center of G(p, n), but is different from all powers of a. Hence, if $G = G(p, n)$, then $G' \neq Z(G)$ and G is not extra-special.

## § 4. *The $\aleph_0$-categoricity of the groups* $G(p, \leqq)$.

Let p be a prime-number, $p \neq 2$. If H is a finite extra-special p-group and if the exponent of H is p, then H has cardinality $p^{1+2r}$ for some integer $r \in \mathbb{N}$, $r \geqq 1$. Moreover for each integer $r \in \mathbb{N}$, $r \geqq 1$, there is one and only one extra-special p-group H such that H has exponent p and cardinality $p^{1+2r}$ (see e.g. Gorenstein [3] p. 204). The following Lemma is implicitely contained in Gorenstein [3]. Notice that in the argument below, the groups $A_n$ have to be finite, since otherwise we do not know how to prove that $H = A_1 \cdot C_H(A_1)$, $B_n = A^*_{n+1} \cdot C_{B_n}(A^*_{n+1})$, where $B_{n+1} = C_{B_n}(A^*_{n+1})$. However the groups $B_n$ need not be finite. This is the reason, why we can prove Lemma 6 not only for finite groups H but also for countably finite groups.

*LEMMA 6:* Let p be a prime, $p \neq 2$, and let H be a non-abelian, at most countable group such that H has exponent p, $H' = Z(H)$ is cyclic and $H/Z(H)$ is elementary abelian. Then there exists a chain of normal subgroups $A_n$ $(1 \leqq n \in \mathbb{N})$ : $A_1 \subseteq \dots \subseteq A_n \subseteq A_{n+1} \subseteq \dots$ such that $H = \cup \{A_n ; 1 \leqq n \in \omega\}$ and each $A_n$ is an extra-special p-group.

*Proof.* Since H is either finite or countably-infinite, we may enumerate the elements of H (possibly with repetitions). Thus $H = \{h_i ; i \in \mathbb{N}\}$, where $h_i = h_j$ with $i \neq j$ is possible. Let $i_0$ be the least positive integer such that $h_{i_0} \notin Z(H)$. Put $x = h_{i_0}$. Hence there exists $y \in H$ such that $z = [x, y] \neq 1$, where $[x, y]$ is the commutator of x and y. Let $A_1$ be the subgroup of H which is generated by $\{x, y, z\}$. Clearly $H' = Z(H)$ implies $H'' = \{1\}$ and hence H is nilpotent. Since H has a finite exponent, H is locally finite (see D. Robinson [7] p. 35). Hence $A_1$ is a finite p-group. p-groups of cardinality $p^2$ are abelian. Hence $A_1$ must have at least $p^3$ elements. But $1 \neq z = [x, y] \in H'$, and as $H'$ is a cyclic group of order p, $H'$ is generated by z. Thus $H' \subseteq A_1'$. But $A_1 \subseteq H$ and therefore $A_1' \subseteq H'$. This shows that $H' = A_1'$. Thus $A_1/A_1' = A_1/H'$ is a subgroup of the elementary abelian p-group $H/H'$. Thus $A_1/A_1'$ has cardinality $p^2$. Since $A_1'$ has cardinality p it follows that $A_1$ has cardinality $p^3$. This implies that $A_1$ is an extra-special p-group of exponent p. Thus $Z(H) = H' = A_1' = Z(A_1)$. From $H' \subseteq A_1 \subseteq H$ it follows that $A_1$ is a normal subgroup of H.

Let $B_1$ be the centralizer (8) of $A_1$ in H, $B_1 = C_H(A_1)$. It follows that $H = A_1 \cdot B_1 = \{ab ; a \in A_1 \text{ and } b \in B_1\}$ (use Gorenstein [3], p. 195, lemma 4.6). We show that $B_1$ has similar properties as H. If $f \in Z(B_1)$, then $f \in B_1 = C_H(A_1)$ and so $fg = gf$ for all $g \in A_1$. Hence it follows from $H = A_1 \cdot B_1$ that $f \in Z(H)$. Thus $Z(B_1) \subseteq Z(H)$. But $z = [x, y] \in H' = Z(A_1)$

implies $z \in C_H(A_1) = B_1$; from $z \in H' = Z(H)$ it follows therefore $z \in Z(B_1)$. But $Z(H)$ is generated by z. Hence $Z(H) = Z(B_1)$ is cyclic.

If $B_1 \neq Z(B_1)$, then there are $u \in B_1$ and $v \in B_1$ such that $w = [u, v] \neq 1$. Hence $w \in B'_1$ and $w \in H'$. But $H'$ is cyclic and therefore generated by w. From $B'_1 \subseteq H'$ it follows hence that $H' = B'_1$. Altogether we have shown that $B'_1 = H' = Z(H) = Z(B_1)$, thus $B'_1 = Z(B_1)$. Further $B_1/Z(B_1) = B_1/Z(H)$ is a subgroup of the elementary abelian subgroup $H/Z(H)$. Thus $B_1/Z(B_1)$ too is elementary abelian.

Now we are in a position, to define $A_2$.
If $B_1 = Z(B_1)$, then $H = A_1$, and we put $A_2 = A_1$, and more generally $A_n = A_1$ for all $1 \leqq n \in \mathbb{N}$.
If $B_1 \neq Z(B_1)$, then $B_1$ satisfies the same requirements as H does. Hence, if $i_1$ is the least positive integer, such that $h_{i_1} \in B_1$ and $h_{i_1} \notin Z(B_1)$ then there is an element $r \in B_1$ such that $s = [h_{i_1}, r] \neq 1$. We let $A_2^*$ be the subgroup of $B_1$ generated by $\{s, r, h_{i_1}\}$. As in the case of $A_1$ it follows that $A_2^*$ is an extra-special p-group of cardinality $p^3$. Putting $B_2 = C_{B_1}(A_2^*)$, it follows as above, that $B_1 = A_2 \cdot B_2$. Put $A_2 = A_1 \cdot A_2^*$. As $A_2$ is the central product of $A_1$ and $A_2^*$, where both, $A_1$ and $A_2^*$, are extra-special p-groups of power $p^3$ and exponent p, it follows that $A_2$ is an extra-special p-group of cardinality $p^5$ and of exponent p.

Continuing this argument, we obtain the desired chain of normal subgroups $A_n$, each being extra-special. It follows from the choice of the elements $h_{i.}$, that H is the union over all these groups $A_n$ (for $1 \leqq n \in \mathbb{N}$). Q.E.D.

*COROLLARY:* Let p be a prime, $p \neq 2$, and let H be a non-abelian at most countable group such that H has exponent p, $H' = Z(H)$ is cyclic and $H/Z(H)$ is elementary abelian. Let $D_1$ and $D_2$ be finite subgroups of H such that both, $D_1$ and $D_2$ be finite subgroups of H such that both, $D_1$ and $D_2$ are extra-special. If $D_1$ and $D_2$ are equipotent, then every isomorphism $\varphi$ from $D_1$ onto $D_2$ can be extended to an automorphism of H.

*Proof.* As it was shown in Lemma 6, $H = D_1 \cdot C_H(D_1)$ and also $H = D_2 \cdot C_H(D_2)$. Clearly, if $H = D_1 = D_2$ we do not have anything to prove. Assume that $H \neq D_1$. Then also $H \neq D_2$. Put $E_1 = C_H(D_1)$ and $E_2 = C_H(D_2)$. It was shown in Lemma 6 that $E_1$ and $E_2$ satisfy the same requirements as H does. Hence $E_1$ is the union of extra-special p-groups $A_n$ $(1 \leqq n \in \mathbb{N})$ and similarly $E_2$ is the union of extra-special p-groups $A^+_n$ $(1 \leqq n \in \mathbb{N})$. Here each $A_n$ and each $A^+_n$ is the central product of copies of M(p), where M(p) is the extra-special p-group of exponent p and cardinality $p^3$ (thus $M(p) = G(p, 2)$). In particular, the construction of the groups $A_n$ and $A^+_n$ was done in such a way, that $A^+_{n+1} \cong A_n \cdot M(p)$ and $A^+_{n+1} \cong A_n \cdot M(p)$ (the product here is always a central product). We may therefore construct an isomorphism $\psi$ from $E_1$ onto $E_2$ as follows: since $A_1 \cong M(p) \cong A^+_1$, let $\psi_1$ be any isomorphism from $A_1$ onto $A_2$ such that the restriction of $\psi_1$ to the center of $A_1$ coincides with the restriction of $\varphi$ to the center of $D_1$ (here $\varphi : D_1 \rightarrow D_2$ is the given isomorphism. Notice that $Z(D_1) = Z(D_2) = Z(E_1) = Z(A_n) = Z(E_2) = Z(A^+_n)$ for all $1 \leqq n \in \mathbb{N}$).

Since $A_2$ $A_1 : M(p) \cong M(p) : M(p)$, $A_2{}^+ \cong M(p) : M(p)$, and more generally $A_n \cong M(p)^n$, $A^+_n \cong M(p)^n$, we can extend $\psi_1$ step by steps to isomorphisms $\psi_n$ from $A_n$ onto $A^+_n$. If $\psi$ is the "limit" of these $\psi_n$ 's, that is $\psi = \cup \{\psi_n ; 1 \leqq n \in \mathbb{N}\}$, then $\psi$ maps $E_1$ isomorphically onto $E_2$. Further $\psi$ and $\varphi$ coincide on $Z(D_1)$. We obtain an isomorphism $\overline{\varphi}: H \rightarrow H$ which extends $\varphi$ as follows. If $x \in H$, then $x = d \cdot e$ for some $d \in D_1$ and some $e \in E_1$. Define $\overline{\varphi}$ by the stipulation $\overline{\varphi}(x) = \varphi(d) . \psi(e)$. The definition does not depend on the particular choice of d and e. Namely, if $x = d \cdot e = u \cdot v \in H = D_1 \cdot E_1$ with $d, u \in D_1$, and $e, v \in E_1$, then $u^{-1} \cdot d = v \cdot e^{-1} \in D_1 \cap C_H(D_1)$, and hence $u^{-1} \cdot d = v \cdot e^{-1} \in Z(D_1)$. Thus $\varphi(u)^{-1} \cdot \varphi(d) = \varphi(u^{-1} \cdot d) = \psi(v \cdot e^{-1}) = \psi(v) \cdot \psi(e)^{-1}$ and hence $\varphi(d) \cdot \psi(e) = \varphi(u) \cdot \psi(v)$. Thus $\overline{\varphi}$ is well-defined, Q.E.D.

*LEMMA 7:* Let p be a fixed prime, $p \neq 2$, and let G and H be two countably-infinite extra-special p-groups such that both, G and H have exponent p. Then G and H are isomorphic.

*Proof.* By the definition $G' = Z(G)$ where $G'$ is cyclic. Clearly $G/\Phi(G)$ is elementary abelian (to see this use theorem 7.3.4 in W.R. Scott [10] p. 160 and notice that Scott uses the finiteness of G only to conclude that G is nilpotent. But in our case G is already nilpotent). But $\Phi(G) = G' = Z(G)$ and so $G/Z(G)$ is elementary-abelian. Similarly $H/Z(H)$ is elementary abelian. By lemma 6 G is the union of an ascending chain of finite normal subgroups $A_n$ (for $1 \leqq n \in \mathbb{N}$) such that $A_n$ is an extra-special p-group of cardinality $p^{1+2n}$. Similarly H is the union of an ascending chain of extra-special p-groups $D_n$ (for $1 \leqq n \in \mathbb{N}$) where $D_n$ has cardinality $p^{1+2n}$. $A_n$ and $D_n$ have exponent p and are hence isomorphic. Let $\varphi_n$ be an isomorphism from $A_n$ onto $D_n$. We define a sequence of mappings $\sigma_n$ from $A_n$ onto $D_n$ as follows. Put $\sigma_1 = \varphi_1$. Suppose that $\sigma_n$ maps $A_n$ isomorphically onto $D_n$. Then $\varphi_{n+1}(A_n) = K_n$ is a subgroup of $D_{n+1}$ and $D_n$ and $K_n$ are isomorphic subgroups. In fact, if $\tau_n$ denotes the restriction ([4]) of $\varphi_{n+1}$ to $A_n$,

$$\tau_n = \varphi_{n+1} \restriction A_n = \{\langle x, y\rangle ; x \in A_n \text{ and } \varphi_{n+1}(x) = y\},$$

then $\sigma_n(\tau_n^{-1}(K_n)) = D_n$. By the corollary to lemma 6, $\sigma_n \circ \tau_n^{-1}$

can be extended to an automorphism $\alpha_{n+1}$ of $D_{n+1}$. Define $\sigma_{n+1}$ as follows: $\sigma_{n+1} = \alpha_{n+1} \circ \varphi_{n+1}$. Clearly $\sigma_{n+1}$ extends $\sigma_n$. It follows from $G = \bigcup \{A_n ; 1 \leqq n \in \mathbb{N}\}$ and $H = \bigcup \{D_n ; 1 \leqq n \in \mathbb{N}\}$ that $\delta = \bigcup \{\sigma_n ; 1 \leqq n \in \mathbb{N}\}$ is an isomorphism from G onto H, Q.E.D.

*COROLLARY 1:* Let H be a countably-infinite extra-special p-group such that H has exponent p. Then H is isomorphic with $G(p, \omega)$.

This follows immediately from lemma 5 and lemma 6. Notice that $\omega$ is as usual the first infinite ordinal; thus $\omega = \mathbb{N}$.

*COROLLARY 2:* Let $S_1$ and $S_2$ be countably-infinite sets and let $\leqq$ be a linear ordering on $S_1$ and $\preccurlyeq$ a linear ordering on $S_2$. Then the groups $G(p, \leqq)$ and $G(p, \preccurlyeq)$ are isomorphic. (5)

*THEOREM 3:* Let p be a prime and let $T_p$ be the first-order theory of $G(p, \omega)$. Then $T_p$ is $\aleph_0$-categorical.

*Proof.* $T_p = Th(G(p, \omega))$, i.e. $T_p$ is the set of all sentences formulated in the first-order language of group-theory, which are true in the group $G(p, \omega)$. We have to show that when H is a countably-infinite model of $T_p$, then $H \cong G(p, \omega)$. In fact $H \models T_p$ means, that H and $G(p, \omega)$ are elementary equivalent: $H \equiv G(p, \omega)$. Put $G = G(p, \omega)$. The center of H is clearly first-order definable. Since $Z(G)$ is cyclic of order p, it follows from $H \equiv G$, that also $Z(H)$ is cyclic of order p. Further $G \models \forall x \forall y\, (x^{-1}y^{-1}xy \in Z(G))$. Thus $H \equiv G$ implies $H' \subseteq Z(H)$. On the other hand $G' = Z(G)$, that is G satisfies the sentence $\exists x \exists y\, (1 \neq x^{-1}y^{-1}xy)$, and hence the same is true in H. Thus $H' \neq \{1\}$. But $H' \subseteq Z(H)$ and $Z(H)$ is cyclic. Thus $H' = Z(H)$. $G/Z(G)$ is an infinite elementary abelian p-group. Since $Z(G)$ is first-order definable, the property that $G/Z(G)$ is elementary abelian can be described by an infinite set of first-order sentences (see J.G. Rosenstein [8] p. 441) (here we have only to express that $G/Z(G)$ is infinite, that $G/Z(G)$ is abelian and that every element of G/Z(G) has order p). It follows from H ≡ G that H satisfies the same set of sentences. Hence also H/Z(H) = H/H' is elementary abelian. Thus H is extra-special and corollary 1 implies that H and G are isomorphic, Q.E.D.

The set $T_p$ has a recursive set of axioms, which we describe now. Notice that we do not refer to the Frattini-subgroup Φ(H), since Φ(H) is in general not first-order definable. We refer instead to the factor-group H/Z(H).

*DEFINITION.* For a prime p, p ≠ 2, let $\Sigma_p$ be the set which consists of the following axioms:

(i) the axioms for group theory,

(ii) $$\exists x_1 \ldots \exists x_p \bigwedge_{i \neq j} (x_i \neq x_j) \wedge \bigwedge_{i=1}^{p} \forall y (y x_i = x_i y) \wedge$$

$$\wedge \forall z (\forall y (zy = yz) \Rightarrow \bigvee_{i=1}^{p} {}^{!}x = z)\Big)$$

(iii) $\forall x \forall y \forall z\, (zx^{-1}y^{-1}xy = x^{-1}y^{-1}xyz)$,

(iv) $\exists x \exists y\, (1 \neq x^{-1}y^{-1}xy)$,

(v) $\forall x\, (x^p = 1)$,

(vi) for every $1 < k \in \mathbb{N}$ the following sentence:

$$\exists x_1 \ldots \exists x_k \Big( \bigwedge_{i \neq j} (\forall z (\forall y (zy = yz) \Rightarrow x_i \neq z x_j))\Big)$$

Axiom (ii) says that the center is a cyclic group of order p. Axiom (iii) says that the derived group is contained in the center and (i) says that the derived group is not trivial. Hence (ii), (iii) and (iv) imply that the center coincides with the derived group. Axiom (vi) says that the factor group modulo the center is infinite. We should perhaps say, that $\bigwedge$ is a symbol for conjunction. Thus $\bigwedge_{i=1}^{p} \Phi_i$ is the same as $\Phi_1 \wedge \Phi_2 \wedge \ldots \wedge \Phi_p$ (where ∧ means "and"). Since all the axioms (i), ..., (vi) are true in G(p, ω), he set $\Sigma_p$ is consistent. If H is a model of $\Sigma_p$, then H is an infinite group (this follows from (i) and (vi): take $z = 1$), $Z(H) = H'$ is cyclic of order p (by (i), (ii), (iii) and (iv)), H has exponent p (by (v)), and by (vi) $H/Z(H)$ is infinite. But $H/Z(H)$ is a homomorphic image of H, where H has exponent p. Thus $H/Z(H)$ has exponent p. But as $H' = Z(H)$, and $H/H'$ is always abelian, $H/H(Z)$ is an elementary abelian p-group. Thus if $H \models \Sigma_p$ and H is countably infinite, then (since obviously $H' = \Phi(H)$ follows) by corollary 1 to lemma 7, H and $G(p, \omega)$ are isomorphic. In particular H and $G(p, \omega)$ are elementarily equivalent. Thus $\Sigma_p$ axiomatizes $T_p$, and $\Sigma_p$ is a complete theory. Clearly $T_p$ is not finitely axiomatizable (any finite subset of $\Sigma_p$ has a finite model, $G(p, n)$ for some $n \in \mathbb{N}$). $T_p$ is clearly decidable.

Theorem 3 has the following consequence: (6)

(*) From the assumption, that G is a nilpotent p-group of exponent p, such that G' is cyclic of order p (and hence a BFC-group) and categorical in $\aleph_0$, we cannot conclude that the center of G is infinite.

The groups $G(p, \omega)$, where p is an odd prime, may also serve to disprove another conjecture concerning categoricity in $\aleph_1$. We shall show in the next section, that $T_p = Th(G(p, \omega))$ is not stable and hence not $\aleph_1$-categorical. But clearly $G(p, \omega)/Z(G(p, \omega))$ is categorical in power $\aleph_0$ and $\aleph_1$. Hence we conclude:

(**) From the assumption that a nilpotent p-group G is the central-extension of a finite group by an $\aleph_1$-categorical group one cannot conclude, that G is $\aleph_1$-categorical.

## § 5. *The groups* $G(p, \omega)$ *are not* $\aleph_1$*-categorical.*

M. Morley showed that a countable $\aleph_1$-categorical theory T is totally-transcendental, and hence $\omega$-stable. A general theory for stable and unstable theories has been developed by S. Shelah. The theory T is unstable, if for every infinite cardinal $\lambda$ we have that T is not $\lambda$-stable.

*LEMMA 8:* For every odd prime p, $T_p = Th(G(p, \omega))$ is unstable.

*Proof.* Put $G = G(p, \omega)$. According to the definition, G is generated by $\{a, b_i ; i \in \omega\}$. In particular $i < j$ implies $b_j b_i = ab_i b_j = b_i b_j a$ (since $a \in Z(G)$). Hence $i < j$ implies $[b_j, b_i] = a$. Let $\Psi(u, v, x, y)$ be the following formula (of the first-order language of group theory):

$$\Psi(u, v, x, y) \Leftrightarrow [x, u] = v = y.$$

For $i \in \omega$ let $\mathfrak{a}_i$ denote the ordered pair of $b_i$ and a, thus $\mathfrak{a}_i = \langle b_i, a\rangle$. We write $\Psi(\mathfrak{a}_i, \mathfrak{a}_j)$ instead of $\Psi(b_i, a, b_j, a)$. We claim that

$\forall i \in \omega\ \forall j \in \omega((G \models \Psi(\mathfrak{a}_i, \mathfrak{a}_j)) \Leftrightarrow i < j)$.

In fact, if $i < j$, then $[b_j, b_i] = a$ is true in G, and hence $\Psi(b_i, a, b_j, a)$ holds in G.

In order to prove the converse, assume that $G \models \Psi(b_i, a, b_j, a)$. $\leqq$ is the usual linear ordering on $\omega$. Hence either $i = j$, or $i < j$ or $j < i$. Clearly i is different from j since otherwise $a = 1$ would follow, a contradiction. Assume that $j < i$. Then as we have seen above: $a = [b_i, b_j]$. But $[b_i, b_j] = [b_j, b_i]^{-1}$, and $G \models [b_j, b_i] = a$ would imply $a = a^{-1}$, a contradiction. Thus we have proved our claim.

But now from S. Shelah [11], theorem 2.13, it follows that $T_p = Th(G)$ is unstable, Q.E.D.

*THEOREM 4:* The theory $T_2 = Th(G(2, \omega))$ is $\aleph_0$-categorical and $\aleph_1$-categorical (and hence categorical in all infinite powers). If p is an odd prime, then $T_p = Th(G(p, \omega))$ is $\aleph_0$-categorical but not $\aleph_1$-categorical; however $Th(G(p, \omega)/Z(G(p, \omega)))$ is $\aleph_0$-categorical and $\aleph_1$-categorical.

*Proof.* $G(2, \omega)$ and $G(p, \omega)/Z(G(p, \omega))$ are elementary abelian q-groups (with $q = 2$ or $q = p$). It is well known that such groups are categorical in all infinite powers $\mathfrak{m}$. Since categoricity in $\aleph_1$ implies $\lambda$-stability for *all* infinite cardinals $\lambda$, the claim follows from lemma 8 and theorem 3, Q.E.D.

In theorem 4 a rather amazing fact in stated: there are groups G such that Th(G) is not $\aleph_1$-categorical, but for some finite cyclic normal subgroups D, Th(G/D) is $\aleph_1$-categorical !

The groups $G(p, \leqq)$ can be considered as symplectic spaces over the finite Galois-field $F_p$ of p elements (namely, if x and y are in $G(p, \leqq)$, then put $f(x, y) = k \in F_p$ iff $[x, y] = a^k$). If one adopts this point of view, then lemma 6 and its corollary can also be proved by applying some results and methods of E. Witt.

An abelian group A is $\aleph_0$-categorical iff A is the direct sum of finitely many vectorspaces over finite fields. It is then perhaps not too much surprising, that the groups $G(p, \leqq)$ are $\aleph_0$-categorical. But a vectorspace over $F_p$ is also $\aleph_1$-categorical. It is perhaps surprising, that $G(p, \leqq)$ is not $\aleph_1$-categorical !

Open problem: do there exist infinite partial orderings $\leqq$ such that the first-order theory of $G(p, \leqq)$ is $\aleph_1$-categorical ? If the answer is positive, then it would be interesting to classify those partial orderings $\leqq$ for which $Th(G(p, \leqq))$ is $\aleph_1$-categorical.

## § 6. *Group-theoretical consequences*

*LEMMA 9:* Let H be an extra-special p-group and assume that H has exponent p. If H is finite or countable, then H has the presentation:

$$Grp\{a, b_i\,;\ \forall i \in S(a^p = b_i^p = 1 \ \&\ ab_i = b_i a) \ \&\ \forall\, i, j \in S \ ([b_j, b_i] = a \Leftrightarrow i < j)\},$$

for some index-set S and some linear-ordering $\leqq$ on S.

Lemma 9 immediately follows from lemma 7 and its corollaries. In fact the second corollary of lemma 7 says, that the presentation does not depend on the order-type (if H is at most countable !). Notice that lemma 9 is not needed in the proof of $\aleph_0$-categoricity (theorem 3). However lemma 9 is essentially used in the proof of non- $\aleph_1$-categoricity (theorem 4).

*LEMMA 10:* (i) Let G and H be extra-special p-groups and assume that both have exponent p and both have the same cardinality. If G is finite or countable, then $G \cong H$.

(ii) For every uncountable cardinal $\mathfrak{m}$ there are exactly $2^{\mathfrak{m}}$ pairwise non-isomorphic extra-special p-groups of exponent p which have cardinality $\mathfrak{m}$.

The proof of (i) follows from lemma 7 if G is countable. For finite groups G and H lemma 10 (i) was known. It follows from lemma 8 that $T_p = Th(G(p, \omega))$ is unstable. Therefore by a result of S. Shelah [12] (see also Shelah [11], p. 283) $T_p$ has exactly $2^{\mathfrak{m}}$ non-isomorphic models of power $\mathfrak{m}$. Models of $T_p$ are models of $\Sigma_p$, and models of $\Sigma_p$ are extra-special p-groups of exponent p. Thus lemma 10 is proved.

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## FOOTNOTES

(0) This paper is a modified version of our paper read at the conference on Model-Theory at Louvain-La-Neuve in spring 1975.

(1) For the convenience of the printer we write m(ν) instead of $m_\nu$. Hence m(n — 1) is a number m with index n — 1 and in particular m(n — 1) should not be read as multiplication of m with n — 1. We use a multiplication-dot when multiplication is meant.

(1a) If Th(G) is $\aleph_0$-categorical and G is a direct sum of finite groups, then [G : Z(G)] is finite. Hence in this particular case the conjecture is true.

(2) G′ is the commutator-subgroup of G (G′ is sometimes also called the derived subgroup of G). Further x ≡ y (mod. p) means that x and y are congruent modulo p.

(3) The centralizer of $A_1$ in H is $C_H(A_1) = \{b \in H; \forall a \in A_1 (ba = ab)\}$.

(4) In order to avoid misunderstandings we emphasize that $\{a, b, c, \ldots\}$ is merely the set which has $a, b, c, \ldots$ as elements. $<a, b>$ is the ordered pair of $a$ and $b$, thus $<a, b> = \{\{a\}, \{a, b\}\}$. In particular $<a, b>$ is *not* the subgroup generated by $a$ and $b$, in distinction to the notation in [3], [5], [7] and [10].

(5) The theory of dense linear orderings without first and without last element is $\aleph_0$-categorical (G. Cantor). Let $\eta_0$ denote the order type of the set of rationals. Since the definition of $G(p, \eta_0)$ is closely connected with the linear-ordering of type $\eta_0$ it seems reasonable to conjecture that $Th(G(p, \eta_0))$ is $\aleph_0$-categorical. The theory of the linear ordering on ω is not $\aleph_0$-categorical, and hence one might conjecture that Th(G(p, ω)) is not $\aleph_0$-categorical. However corollary 2 states that the isomorphism-type of G(p, <) (if G(p, <) is at most countable) does not depend on the isomorphism-type of the linear-ordering used to define G(p, <). We do not know whether a similar result holds for uncountable groups G(p, <).

(6) A group H is called $\aleph_\alpha$-categorical if Th(H) is $\aleph_\alpha$-categorical.