IDEALS IN SOME NONSTANDARD DEDEKIND RINGS

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Introduction

We will discuss the classification of maximal and prime ideals and the associated residue class rings in a class of nonstandard Dedekind rings. This material was treated previously in [1] in the context of nonstandard rings of algebraic integers. In the present article we adopt a more general perspective. We intend to discuss the following classes of nonstandard Dedekind rings:

1. "*Strong*" nonstandard Dedekind rings, in which nonstandard higher order structure is present (these are the nonstandard Dedekind rings of nonstandard analysis).
2. Nonstandard rings of algebraic integers as in [1]. These are defined as follows: let $\mathbb{Z}^*$ be an elementary extension of the ordinary ring of integers $\mathbb{Z}$, and let K be a finite demensional extension field of $\mathbb{Q}^*$ (the quotient field of $\mathbb{Z}^*$). Let A be the integral closure of $\mathbb{Z}^*$ in K. A is called a *nonstandard ring of algebraic integers.*
3. The function field analog of the preceding case: let R be the ring k[x] of polynomials over a finite field k, let R^* be an elementary extension of R, and let A be the integral closure of R^* in a finite separable algebraic extension K of the quotient field of R^*.
4. Existentially complete models of arithmetic. These are "*weak*" nonstandard models, satisfying a fragment of complete arithmetic. See [4].

Our methods involve heavy use of a function exp(i, j) defined for finitely generated ideals i, j with j maximal, and satisfying:

$$i = \prod_j j^{\exp(i,j)}$$

in some standard or nonstandard sense. We will see that such a function is available in cases 1.4 above, either by assumption (case 1) or by the use of definability theorems (case 2: [2]; case 3: [5,6]; case 4: [3,4]).

Our treatment of these four cases will be quite uniform, and we accordingly adopt an axiomatic approach, in which the existence of a suitable function exp is assumed. An important feature of our axiomatic approach will be the assumption that a nonstandard Chinese Remainder Theorem is valid. Generally speaking, a *Chinese Remainder Theorem* for a ring R is an assertion to the effect that for suitable sets C of maximal ideals and for suitable functions h from C into R there is an element $a \in R$ satisfying the simultaneous congruences:

$$a \equiv h(x) \quad \mod (x^n) \text{ for } x \in C.$$

Here n is an arbitrarily large exponent (possibly nonstandard).

As we will see in § 2, the following Chinese Remainder Theorems are available:

0. In standard Dedekind rings: C is an arbitrary finite set and h is an arbitrary function.
1. In strong nonstandard Dedekind rings: C is a *-finite internal set and h is any internal function.

2-3. In nonstandard rings of algebraic integers (or in the function field analog): C is a definable bounded set of finitely generated maximal ideals and h is a definable function.

4. In existentially complete models of arithmetic: C is a bounded recursive set of principal maximal ideals and h is a recursive function.

In § 1 we carry out the investigation of prime and maximal ideals in our axiomatic setting. Thus throughout § 1 we assume we have a function exp such that the Chinese Remainder Theorem holds with respect to a suitable class of functions on R. In § 1.5 these functions will be used to form ultrapowers (e.g. definable ultrapowers or recursive ultrapowers). In § 2 we will review our intended examples (cases 1-4) individually, showing in each case how the considerations of § 1 apply.

§ 1. *Prime and Maximal Ideals in Rings with Chinese Remainder Theorem*

1.1. An Axiomatic Setting

We assume given an integral domain R. Let $\mathcal{M}$ be the space of maximal ideals of R and let $\mathcal{F}$ be the subspace of $\mathcal{M}$ consisting of the finitely generated maximal ideals. We assume:
Ho: There is an integer k *such that every finitely generated ideal is generated by* k *elements.*

In practice k will of course be 1 or 2. For our purposes the value of k is irrelevant.

Our next hypothesis concerns the topology on $\mathcal{M}$ and $\mathcal{F}$. A canonical basis of closed sets for the topology of $\mathcal{M}$ is provided by the sets $\{C'_i\}$ which are defined by:

$C'_i = \{M \in \mathcal{M} : i \subseteq M\}$ when i is a nonzero finitely generated ideal.

Similarly a canonical basis for the relative topology on $\mathcal{F}$ is provided by the sets

$$C_i = C'_i \cap \mathcal{F}$$

as i varies over all nonzero finitely generated ideals.
Notice that the basis $\{C_i\}$ forms a lattice, since

$$C_i \cap C_j = C_{(i,j)}$$

$$C_i \cup C_j = C_{i.j}.$$

In the case of standard Dedekind rings the sets C_i are arbitrary finite sets. We will assume:

H1 : 1. *The lattice* $\{C_i : O \neq i$, i *finitely generated*$\}$ *is relatively complemented.*

2. *Every finitely generated ideal is contained in a maximal finitely generated ideal (compare* H 2.5 *below).*

It follows from H1 that the map $C'_i \to C_i$ is an isomorphism of lattices. It also follows from H1 that $\mathcal{F}$ is dense in $\mathcal{M}$ since every basic open set is an union of basic closed sets.

Our next hypothesis concerns an exponential function exp as described in the introduction.

H2 : *There is a fixed ordered abelian semigroup* $\langle S, \oplus, < \rangle$ *and a partial function* $\exp : R^{2k} \to S$ *satisfying:*

0. $\exp(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k)$ *is defined for* $\overline{\alpha} \in R^k, \overline{\beta} \in R^k$ *iff* $(\overline{\alpha}) \neq O$ *and* $(\overline{\beta}) \in \mathcal{F}$.
1. S *contains an identity* O, *no negative element, and a minimal positive element 1.*
2. $\exp(\overline{\alpha}, \overline{\beta})$ *depends only on the ideals* $(\overline{\alpha}), (\overline{\beta})$. *(We will therefore occasionally write:* $\exp((\overline{\alpha}), (\overline{\beta}))$.)
3. *The sets* $(\overline{\beta})^n = \{a : \exp((a), \overline{\beta}) \geqslant n\}$ *for* $n \in S$ *are nonzero ideals of* R *and* $(\overline{\beta})^{n+1} \neq (\overline{\beta})^n$ *for any* n.
4. $(\overline{\alpha})^1 = (\overline{\alpha})$ *and* $\exp(ab, (\overline{\alpha})) = \exp(a, (\overline{\alpha})) + \exp(b, (\alpha))$. $C_{(\alpha)^n} = \{(\overline{\alpha})\}$
5. *Given finitely generated ideals* $\overline{\alpha}, \overline{\alpha}'$ *we have:* $(\overline{\alpha}) \subseteq (\overline{\alpha}')$ *iff for all* $(\beta) \in \mathcal{F}$ $\exp(\overline{\alpha}, \overline{\beta}) \geqslant \exp(\overline{\alpha}', \overline{\beta})$.

H. 2.5 supplies much of the force of unique factorization for ideals.

We assume a Chinese Remainder Theorem in the following form:

H3 : *There is a fixed class* $\mathcal{G}$ *of functions* $h : \mathcal{F} \to R$ *with the following properties:*

1. *There are basis functions* $h_1, \ldots, h_k \in \mathcal{G}$ *such that for all* $x \in \mathcal{F}$: $x = (h_1(x), \ldots, h_k(x))$.

2. *$\mathcal{J}$ contains the constant functions and is closed under:* $+, -, \cdot$. *If* $h_1, h_2 \in \mathcal{J}$ *have range in* S *then* $h_1 \oplus h_2 \in \mathcal{J}$. *If* $h_1, \ldots, h_{2k} \in \mathcal{J}$ *then* $\exp(h_1, \ldots, h_{2k}) \in \mathcal{J}$.
(The partial function exp *should be treated as a total function by identifying "undefined" with a fixed element not in* S.)
3. *If* $j \neq O$ *is a finitely generated ideal and* $h, h_1, h_2 \in \mathcal{J}$ *then the following sets belong to the lattice* $\{C_i\}$:
$C_j \cap \{x : h(x) = O \text{ in } R\}$.
$C_j \cap \{x : h_1(x), h_2(x) \in S,\ h_1(x) \geqslant h_2(x)\}$.
(Using the closure properties H3.2 on $\mathcal{J}$ it follows that the sets corresponding to general atomic formulae lie in $\{C_i\}$).
4. *Chinese Remainder Theorem: for* $h \in \mathcal{J}$, $C \in \{C_i\}$, $n \in S$ *there exists* $a \in R$ *satisfying the simultaneous congruences:*
$a \equiv h(x)$ *(mod* x^n) *for* $x \in C$.
5. *For* $h \in \mathcal{J}$ *with range in* S *and for* $C \in \{C_i\}$ *there is* $n \in S$ *satisfying:*
$h(x) \leqslant n$ *for* $x \in C$.
6. *For* $h_1, h_2 \in \mathcal{J}$, $C \in \{C_i\}$ *there is* $h \in \mathcal{J}$ *satisfying:*

$$h(x) = \begin{cases} h_1(x) & x \in C \\ h_2(x) & x \notin C. \end{cases}$$

In § 1.5 we will use the functions in $\mathcal{J}$ to carry out an ultrapower construction. For this purpose we need at least the following supply of Skolem functions:

H4 : Let F (x, y) *be a quantifier-free formula in the predicates and functions* $+, \cdot, S, \oplus, <$, exp. *Let* $h_i \in \mathcal{J}$ *be functions such that* R *satisfies:*

$\exists \bar{y}\, F(\bar{h}(x), \bar{y})$ *for all* $x \in \mathcal{F}$.

Then there are functions $\bar{f} \in \mathcal{J}$ *such that* R *satisfies*

$F(\bar{h}(x), \bar{f}(x))$ *for all* $x \in \mathcal{F}$.

The hypotheses H3-4 must be regarded as a first approximation, sufficient for treating generalities. Whatever difficulties exist in this subject reside in the definability theory connected with the Chinese Remainder Theorem, and at a certain level we are forced to treat the cases individually (as in § 2). The letter $\mathcal{I}$ is intended to suggest the word "internal".

In the next paragraph we make a few remarks on the classification of maximal ideals, using the hypotheses HO-1. In the classification of prime and primary ideals contained in some finitely generated maximal ideal (§§ 1.3-4) we use hypotheses HO-2. Finally in the classification of general prime and primary ideals we use hypotheses HO-4, and more generally hypotheses H3-4 play a role in the reduction of problems connected with general maximal ideals to analogous problems concerning finitely generated maximal ideals.

1.2 Maximal Ideals of R

We retain the notation R, $\mathcal{M}$, $\mathcal{F}$, C_i and assumptions HO-1 from 1.1. Let L(R) be the lattice of basic closed sets $\{C_i\}$ in $\mathcal{F}$. Then L(R) is distributive and relatively complemented. Let B(R) be the Boolean algebra generated by L(R) (in the algebra of subsets of $\mathcal{F}$). Then either B(R) = L(R) or B(R) is the disjoint set-theoretic union of L(R) and the set of complements of elements of L(R). We will correlate maximal ideals of R with ultrafilters of B(R) which meet L(R). (The introduction of B(R) is obviously unnecessary, but we prefer to discuss ultrafilters in the familiar context of a Boolean algebra.)

Definition 1. A filter D on B(R) is *bounded* iff it meets L(R). Thus the space of bounded ultrafilters on B(R) is an open subspace of the Stone space of B(R), containing the principal ultrafilters (points of $\mathcal{F}$).

Definition 2. Let I be an ideal of R. Define:

$$D(I) = \{X \in B(R) : \exists\, i \subseteq I \text{ finitely generated with } C_i \subseteq X\}$$

Let D be a filter in B(R). Define:

$M(D) = \{a : C_a \in D\}$.

Lemma 3. If I *is a nonzero ideal of* R *then* D(I) *is a proper bounded filter in* B(R). *If* D *is a bounded filter in* B(R) *then* M(D) *is a nonzero ideal of* R. *Furthermore:* $I \subseteq M(D(I))$ *and* $D \subseteq D(M(D))$.

Proof: Only the last inclusion requires argument. Choose $C \in D \cap L(R)$, $X \in D$. We claim $X \in D(M(D))$. It suffices to show that $X \cap C \in D(M(D))$. Certainly $X \cap C \in D \cap L(R)$, so write $X \cap C = C_i$ for some finitely generated ideal i. It follows that $i \subseteq M(D)$ and $C_i \in D(M(D))$ as desired.

Theorem 4

1. *If* P *is a nonzero prime ideal then* D(P) *is a bounded ultrafilter on* B(R).
2. *If* D *is a bounded ultrafilter on* B(R) *then* M(D) *is a maximal ideal.*
3. *The maps* $D \to M(D)$, $M \to D(M)$ *constitute a 1 - 1 correspondence between maximal ideals of* R *and bounded ultrafilters of* B(R).
4. *If* P *is prime then* M(D(P)) *is the unique maximal ideal containing* P.

Proof : 1. One shows that $C_i \cup C_j \in D(P) \Rightarrow C_i \in D(P)$ or $C_j \in D(P)$ (since B(R) is a Boolean algebra, a prime filter is automatically an ultrafilter).

2. Fixing $a \in R - M(D)$, we show that $(M(D),a) = R$. Since $C_a \notin D$, there is an $X \in D$ such that $X \cap C_a = \emptyset$. Since D is bounded we may assume $X \in L(R)$, say $X = C_i$. Then $i \subseteq M(D)$ and $C_{(a,i)} = \emptyset$ By H1 we have $(a, i) = R$, so $(a, M(D)) = R$.
Note that $M(D) \neq R$ since $C_{(i)} = \emptyset$.

3. It suffices to show that $D = D(M(D))$, $M = M(D(M)$. We already know that $D \subseteq D(M(D))$, $M \subseteq M(D(M))$;
here we have respectively an ultrafilter contained in a proper filter, and a maximal ideal contained in a proper ideal. Thus both inclusions are equalities, as desired.

4. If P is a prime ideal contained in a proper ideal I, we have: $D(P) \subseteq D(I)$.
Since $D(P)$ is an ultrafilter and $D(I)$ is a proper filter, $D(P) = D(I)$. Thus $I \subseteq M(D(I)) = M(D(P))$. In particular if I is maximal then $I = M(D(P))$.

Remark. We may and will identify $\mathcal{M}$ with an open subset of the Stone space B(R). However the topology on $\mathcal{M}$ in the Stone space is finer than the original topology on $\mathcal{M}$. The basis of closed sets $\{C_i'\}$ is now a basis of *clopen* sets. In particular this finer topology is Hausdorff.

1.3 M-ideals for $M \in \mathcal{F}$

We have previously assumed hypotheses HO-1; from this point on we also assume H2.
Definition 5 Let M be a maximal ideal. An ideal I of R is an *M-ideal* iff the set of maximal ideals containing I is precisely $\{M\}$. The relevance of this notion arises from Theorem 4, according to which all prime ideals P are M-ideals, with $M = M(D(P))$.

In the present paragraph we will primarily consider M-ideals relative to a finitely generated maximal ideal M. Our main result will be:

Theorem 6. Fix $M \in \mathcal{F}$. There is a 1 - 1 correspondence between M-ideals I and Dedekind cuts d in S, given by

$$I(d) = \bigcap_{n \leqslant d} M^n = \bigcup_{n \geqslant d} M^n = \{a \in R : \exp\ (a, M) \geqslant d\}.$$

$$d(I) = \sup\ \{n : I \subseteq M^n\} = \inf\ \{n : I \supseteq M^n\} = \inf\ \{\exp(a, M) : a \in I\}.$$

Here of course M^n means $(\alpha)^n$ where $M = (\alpha)$.
Notation For the duration of the present paragraph it is convenient to work with the notation I(d), d(I). Subsequently we will write M^d rather than I(d) and exp(I, M) rather than d(I).

Lemma 7. Let M *be a maximal ideal,* $I \subseteq M$ *an ideal. Then the following are equivalent:*

1. I *is an* M*-ideal.*
2. $D(I) = D(M)$.

   If in addition $M \in \mathcal{F}$ *then* 1,2 *are equivalent to:*
3. *For some* $n \in S$ $M^n \subseteq I$.

Proof: 2 ⇒ 1 : See the proof of Theorem 4.4.

1 ⇒ 2 : If D(I) is not an ultrafilter in B(R) then it is contained in at least two distinct ultrafilters D_1, D_2.
Then $I \subseteq M(D(I)) \subseteq M(D_1), M(D_2)$.
Thus if 2 fails, so does 1.

Now assume $M \in \mathcal{F}$.

3 ⇒ 2 : If $M^n \subseteq I \subseteq M$ then by H2.4 $C_M n = \{M\}$, so I is an M-ideal.

2 ⇒ 3 : In the following sublemma we prove something stronger.

Sublemma 8. Let I *be an* M*-ideal,* $M \in \mathcal{F}$, $a \in I$, $\exp(a, M) = n$. *Then* $M^n \subseteq I$.

Proof: Fix $b \in R$, $\exp(b, M) \geqslant n$. We claim that $b \in I$.
Since $D(I) = D(M)$ fix $i \subseteq I$ finitely generated with $C_i = \{M\}$.
Then $\exp(i, M') = O$ for $M' \neq M$. (For if $\exp(i, M') \geqslant 1$ then by H2.5 for each $a \in i$ $\exp(a, M') \geqslant 1$, so $a \in (M')^1 = M'$, proving $i \subseteq M'$). In particular $\exp((i, a), M') = O$ for $M' \neq M$.
Furthermore $\exp((i, a), M) \leqslant \exp(a, M) = n$. Thus

$$\exp((i, a), M') \leqslant \exp(b, M')$$

for all $M' \in \mathcal{F}$, so by H2.5 $b \in (i, a) \subseteq I$, as desired.

Proof of Theorem 6. One sees easily that $\bigcap_{n \leqslant d} M^n$ and $\bigcup_{n \geqslant d} M^n$ both coincide with
$\{a : \exp(a, M) \geqslant d\}$. We will argue that:

$$(*) \ \inf \{\exp(a, M) : a \in I\} \geqslant \inf \{n : M^n \subseteq I\} \geqslant \sup \{n : I \subseteq M^n\} \geqslant \inf \{\exp(a, M) : a \in I\},$$

proving the second set of equalities.

We know that if $\exp(a, M) = n$ with $a \in I$ then $M^n \subseteq I$, proving the first inequality in (*). The middle inequality follows from the observation that $M^n \subseteq I \subseteq M^m$ implies $n \geqslant m$ (otherwise $n < m$, $M^m \subseteq M^n \subseteq M^m$, a contradiction.) For the final inequality, suppose $m \leqslant \inf \{\exp(a, M) : a \in I\}$ Then $I \subseteq M^m$ by definition.

Hence I(d), d(I) are well defined. To conclude we must show that $I(d(I)) = I$, $d(I(d)) = d$.

$$I(d(I)) = \cup \{M^n : M^n \subseteq I\} \subseteq I$$
$$I(d(I)) = \cap \{M^n : M^n \supseteq I\} \supseteq I.$$

Thus $I(d(I)) = I$.

$$d(I(d)) = \inf\{n : M^n \subseteq \bigcup_{m \geqslant d} M^m\} = \inf\{n : n \geqslant d\} = d.$$

1.4 Prime and Primary Ideals below $M \in \mathcal{F}$

Remark If Q is a primary ideal then the radical P of Q is a prime ideal, and hence an M-ideal for some M. It follows that Q is also an M-ideal; indeed every prime ideal containing Q contains P and hence is contained in M.

In this subsection we specialize the classification in § 1.3 to prime and primary ideals.

Definition 9 A Dedekind cut d is S-*additive* iff $m, n < d \Rightarrow m + n < d$. The cut d is *primary* iff $m < d, n \ll d \Rightarrow m + n < d$ (here $n \ll d$ means that for all natural numbers k, $kn < d$).

Theorem 10. In the 1.1 correspondence between Dedekind cuts and M*-ideals, the prime ideals correspond to additive cuts and the primary ideals to primary cuts.*

Proof: Straightforward.

When S is a nonstandard model of arithmetic the structure of the additive cuts can be made rather transparent [1]. The primary cuts have not been analyzed. As an example let ω be an infinite integer and let $d = \{n : \forall k \text{ finite } n < (1 + 1/k)\omega\}$. Let ω^- be the cut $\{n : \forall k \text{ finite } n < \omega/k\}$. Then ω^- is an additive cut and for all n

$$n \ll d \text{ iff } n < \omega^-.$$

M^d is primary: indeed if $m < d$, $n \ll d$, then for any finite k $m < (1 + 1/2k)\,\omega$, $n < \omega/2k$, so $m + n < (1 + 1/k)\,\omega$, i.e. $m + n < d$, as desired. On the other hand $M^d \subseteq M^{\omega} \subseteq M^{\omega^-}$ and M^{ω} cannot be primary (if $\exp(a, M) = \omega - 1$ and $\exp(b, M) = 1$ then $ab \in M$ but $a \notin M$, $b \notin \text{rad}\ (M)$).
Corollary 11. Every finitely generated prime ideal P *is maximal. Every finitely generated primary ideal* Q *is a finite power of a maximal. (Note that by H1 if* M *is a maximal ideal and* I *is a finitely generated* M*-ideal then* $M \in \mathcal{F}$.)

1.5 $\mathcal{G}$-Ultrapowers

Before proceeding further we describe some machinery which can be useful in reducing questions concerning arbitrary maximal ideals M to similar questions concerning finitely generated maximal ideals. This is the machinery of definable or recursive ultrapowers. In an axiomatic setting we speak of $\mathcal{G}$-ultrapowers, where $\mathcal{G}$ is the set of functions refered to in H3.

Definition 12. Let D be a bounded ultrafilter on B(R). Let R_o be the *$\mathcal{G}$-ultrapower* formed by considering the given family $\mathcal{G}$ of functions $h : \mathcal{F} \to R$ (H3) modulo the equivalence relation: $h_1 \sim h_2$ iff $\{x : h_1(x) = h_2(x)\}$ contains a element of D. Under hypothesis H3 R_D is a ring containing the semigroup S_D corresponding to $\{h \in \mathcal{G} : \text{range } h \subseteq S\}$ and equipped with the function exp (or for emphasis, $\exp_D$) defined by

$$\exp_D(\overline{h}) = (\exp(\overline{h}))_D$$

(on the right $\exp(\overline{h}) \in \mathcal{G}$ and $\exp(\overline{h})_D$ denotes the corresponding equivalence class in R_D.

The relationship of R to R_D is of course controlled by the structure of $\mathcal{G}$. For the most part $\mathcal{G}$ will consist of all definable functions, R_D will be a definable ultrapower of R, and-in the presence of an adequate supply of definable Skolem functions-the canonical embedding $R \to R_D$ will be an elementary embedding. When $\mathcal{I}$ consists of the recursive functions R_D is a recursive ultrapower and stands in a somewhat looser relationship to R. We intend to reduce questions concerning general maximal ideals in R to questions concerning certain finitely generated maximal ideals in R_D; this of course constitutes a "reduction" of the problem only if R_D greatly resembles R.

Hypothesis H4 says:

Lemma 13. Let $F(x)$ *be an existential formula (with predicates and functions as in H4). For* $h_{iD} \in R_D$ *the following are equivalent:*

1. R_D *satisfies* $F(h_D)$.
2. $\{m : R$ *satisfies* $F(h(m))\}$ *contains an element of* D.

Proof: Say $F(x) = \exists\, y\, F'(x, y)$. The nontrivial implication is $2 \Rightarrow 1$: Let $C \in D$ be contained in $\{m : R$ satisfies $\exists\, y\, F'(h(m), y)\}$. In particular one may choose elements r, q in R such that $F'(r, q)$ holds in R. Define

$$h'(m) = \begin{cases} h(m) & m \in C. \\ r & m \notin C \end{cases}$$

Then $h'_D = h_D$ and R satisfies $\exists\, y\, F'(h'(m), y)$ for all $m \in \mathcal{F}$.

Thus there are functions $f \in \mathcal{I}$ such that $F'(h'(m), f(m))$ holds in R for all m, and thus R_D satisfies $F'(h'_D, f'_D)$ as desired.

Corollary 14. R_D *is a model of the* A_2 *part of the theory of* R.
Corollary 15. R_D *satisfies* Ho.
Corollary 16. R_D *satisfies* H2.

Proof: We know immediately that S_D is an ordered abelian semigroup and that $\exp_D$ is a function into S_D, defined for 2k-tuples $(\overline{\alpha}, \overline{\beta}) \in R^k_D \times R^k_D$ such that the set

$\{x : (\overline{\beta}(x)) \in \mathcal{F}\}$

contains an element of D. We should verify that the set of such $(\overline{\beta})$ coincides with $\{\overline{\beta} \in R^k_D : (\overline{\beta})$ is maximal in $R_D\}$.

Assume first that $\overline{\beta}$ is not maximal in R_D and choose $h_D \in R_D$ such that $h_D \notin \overline{(\beta)}$ and $1 \notin \overline{(\beta, h_D)}$. By Lemma 13 $h_D(x) \notin (\overline{\beta}(x))$ and $1 \notin (\overline{\beta}(x), h_D(x))$ for a set in D, so $(\overline{\beta}(x)$ is not maximal on a set in D.

Suppose on the other hand that $X = \{x : (\beta(x)) \text{ is maximal}\}$ contains no member of D. Intersecting X with an arbitrary bounded element of D and applying H3.3 we may assume that $X \in L(R)$, $X \notin D$. In other words there is an element $C \in D$ such that $(\overline{\beta}(x))$ is not maximal for $x \in C$.
Then for $x \in C$ we have $M_1 \neq M_2$ such that

$$\exp(\overline{\beta}(x), M_1) + \exp(\overline{\beta}(x), M_2) \geqslant 2.$$

Lemma 13 implies that the same is true for $\overline{\beta}$ in R_D: there are M_1, M_2 such that $(M_1, M_2) = R_D$ and

$$\exp_D(\overline{\beta}, M_1) + \exp_D(\overline{\beta}, M_2) \text{ is defined and } \geqslant 2.$$

One then sees easily that $\overline{(\beta)}$ is not maximal.

Thus $\exp_D$ is defined on $\mathscr{F}_D$ (the set of finitely generated maximal ideals of R_D) and evidently has properties H2.0-5 by Corollary 14.

Corollary 17. R_D *satisfies H1.*

Proof: H1.1 : It suffices to show that R_D satisfies:

$$\forall i, j \ \exists \ell, n \ (i \leqslant \ell \ \& \ (\ell, j) = R \ \& \ (\ell, j^n) \subseteq i).$$

This is clearly an A_2 sentence (for example express $(\ell, j^n) \subseteq i$ in terms of bases $\overline{\ell}, \overline{j}, \overline{i}$ of ℓ, j, i by writing:

$$\exists \alpha_{st}, \beta_s \ \Sigma\alpha_{st} \ell_t + \beta_s = i_s \ \& \ \exp(\beta_s, j) \geqslant n.$$

H1.2: It suffices to prove that

$$\forall i \ \exists j \ (i \subseteq j \in \mathscr{F})$$

writing "exp(j, j) = 1" for "$j \in \mathscr{F}$" we see that this is an A_2 assertion.

Thus we have recovered for use in R_D those axioms which we previously used in our discussion of R. Of course we are presently operating at a rather general level, and in concrete cases we are concerned with transferring more precise information from R to R_D.

The relationship between R and R_D is quite complex in the case of weak nonstandard models (e.g. existentially complete models for arithmetic).

We will apply this construction to the analysis of general M-ideals in § 1.7.

1.6 Residue Fields of R.

Consider a maximal ideal M of R and the residue field $R = R/M$. We know nothing about these fields *a priori*. Nevertheless in our intended examples (§ 2) we know a great deal about R/M for M finitely generated (e.g. that it is a pseudo-finite field [7]), and the problem lies in obtaining similar information concerning those M which are not finitely generated.

Definition 18. Let M be a maximal ideal of R, D the corresponding bounded ultrafilter, and R_D the corresponding $\mathcal{J}$ ultrapower. Let $h_1, \ldots, h_k \in \mathcal{J}$ be the *basis functions* of H3.1 and let $\mathbf{1}_D$ be the k-tuple $<h_{iD}>$. Then $\mathbf{1}_D$ generates an ideal $(\mathbf{1}_D)$ in R_D.

Theorem 19. Let $M, D, R_D, \mathbf{1}_D$ *be as above. Then* $(\mathbf{1}_D)$ *is a maximal ideal,* $(\mathbf{1}_D) \cap R = M$ *and the induced homomorphism* $R = R/M \rightarrow R_D/(\mathbf{1}_D)$ *is an isomorphism.*

Proof: We compute $(\mathbf{1}_D) \cap R$ as follows. Fix $r \in R$. Then $r \in (\mathbf{1}_D)$ iff $\exp_D(r, \mathbf{1}_D) > O$ iff $\{x : \exp_D(r, x) > O\} \in D$ iff $C_r \in D$ iff $r \in M(D) = M(D(M)) = M$.

Thus there is an induced monorphism

$$\tau : R \rightarrow R_D/(\mathbf{1}_D).$$

To see that it is surjective fix $h \in R_D$, $C \in D$ bounded and seek r satisfying

$$r - h \in \mathbf{1}_D; \text{ it suffices to have}$$
$$r \equiv h(x) \bmod(x) \text{ for } x \in C.$$

This is a special case of the Chinese Remainder Theorem.

Corollary 20. If T *is the theory of all residue fields of* R *modulo finitely generated maximal ideals and* R *is a residue field of* R *modulo an arbitrary maximal ideal then* R *is a model of the inductive part* T_{A2} of T.

In our examples stronger results are true. To a certain extend the relation between R and R_D requires a case - by-case analysis.

1.7 General M-ideals

We now fix an arbitrary maximal ideal M in R and consider the classification of M-ideals in R. Let D = D(M). We already know the complete classification of $(\mathbf{1}_D)$-ideals. They have the form $(\mathbf{1}_D)^d$ where d is a Dedekind cut in S_D.

Definition 21. We define a contraction map c from $(\mathbf{1}_D)$-ideals I of R_D to ideals of R by $c(I) = I \cap R$.

We define an extension map e from M-ideals J of R to ideals of R_D by first defining the cut

$d(J) = \inf \{n_D \in R_D : \exists\, a \in J \exp(a, .)_D \leqslant n_D\}$ in S_D and setting $e(J) = (\mathbf{1}_D)^{d(J)}$.

Theorem 22. The contraction and extension maps c, e *defined above set up a 1 - 1 correspondence between* $(\mathbf{1}_D)$*-ideals* I *and* M*-ideals* J. *Under this correspondence prime* $(\mathbf{1}_D)$*-ideals correspond to prime* M*-ideals and primary* $(\mathbf{1}_D)$*-ideals correspond to primary* M*-ideals.*

Proof: Suppose I is a $(\mathbf{1}_D)$-ideal. We claim c(I) is an M-ideal, i.e.

$$D(c(I)) = D.$$

Certainly $I \cap R \subseteq (\mathbf{1}_D) \cap R = M$, and thus $D(c(I)) \subseteq D$. Suppose conversely that $C \in D$. We must show that $C \in D(c(I))$, and we may take C bounded. Since I is a $(\mathbf{1}_D)$-ideal we have

$$n_D \in S_D \text{ such that } (\mathbf{1}_D)^{n_D} \subseteq I. \text{ Let}$$
$$n \geqslant \sup_{x \in C} n_D(x) \qquad \text{(H3.5)}.$$

By the Chinese Remainder Theorem and Lemma 13 we can find an element a of R with

$$\exp(a, x) \geqslant n \text{ for all } x \in C.$$

Again by the Chinese Remainder Theorem, Lemma 13, and H3.6, we can find an element b of R with

$$\exp(b, x) \geqslant n \text{ for all } x \in C$$
$$\exp(b, x) = O \text{ for } x \in C_a - C.$$

Then $\exp_D(a, \mathbf{1}_D) \geqslant n \geqslant n_D$ in R_D, so $a \in (\mathbf{1}_D)^{n_D} \subseteq I$.
Similarly $b \in I$. Thus $(a, b) \subseteq c(I)$, $C_{(a,b)} = C$, and so $C \in D(c(I))$ as desired.

Thus c carries $(\mathbf{1}_D)$-ideals to M-ideals. Clearly e carries M-ideals to $(\mathbf{1}_D)$-ideals. We come to the major step: the verification that

1. ec (I) = I for $(\mathbf{1}_D)$-ideals I.
2. ce (J) = J for M-ideals J.

1: Let $I = (\mathbf{1}_D)^d$ be a (1_D)-ideal in R_D. We claim that $d(I \cap R) = d$, so that $e(c(I)) = I$. In other words we claim:

$$\inf \{\exp_D(a, \mathbf{1}_D) : a \in (\mathbf{1}_D)^d \cap R\} = \mathbf{d} \text{ in } S_D.$$

Clearly $\exp_D(a, \mathbf{1}_D) \geqslant \mathbf{d}$ for $a \in (\mathbf{1}_D)^d \cap R$. The problem is: given $n_D \geqslant d$, find a in R such that $\exp_D(a, \mathbf{1}_D) = n_D$. There is a function $h \in \mathcal{J}$ such that

$$h(x) \in x^{n_D(x)} - x^{n_D(x)+1} \text{ for all } x \in \mathcal{F} \text{ (Hypothesis H4).}$$

By the Chinese Remainder Theorem (H3.4) there is $a \in R$ satisfying

$$a \equiv h(x) \pmod{x} \quad x \in C$$

for any fixed bounded set $C \in D$. Then $\exp_D(a, \mathbf{1}_D) = n_D$, as desired.

2: Given an M-ideal J in R we must verify that $J = (\mathbf{1}_D)^{d(J)} \cap R$. Clearly $J \subseteq (\mathbf{1}_D)^{d(J)} \cap R$. Suppose conversely that $a \in R$, $\exp_D(a, \mathbf{1}_D) \geqslant d(J)$. This means that there is $b \in J$ such that for all x in some set $C \in D$

$$\exp(a, x) \geqslant \exp(b, x).$$

Since J is an M-ideal there is a finitely generated ideal $j \subseteq J$ such that $C_j \subseteq C$. Thus $a \in (b, j)$ by H2.5. Thus $a \in J$, as desired.

Finally we must verify the statement concerning prime and primary ideals. Certainly the intersection of a prime or primary ideal of R_D with R remains prime or primary respectively. Suppose now that P is a prime or primary ideal of R.

$$\text{Let } I = e(P), \text{ so that } P = I \cap R.$$

Suppose $a_D b_D \in I$, $a_D \notin I$. Fix a bounded set $C \in D$ and choose $a, b \in R$ such that

$$a \equiv a_D(x) \pmod{x^{\exp(a_D(x),x) + 1}}$$

$$b \equiv b_D(x) \pmod{x^{\exp(a_D(x),x) + 1}} \text{ for } x \in C.$$

Then

$$\exp(a, x) = \exp(a_D(x),x)$$
$$\exp(b, x) = \exp(b_D(x),x)$$

for $x \in C$, and thus $\exp_D(a, \mathbf{1}_D) = \exp(a_D, \mathbf{1}_D)$ and $\exp_D(b, \mathbf{1}_D) = \exp_D(b_D, \mathbf{1}_D)$. It follows that $ab \in I \cap R$, $a \notin I \cap R$. Thus for some finite n $b^n \in P$ ($n = 1$ if P is prime) and since $\exp_D(b^n, \mathbf{1}_D) = \exp_D(b^n_D, \mathbf{1}_D)$ it follows that $b^n_D \in I$, as desired.

Corollary 23. The prime M*-ideals of* R *are classified by additive Dedekind cuts* d *in* $S_{D(M)}$. *That is, each such ideal* P *may be written uniquely* $P = (\mathbf{1}_D)^d \cap R$ *where* d *is an additive Dedekind cut in* $S_{D(M)}$.

Definition 24. Let M be a maximal ideal of R, D = (D(M). Let d be a Dedekind cut of S_D. Define

$$M^d = \{a \in R : \exp_D(a, \mathbf{1}_D) \geqslant d\}.$$

Remark. $M^d = (\mathbf{1}_D)^d \cap R$. Thus the M-ideals J are uniquely expressible in the form

$$J = M^d$$

as d varies over Dedekind cuts of S_D. Thus from a purely formal point of view general M-ideals and M-ideals associated to an $M \in \mathcal{F}$ may be placed on the same footing.

1.8 Residue Class Rings Modulo Prime Ideals

Let P be a prime ideal of R lying below the maximal ideal M. Let D = D(M). Much as in § 1.6 one can prove:

Theorem 25. $R/P \sim R_D/e(P)$.

Here $e(P) = (\mathbf{1}_D)^{d(P)}$ as in § 1.7. Thus in considering residue rings R/P we may assume that P is an M-ideal for some finitely generated M (at the cost of replacing R by R_D).

One may prove easily:

Theorem 26. Let $P = M^d$ *with* $d > 1$ *be a nonmaximal prime* *ideal. Then* R/P *is a valuation domain with residue class field* R/M. *The value group of the quotient field* K *is* $\{\pm n : n \in S_D, n < d\}$.

The field K may be considered as a valued field with valuation ring R/P. In the examples K is a Hensel field. This is because in the examples S_D is a weak nonstandard model of artihmetic satisfying an axiom of induction strong enough to support the following argument: let $p(x) \in R[x]$ have a root α modulo M, with $p'(\alpha) \notin M$. Then for every n p(x) has a root modulo M^n (here we use induction, together with Theorem 25 which allows us to assume M is finitely generated). Taking $P = M^d, n > d$ we see that p(x) has a root in R/P, as required for Hensel's Lemma.

Theorem 26, supplemented by Hensel's Lemma in K, takes its point from the Ax-Kochen-Ershov theorem (cf.[9]). In particular if K is an unramified Hensel field then its first order theory is determined by the theory of R/P and the theory of the value group.

§ 2. *Examples*

2.1 Strong Nonstandard Dedekind Rings

A *strong nonstandard* Dedekind ring is a ring R in a nonstandard model $\mathcal{N}$ of a suitable set theory (typically ZFC or finite type theory with urelements) such that $\mathcal{N}$ satisfies:

"R is a Dedekind ring".

These are the nonstandard Dedekind rings used in nonstandard analysis (also called nonstandard arithmetic in this context); $\mathcal{N}$ is usually taken to be card$(R)^+$-saturated.

In this context the hypothesis HO-2 are clearly valid. H3 will be valid as well, taking for $\mathcal{I}$ the set of all functions from $\mathcal{F}$ to R in $\mathcal{N}$ (the "internal" functions). To obtain H4 assume that $\mathcal{N}$ satisfies the axiom of choice, or at least:

"R can be well-ordered."

In this context all results from § 1 take on a somewhat sharper form. We recognize B(R) as the finite-cofinite algebra on $\mathcal{F}$ in the sense of $\mathcal{N}$. The semigroup S is a strong nonstandard model of arithmetic. $\mathcal{G}$-ultrapowers are "internal" ultrapowers; but note that the nonprincipal bounded ultrafilters are automatically external, so our ultraproduct construction cannot be carried out within $\mathcal{N}$. Using the axiom of choice (or a well-ordering of R in $\mathcal{N}$) we see that the canonical embedding $R \rightarrow R_D$ is an elementary embedding. In fact we can embed $\mathcal{N} \rightarrow \mathcal{N}_D$ as an elementary substructure (given the full axiom of choice) so that R_D is again a strong nonstandard Dedekind ring.

If $\mathcal{N}$ satisfies:

"The residue fields of R modulo maximal ideals are finite" then the residue fields of R modulo finitely generated maximal ideals are * - finite (finite in the sense of $\mathcal{N}$) and hence pseudofinite (1) By Theorem 19 it follows that *all* residue class fields of R are isomorphic with * - finite fields (relative to a suitable $\mathcal{N}_D$) and are hence pseudo-finite.

Turning to § 1.8, we see that the valued fields described there are Hensel fields. We will not discuss the question of ramification here, but the discussion in (1) is adequate for the treatment of finite ramification.

2.2 Nonstandard Rings of Algebraic Integers

This case greatly resembles the preceding, since no definability problems arise (given Julia Robinson's definability theorem [2]). In any nonstandard ring of algebraic integers A there is a canonical model $\mathbb{N}^*$ of full artihmetic, and a definable bijection $\Phi : \mathbb{N}^* \leftrightarrow A$ with respect to which all relevant structure on A is definable over $\mathbb{N}^*$. In particular a suitable function exp is at hand. As for hypotheses H3-4, we let $\mathcal{G}$ consist of all definable functions from $\mathcal{F}$ into $\mathbb{N}^*$. ($\mathcal{F}$ may be identified with a subset of A^k or of $\mathbb{N}^*$.) H4 is valid because $\mathbb{N}^*$

has definable Skolem functions, i.e. $\mathbb{N}^*$ is well-ordered relative to definable subsets.

L(R) consists of the bounded definable subsets of R. $\mathcal{J}$-ultra powers are definable ultrapowers. Of course as far as the construction of ultrapowers relative to bounded ultrafilters is concerned, we are not interested in functions in $\mathcal{J}$, but in their restrictions to bounded definable sets. It follows that one may replace $\mathcal{J}$ equally well by the recursive functions (in the sense of $\mathbb{N}^*$), or some intermediate class of functions.

As in § 2.1 we see that the embeddings $A \to A_D$ are elementary. For $M \in \mathcal{F}$ the residue field A/M will be finite or pseudofinite of characteristic zero. By Theorem 19 the same applies to all residue fields A/M. Again the valued fields of § 1.8 are Hensel fields, and there is a uniform bound on ramification.

2.3 Nonstandard Function Fields over Finite Fields

The treatment of number fields sketched in § 2.2 has a slightly more complicated analog for nonstandard function fields over finite fields (of characteristic not 2).

Theorem 27. Let k *be a finite fields of characteristic not* 2 *and of order* q, *let* $\mathbb{N}^*$ *be a nonstandard model of arithmetic, and let* T_n *be the theory of algebraic extensions of the function field* k(x) *in one variable having degree at most* n *over* k(x).

1. *Let* R *be the set of definable maps* $p : \mathbb{N}^* \to k$ *such that* $p(n) = O$ *for sufficiently large* n *(we identify* k *with the set* $\{1, \ldots, q\} \subseteq \mathbb{N}^*$*). Identify* p *with the formal expression* $p(x) = \Sigma p(n) X^n$.
   Defining $+, .$ *on* R *in the obvious way, we assert that* R *is an elementary extension of the polynomial ring* k[x] *(embedding in* R *in the obvious way).*
   Let K *be the quotient field of* R *and let* L *be an algebraic extension of* K *of dimension at most* n. *Then* L *is a model of* T_n.
2. *Conversely let* L *be an arbitrary model of* T_n.
   Then L *contains a subring* R *such that* L *has dimension*

at most n *over the quotient field of* R, *and* R *is isomorphic with the nonstandard polynomial ring contructed from a certain nonstandard model of arithmetic* $\mathbb{N}^*$. *The ring* $\mathbb{N}^*$ *may be identified with a definable subset of* L *in such a way that the operations* $+, .$ *on* $\mathbb{N}^*$ *are* L-*definable, and there is an* L-*definable isomorphism between the nonstandard polynomial ring over* $\mathbb{N}^*$ *(coded by elements of* $\mathbb{N}^*$*) and the ring* R.

The first half of this theorem is evident. Granted Theorem 27.2 as well, we easily obtain hypotheses HO-4, taking for $\mathcal{G}$ all definable functions as in § 2.2.

To prove Theorem 27.2 it suffices to consider an arbitrary standard function field of degree n over k(x), and to define a standard model $\mathbb{N}$ of arithmetic and an isomorphism $R(\mathbb{N}) \sim k[x]$ using the parameter x and first order definitions which depend at most on the degree n. It then will follow that in any nonstandard function field a similar procedure will produce a nonstandard model of arithmetic and the desired isomorphhism.

The first step is to define k[x] within L. The method is the same as that used in (2) in the case of number fields; compare (3), in which the definition of k[x] within k(x) is sketched. One proceeds as follows: using the theory of ternary quadratic forms one defines a ring R_0 in L, intermediate between k[x] and the integral closure of k[x] in L. For this one needs some facts about the Hilbert symbol $(a, b)_p$, bounds on ramification and the number of primes at infinity, and the Dirichlet theorem (see Tate's discussion and Hasse's remarks in (10) for the last point; cf. (2)). As in the first section of (2) we pass easily from a definition of R_0 to a definition of the integral closure of k[x] in L, and then to a definition of k[x].

The second step (and the more important one for our purposes) is the study of definability in the polynomial ring k[x]. In (5) R.M. Robinson showed how to define a model of arithmetic $\mathbb{N}$ over k[x] depending on the parameter x. The elements of $\mathbb{N}$ are the powers $\{x^n : n \geqslant O\}$. For the proof of Theorem 30.2 we require a first order definition for the canonical isomorphism between k[x] and the corresponding ring of finitely supported functions from $\mathbb{N}$ to k. We consider the predicate:

$$\text{Val}(a, b, c) = \text{"If } a = \Sigma a_i x^i \text{ with } a_i \in k \text{ then for some } j\ b = x^j \text{ and } a_j = c."$$

The proof of Theorem 27 concludes with the following result:
Lemma 28. Val (a, b, c) *is first order definable over* k[x] *in the parameter* x.

Proof: We use two auxiliary predicates:
Degree (a, b) = "There is a j such that degree a = j and $b = x^j$".
Occurs (a, c) = "If $a = \Sigma a_i x^i$ with $a_i \in k$ then there is a j such that $c = a_j x^j \neq O$."

We now supply first order definition for Degree, Occurs, and Val:

1. Degree (a, b) iff $b \in \mathbb{N}$, $a \neq O$, $a/b \in k[1/x]$ and $ax/b \notin k[1/x]$. By [5] or our preceding remarks, k[1/x] is definable over k(x), which is definable over k[x].
2. Occurs (a, c) iff $\exists\, b \in N$, $\alpha \in k^*$ such that $c = \alpha b$ and $\exists\, d_1, d_2$ such that degree $(d_1) <$ degree (b) and $a = d_1 + c(1 + xd_2)$.
3. Val (a, b, c) iff $b \in N$ and $c \in k$ and either:
   $c \neq O$, cb occurs in a or
   $c = O$ and for all $\alpha \in k$ αb does not occur in a.

It follows that definability in function fields over finite fields may be handled as easily as definability over number fields.

2.4 Existentially Complete Models for Arithmetic

We assume some familiarity with existentially complete models for arithmetic and the Π_2 part of arithmetic, called T_{Π_2}. The relevant background is found in the first two chapters of [4]. The main facts we need are that T_{Π_2} is adequate for the development of recursion theory and that the existentially complete models for arithmetic are models of T_{Π_2}. As in [4] we say that a set or relation is *r.e.* iff it is existentially definable with parameters. A set or relation is *recursive* iff both it and its complement are r.e. Functions are identified with their graphs. We note that in the literature models of arithmetic usually consist solely of nonnegative elements. For us models of arithmetic are rings (the discrepancy is harmless).

The function $\exp(a, p) = \sup \{n : p^n \text{ divides } a\}$ is recursive and hence behaves well in all models $\mathbb{Z}^*$ of T_{Π_2}. Thus hypotheses HO-2 present no dificulties. To obtain H3-4 we take $\mathcal{I}$ to be the set of all recursive functions on $\mathbb{Z}^*$. We do not *in general* have definable Skolem functions, but H4 is correct.

In § 1.1-1.5 we find ourselves consrtucting recursive ultrapowers $\mathbb{Z}^*_D$ and studying the embedding $\mathbb{Z}^* \to \mathbb{Z}^*_D$. The following fact was proved in [4, pp. 50 - 51] in a slightly narrower context:

Theorem 29. If $\mathbb{Z}^*$ satisfies T_{Π_2}, so does $\mathbb{Z}^*_D$ In fact $\mathbb{Z}^*_D$ satisfies the Π_2 part of the theory of $\mathbb{Z}^*$ (in a language naming all elements of $\mathbb{Z}^*$).

Notice however that even if $\mathbb{Z}^*$ is an existentially complete model for arithmetic there is no guarantee that $\mathbb{Z}^*_D$ will remain existentially complete. We pause to develop this point further.

Theorem 30. Let $\mathbb{Z}^*$ be a model of T_{Π_2}, D a recursive ultrafilter on $\mathbb{Z}^*$. then the following are equivalent:

1. *$\mathbb{Z}^*_D$ is an existentially complete model for arithmetic.*

2. $\mathbb{Z}^*$ *is an existentially complete model for arithmetic and for every r.e. subset* A *of* $\mathbb{Z}^*$ *there is a set* $C \in D$ *such that* $C \subseteq A$ *or* $C \cap A = \emptyset$.

Proof: 1 $\Rightarrow$ 2: Evidently $\mathbb{Z}^*$ is existentially complete in $\mathbb{Z}^*_D$. Thus if $\mathbb{Z}^*_D$ is existentially complete, so is $\mathbb{Z}^*$. Now let A be an r.e. subset of $\mathbb{Z}^*$. Let $\mathbf{1} : \mathbb{Z}^* \to \mathbb{Z}^*$ be the identity function. Let $A = \{x : \exists \overline{y}\, F(x, \overline{y}, a)\}$ where a is a possible parameter from $\mathbb{Z}^*$.

If $\mathbf{1}_D \in A$ then there are recursive functions $\overline{h} : \mathbb{Z}^* \to \mathbb{Z}^*$ such that $\mathbb{Z}^*_D$ satisfies $F(1_D, \overline{h}_D, a)$. Thus

$$C = \{x : F(x, \overline{h}_D(x), a)\} \in D,$$

and $C \subseteq A$.

If $\mathbf{1}_D \notin A$ then since $\mathbb{Z}^*_D$ is existentially complete, there is an existential sentence $G(x, b)$ such that $T_{\Pi_2} \vdash G(x, b) \Rightarrow \neg \exists y\, F(x, \overline{y}, a)$ and $\mathbb{Z}^*_D$ satisfies $G(\mathbf{1}, b)$. Writing

$$G(x, b) = \exists \overline{z}\, G'(x, \overline{z}, b)$$

we have recursive functions $h : \mathbb{Z}^* \to \mathbb{Z}^*$ such that

$$C = \{x : G'(x, \overline{h}(x), b)\} \in D.$$

Evidently $C \cap A = \emptyset$, as desired.

2 $\Rightarrow$ 1. According to the second clause of 2 we can extend D in a unique way to an ultrafilter D' on the r.e. sets. Then $\mathbb{Z}^*_D$ can be identified with the r.e. ultrapower $\mathbb{Z}^*_D$ formed from all r.e. partial functions defined on a set in D'. Now consider $\mathbb{Z}^*_D$ as a submodel of any model $\mathbb{Z}'$ of arithmetic.

As Hirschfeld observes in [4], to be existentially complete in $\mathbb{Z}'$ it is necessary and sufficient to be closed under partial recursive functions in $\mathbb{Z}'$. Since $\mathbb{Z}^*$ is existentially complete in $\mathbb{Z}'$, it is closed under partial recursive functions in $\mathbb{Z}'$, and the same then applies to $\mathbb{Z}^*_{D'}$, which has been identified with $\mathbb{Z}^*_D$.

In [4] examples were given to show that in fact recursive ultrapowers of the standard model may or may not be existentially complete. Of course we are here concerned with non-principal bounded ultrafilters, which do not exist over the standard model. Suitable constructions along Hirschfeld's lines may be carried out over any bounded infinite interval in a countable model of T_{Π_2} (although I have no idea what becomes of the notion of "maximal set", used in [4]). In view of Theorem 30, our claim is:

Theorem 31. Let $[m, n]$ *be an infinite interval in a countable model of* T_{Π_2}. *Then an ultrafilter* D *on the recursive subsets of* $[m, n]$ *may be constructed with or without the following property (as desired):*
(*) *For all r.e.* A *there is a recursive* $C \in D$ *such that* $A \subseteq C$ *or* $A \cap C = \varnothing$.

Proof: To obtain (*) one constructs a descending sequence $\{C_n : n = 0, 1 \ldots\}$ (n varies over standard integers, as this construction occurs in the real world) so that $\{C_n\}$ generates the desired filter. See [4].

To avoid (*) one fixes a particular non-recursive r.e. set $A \subseteq [m, n]$ (e.g. $\{k : k - m \text{ is infinite}\}$). Let D be a filter of recursive sets maximal with respect to:
(†) For all $C \in D$, $C \cap A$ is nonrecursive.
D certainly lacks property (*). To see that D is an ultrafilter, suppose that C_1, C_2 partition an element C of D, while $C_1, C_2 \notin D$. Then there are elements X, Y of D such that

$$C_1 \cap X \cap A, \; C_2 \cap Y \cap A \text{ are recursive.}$$

Then

$C_1 \cap X \cap Y \cap A$, $C_2 \cap X \cap Y \cap A$ are recursive, so
$C \cap X \cap Y \cap A$ is recursives contradicting (†).

Thus if we intend to analyze existentially complete models of arithmetic using bounded recursive ultrapowers as in § 1.1-1.5 we should work in the more extensive category of models of T_{Π_2}.

Turning now to the residue class rings $\mathbb{Z}^*/M$ and $\mathbb{Z}^*/P$ relative to maximal and nonmaximal prime ideals, we see once more that the residue fields $\mathbb{Z}^*/M$ are pseudofinite. It is only necessary to check that the statement

"All residue fields $\mathbb{Z}^*/(p)$ are pseudofinite"

is a consequence of T_{Π_2}. For a discussion of effectivity in this context compare [8].

The valued fields mentioned in § 1.8 are of course Hensel fields (the construction usually used to prove Hensel's Lemma is clearly effective) and there is no ramification.

The analog of the nonstandard rings of algebraic integers considered in (1) is simply the class of rings R obtained by taking an arbitrary existentially complete model $\mathbb{Z}^*$ for arithmetic and forming the integral closure of $\mathbb{Z}^*$ in a finite algebraic extension K of the quotient field $\mathbb{Q}^*$ of $\mathbb{Z}^*$. For the purposes of definability theory it is advisable to take $\mathbb{N}^*$ as a distinguished predicate in R (Julia Robinson's definability theorem has not been proved in this general context; in particular nothing seems to be known concerning the existentially complete models for the complete theory of the rational field $\mathbb{Q}$, if the ring of integers $\mathbb{Z}$ is *not* treated as a distinguished predicate).

We may of course generalize a little more by taking an arbitrary model $\mathbb{Z}^*$ of T_{Π_2} and the integral closure R of $\mathbb{Z}^*$

in a finite algebraic extension of the quotient field. As long as we retain $\mathbb{Z}^*$ as a distinguished predicate our analysis of $\mathbb{Z}^*$ applies *mutatis mutandis* to R.

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Note: My attention has been drawn to similar results published by Norbert Klingen, «Zur Idealstruktur in Nicht standard modellen von Dedekindringen,» J. Reine Angew. Math. 274/275 (1975), 38-60.