FINITE-DIMENSIONAL MODELS OF CATEGORICAL SEMI-MINIMAL THEORIES

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A semi-minimal theory is a complete first-order theory with infinite models in which $v_0 = v_0$ is a semi-minimal formula: for any formula φ, there are strongly φ-minimal formulas $\psi_1, \ldots, \psi_k$ such that in any model $\mathcal{A}$ of the theory, $\{\psi_i(\mathcal{A})\,;\ 1 \leqslant i \leqslant k\}$ is a partition of A. Strong φ-minimality, a notion first defined in [S 2], means that ,if φ has $1 + n$ free variables, for any sequence a of n parameters from A, the set $\psi_i(\mathcal{A})$ is included in, or disjoint from the set $\{x \in A;\ \mathcal{A} \models \varphi\,[x, a]\}$, up to finite sets. Semi-minimal theories and their infinite-dimensional models (those which have an infinite indiscernible set) are studied in [A 2], with which the reader is assumed to be acquainted; all definitions and notations are carried over from that paper. In the classification induced by Shelah's degree [S 1], semi-minimal theories are the simplest theories with infinite models: $v_0 = v_0$ has degree 1. They are, however, quite a bit more complex than, for example, strongly minimal theories (see [Ma], [BL]), and include all known examples of 'nontrivial' *uncountable* theories categorical in large powers (those which are not inessential extensions of a countable theory categorical in uncountable powers). On the other hand, there is at present no reason to believe this is not due to lack of imagination, while there are examples of semi-minimal heories which are not categorical.

This paper deals with the finite-dimensional models of an uncountable categorical semi-minimal theory. Such a theory, it is shown, has either a prime model or a finite-dimensional model. If, like all known examples, it enjoys a certain property called 1-reduction, it has exactly $\aleph_0$ nonisomorphic finite-dimensional models; in fact such a theory T possesses a well-ordered elementary chain $<\mathcal{A}_\beta>$ such that any model $\mathcal{A}$ of

T is isomorphic to $\mathcal{A}_{\varkappa}$ where $\varkappa$ is the (finite or infinite) type-dimension of $\mathcal{A}$; models of T are characterized, up to isomorphism, by their type-dimension.

T will denote throughout an uncountable semi-minimal theory in language L, categorical in one (hence any) power greater than $|T|$; $\mathcal{P}$ is a fixed partition sheaf for $v_0 = v_0$, $\mathcal{I}$ is the class of models of T of power at least $|T|$, $\mathcal{M}$ the class of all models of T. $\mathcal{I}$ need not exhaust $\mathcal{M}$, but assuming that it does is a reasonable simplification. A finite-dimensional model is one without an infinite indiscernible set.

1. *Omissible types and the density condition*

A type of T which is omitted in some $\mathcal{A}$ in $\mathcal{M}$ (resp. in $\mathcal{I}$) will be called omissible (resp. $\mathcal{I}$-omissible). As infinite-dimensional models are $\aleph_0$-saturated ([A 2, 3.14]), a type can only be omitted in a finite-dimensional model. Hence the problem of constructing a finite-dimensional model under the assumption that no type is omissible, problem which can also be relativized to $\mathcal{I}$. The first goal is to construct a $\mathcal{I}$-prime model (i.e. a model elementarily embeddable in any $\mathcal{A} \in \mathcal{I}$). This can be done in fact under a weaker assumption, as we now show.

T will be said to satisfy the *density condition* if any 1-formula of L is contained in some non-$\mathcal{I}$-omissible complete $\mathcal{P}$-type (this property is obviously independant of the partition sheaf $\mathcal{P}$; in fact it is equivalent to: non-$\mathcal{I}$-omissible complete 1-types of T are dense in S^T_1, the Stone space of all complete 1-types of T).

2. *Strongly minimal formulas and powerful sequences*

Among nonalgebraic complete $\mathcal{P}$-types, some may contain a strongly minimal formula of T, while the others are nonisolated (an isolating formula would be strongly minimal). The two cases often require separate treatment.

LEMMA 1 — *Let* ϑ *be a strongly minimal formula of* T. *There is a minimal number* n *such that for any model* $\mathcal{A} \in \mathcal{T}$, *any free set of n elements of* $\vartheta(\mathcal{A})$ *generates in* $\mathcal{A}$ *infinitely many elements of* $\vartheta(\mathcal{A})$.

* It is enough to show that for some n there exists a model $\mathcal{A} \in \mathcal{T}$ and n free solutions to ϑ in $\mathcal{A}$ generating in $\mathcal{A}$ infinitely many solutions to ϑ. Let $\mathcal{A}$ be a model of T of power $|T|$ and infinite type-dimension $\mu < |T|$. By Shelah's two-cardinal theorem for stable theories (see[S 1]), the categoricity of T forces $|\vartheta(\mathcal{A})| = |A| = |T|$. On the other hand, letting p be the unique nonalgebraic complete type containing ϑ, if X is a basis of $\vartheta(\mathcal{A})$, X is a basis of $p(\mathcal{A})$; so $|X| = \mu$, and X^*, the set of finite sequences in X, has power μ also. Therefore some finite subset Y of X generates infinitely many solutions to ϑ, because of the finite character of algebraic closure. If n is the cardinality of a minimal such finite $Y \subseteq X$, n fulfills our requirements. *

Given $\mathcal{A}$, any sequence of n free elements of $\vartheta(\mathcal{A})$, with n as above, will be called a *powerful sequence* of ϑ in $\mathcal{A}$ (the case $n = 0$ may occur). If $S \subseteq A$, a powerful sequence of ϑ *over S* in $\mathcal{A}$ will be a powerful sequence of ϑ in $[\mathcal{A}, S]$ (Th$[\mathcal{A}, S]$ is also semi-minimal and categorical, and has ϑ as a strongly minimal formula). We can now state a criterion for elementary substructures:

LEMMA 2 — *Let* $\mathcal{A}$ *be a model of* T, A_0 *a subset of* A. $\mathcal{A} \upharpoonright A_0$ *is an elementary substructure of* $\mathcal{A}$ *if* A_0 *is algebraically closed in* $\mathcal{A}$ *and contains, for each strongly minimal formula* ϑ *of* T, *a powerful sequence* s *of* ϑ *over* $A_0 - s$, *and for each finite* $\mathcal{P}$*-type* p, *an element realizing* p *in* $\mathcal{A}$.

* By [A 2, 1.13], we need only show that A_0 contains infinitely many solutions to every nonalgebraic formula ψ in $\mathcal{P}$. If ψ is almost implied by a strongly minimal formula ϑ, as A_0 is algebraically closed and contains a powerful sequence s of ϑ over $A_0 - s$ in $\mathcal{A}$, A_0 contains the infinitely many solu-

tions to ϑ generated by A_0. If ψ is not almost implied by any strongly minimal formula, one can define a sequence $(\psi_i ; i \in \omega)$ of formulas in $\mathcal{P}$ such that $\psi_0 = \psi$ and for each i, $\psi_0 \wedge \ldots \wedge \psi_i \wedge \psi_{i+1}$ and $\psi_0 \wedge \ldots \wedge \psi_i \wedge \neg \psi_{i+1}$ are both nonalgebraic. Now A_0 contains by hypothesis, for each i, an element x_i satisfying $\psi_0 \wedge \ldots \wedge \psi_i \wedge \neg \psi_{i+1}$ clearly $x_i \neq x_j$ for $i < j < \omega$. Hence A_0 contains infinitely many solutions to ψ. *

3. *Construction of a $\mathcal{T}$-prime model $\mathcal{A}_0$*

Assuming that T satisfies the density condition, we let $\mathcal{A}$ be a saturated model of T of power at least $|T|$, and proceed to construct a set $A_0 \subseteq A$ satisfying the above criterion in a fairly thrifty way. A_0 will be optimal in that respect if it turns out to have power $|T|$, in particular if $\mathcal{M} = \mathcal{T}$.

Let $\langle p_\alpha ; \alpha < \lambda_0 \leqslant |T| \rangle$ be a well-ordering of the nonalgebraic complete $\mathcal{P}$-types, and s_α the finite sequence defined by induction as follows (we set $S_\alpha = \bigcup_{\beta<\alpha} S_\beta$).

Case A: p_α is not isolated;

— if $cl^{\mathcal{A}} S_\alpha$ contains a point realizing p_α, or p_α is $\mathcal{T}$-omissible, $s_\alpha = \varnothing$;

— if not, $s_\alpha = \langle x_\alpha \rangle$, where x_α is any element of A realizing p_α.

Case B: p_α is isolated by the strongly minimal formula ϑ;

— s_α is a powerful sequence of ϑ over S_α in $\mathcal{A}$

(note that $s_\alpha = \varnothing$ iff $cl^{\mathcal{A}} S_\alpha$ contains infinitely many solutions to ϑ).

We now let $S = S_{\lambda_0}$ and $A_0 = cl^{\mathcal{A}} S$. Clearly A_0 satisfies the conditions of Lemma 2, therefore $\mathcal{A}_0 = \mathcal{A} \upharpoonright A_0$ is an elementary substructure of $\mathcal{A}$. Furthermore, $|A_0| \leqslant |T|$.

Now we show that $\mathcal{A}_0$ is a $\mathcal{T}$-prime model. Let us first observe that S is a basis of A_0 in $\mathcal{A}_0$. Now let $\mathcal{A}$ be any model in $\mathcal{T}$. We proceed to construct a $\mathcal{P}$ - $(\mathcal{A}_0, \mathcal{A})$ - elementary map f of S into a basis of $\mathcal{A}$. Proceeding by induction on $\alpha < \lambda_0$, we assume f has been defined on S_α and we must show: $\dim^{\mathcal{A}}_{fS_\alpha} p_\alpha(\mathcal{A}) \geqslant \ell(s_\alpha)$. In case A, if $cl^{\mathcal{A}_0} S_\alpha \cap p_\alpha(\mathcal{A}_0) = \varnothing$, then by induction hypothesis $cl^{\mathcal{A}} fS_\alpha \cap p_\alpha(\mathcal{A}) = \varnothing$, and if p_α is not $\mathcal{T}$-omissible, then $p_\alpha(\mathcal{A}) \neq \varnothing$, so there is in $\mathcal{A}$ an element y realizing p_α and free over fS_α: let fs = y; if $cl^{\mathcal{A}_0} S_\alpha \cap p_\alpha(\mathcal{A}_0) \neq \varnothing$ or p is $\mathcal{T}$-omissible, there is nothing to do as $s_\alpha = \varnothing$. In case B, if n is the length of a powerful sequence of ϑ over S_α in $\mathcal{A}_0$ $(n \geqslant 0)$, then on one hand $\ell(s_\alpha) = n$, on the other hand there is a powerful sequence t of ϑ over fS_α in $\mathcal{A}$, of length n; so we can extend f by any 1-1 map of s_α onto t. This completes the induction and shows that $\mathcal{A}_0$ can be elementarily embedded into $\mathcal{A}$.

Finally, if $\mathcal{A}$ and $\mathcal{B}$ are two $\mathcal{T}$-prime models of power $|T|$, they can be mutually embedded into each other. If one of them is finite-dimensional, these embeddings must be onto, for clearly no finite-dimensional model can be isomorphic to an elementary substructure of itself. If one of them is infinite-dimensional, the other one is as well, and both have type-dimension $\aleph_0$, so again they are isomorphic. Therefore, if there is a $\mathcal{I}$-prime model in $\mathcal{I}$, it is unique up to isomorphism. Summing up:

THEOREM 3 — *If the set Σ of non-$\mathcal{I}$-omissible complete 1-types of $\mathcal{I}$ is dense in S^T_1, then T has a $\mathcal{I}$-prime model $\mathcal{A}_0$.*

If $\mathcal{A}_0$ is in $\mathcal{I}$ (in particular if $|\Sigma| = |T|$ or $\mathcal{M} = \mathcal{I}$), $\mathcal{A}_0$ is unique up to isomorphism.

4. Pseudo-minimality of $\mathcal{A}_0$.

Let $\mathcal{A}$ be a model of T. We shall call $\mathcal{A}$ *pseudo-minimal* if A has a basis S in $\mathcal{A}$ such that if $R \subset S$, $\mathcal{A} \restriction \mathrm{cl}^{\mathcal{A}} R$ is not an elementary substructure of $\mathcal{A}$. Unfortunately that property seems to be weaker than minimality proper. We proceed nonetheless to show that $\mathcal{A}_0$ is pseudo-minimal under some (reasonable) additional assumption. Minimality proper will be discussed in § 6.

THEOREM 4 — *Let $\mathcal{A}_0 = \mathcal{A} \restriction \mathrm{cl}^{\mathcal{A}} S$ be the $\mathcal{I}$-prime model constructed in § 3. If $|S| = |T|$, or if $\mathcal{M} = \mathcal{I}$, or if any omissible 1-type of T is $\mathcal{I}$-omissible, then $\mathcal{A}_0$ is pseudo-minimal.*

* Throughout the proof we write $V^{a,\ldots,n}_{x,\ldots,z}$ for $(V \cup \{x, \ldots, z\}) - \{a, \ldots, n\}$. Let x_0 be an element of S. We shall show that $\mathcal{A}_0 \restriction \mathrm{cl}^{\mathcal{A}_0} S^{x_0} = \mathcal{B}$ is not an elementary sub-

structure of $\mathcal{A}_0$. This implies that for any $R \subset S$, $\mathcal{A}_0 \upharpoonright \mathrm{cl}^{\mathcal{A}_0} R$ is not an elementary substructure of $\mathcal{A}_0$. The idea is to show that no element of B can "fill in" for x_0. We suppose for contradiction that $\mathcal{B} \preccurlyeq \mathcal{A}_0$. Let p_{α_0} be the complete $\mathcal{P}$-type realized by x_0 in $\mathcal{A}_0$ and put $S' = S^{x_0}_{\alpha_0+1}$.

Step 1, Case A: p_{α_0} is nonisolated. If $|S| = |T|$ or if T has no model of power less than $|T|$, then $\mathcal{B} \in \mathcal{T}$. In both these cases $\mathcal{B}$ does not omit p_{α_0}; it is also true in the case where any omissible 1-type is $\mathcal{T}$-omissible. Therefore some $t \in B$ realizes p_{α_0}. On the other hand no element of $\mathrm{cl}^{\mathcal{A}_0} S_{\alpha_0}$ realizes p_{α_0} in $\mathcal{A}_0$, or else x_0 would not have been thrown in.

Step 1, Case B: p_{α_0} is isolated by the strongly minimal formula ϑ. Because $\mathrm{cl}^{\mathcal{A}_0} S'$ contains but finitely many solutions to ϑ (the reason why x_0 was thrown in), some element t of B satisfies ϑ and is not generated in $\mathcal{A}_0$ by S'.

Step 2: in either case, there are elements $x_1, \ldots, x_n$ of S such that, p_{α_i} being the complete $\mathcal{P}$-type realized by x_i $(1 \leqslant i \leqslant n)$, $\alpha_0 < \alpha_i \leqslant \alpha_j$ for $1 \leqslant i < j \leqslant n$, and $t \in \mathrm{cl}^{\mathcal{A}_0} S' x_1 \ldots x_n - \mathrm{cl}^{\mathcal{A}_0} S' x_1 \ldots x_{n-1}$. From this we infer:

(a) by the exchange principle, that $x_n \in \mathrm{cl}^{\mathcal{A}_0} A'wt$, with $w = \langle x_1, \ldots, x_{n-1} \rangle$

(b) because t and x_0 are free over S'w and realize the same $\mathcal{P}$-type, that $\tau(t, \mathcal{A}_0, S'w) = \tau(x_0, \mathcal{A}_0, S'w)$.
By (a), let $\pi(v_0, b, t)$, with $b \in (S'w)^*$, be an irreducible polynomial for x_n over S'wt, with say m solutions.

Step 3, Case A: P_{α_n}, the $\mathcal{P}$-type of x_n, is nonisolated. For all $\varphi \in p_{\alpha_n}$: $(\mathcal{A}_0, b, t) \models \exists^m v\pi(v, \overline{b}, \overline{t}) \wedge \forall v[\pi(v, \overline{b}, \overline{t}) \rightarrow \varphi(v)]$.

By (b), substituting x_0 for t in the above sentences preserves their validity in $\widetilde{\mathcal{A}_0}$. Yet together they imply that $cl^{\mathcal{A}_0} S_{\alpha_n}$ contains an element realizing p_{α_n}, which contradicts the fact that x_{α_n} was thrown in S.

Step 3, Case B: p_{α_n} is isolated by the strongly minimal formula ϑ. Let k, k + i, k + j be the lengths of powerful sequences of ϑ' over $S'wx_0x_n$, $S'wx_n$, $S'wx_0$ respectively. Naturally i and j are nonnegative; further, as it was necessary to throw in x_n at the α_n-th stage, j is strictly positive (in fact j = 1, for reasons of dimension).

Case α: i = 0. Let $\langle a_1, \ldots, a_k \rangle$ be a powerful sequence of ϑ' over $S'wx_n$, and for each m, π_m be a formula and s_m a sequence in S'w such that $\pi_m(v_0, \overline{a}_1, \ldots, \overline{a}_k, \overline{s}_m, \overline{x}_n)$ has m' solutions $(m \leqslant m' < \aleph_0)$ satisfying ϑ' in $\widetilde{\mathcal{A}_0}$. Then x_n satisfies in$[\mathcal{A}_0, S'w]$ every formula $\varphi_{m'}(u)$, where $\varphi_m(u)$ is:

$$\exists z_1 \ldots \exists z_k [\exists^m v\, \pi_m(v, z_1, \ldots, z_k, \overline{s}_m, u) \wedge \vartheta'(u) \wedge \bigwedge_{1 \leqslant \ell \leqslant k} \vartheta'(z_\ell)].$$

(c) for infinitely many m, $[\mathcal{A}_0, S'wt] \models \forall v\, [\pi(v, \overline{b}, \overline{t}) \rightarrow$

$\varphi_m(v)]$. By (b) let x'_n be a solution in $(\mathcal{A}_0, b, x_0)$ to the polynomial $\pi(v_0, b, x_0)$. Then x'_n generates over S'w together with k well chosen solutions to ϑ' infinitely many more; for if not, the number of solutions to ϑ' obtained with the help of k solutions would be bounded, in contradiction with (c) where x_0 is legitimately substituted to t (by (b)). But now $S'wx_0$ generates x'_n therefore k is the length of a powerful sequence of ϑ' over $S'wx_0$ in $\mathcal{A}_0$, contradicting $j > 0$.

Case β: $i > 0$. Let $z_1, \ldots, z_k$ be k elements of B which are solutions to ϑ' in $\mathcal{A}_0$ and are free over $S'wx_n$: we know that not only such elements exist, but that they are not sufficient to generate infinitely many solutions to ϑ', because $i > 0$ (while $\mathcal{B}$ must have infinitely many solutions to ϑ', being an elementary substructure of $\mathcal{A}_0$). By the exchange principle, and because S is free, $\{z_1, \ldots, z_k\}$ is a free set over $S'wx_nx_0$. So $\{z_1, \ldots, z_k\}$ generates over $S'wx_nx_0$ infinitely many solutions to ϑ', one of which, say u, is not generated by $\{z_1, \ldots, z_k\}$ over $S'wx_n$. Thus: $u \in cl^{\mathcal{A}_0} S'wx_nx_0z_1 \ldots z_k \; - \; cl^{\mathcal{A}_0} S'wx_nz_1 \ldots z_k$. Exchanging: $x_0 \in cl^{\mathcal{A}_0} S'wx_nz_1 \ldots z_k u \subseteq cl^{\mathcal{A}_0} S^0$, contradicting the freedom of S. *

COROLLARY 5 — *If the nonomissible complete 1-types of* T *are dense in* S_1^T, *then* T *has a prime model which is pseudominimal and unique up to isomorphism.*

* The first point is that, if we strengthen the density condition by replacing "non-$\mathcal{T}$-omissible" by "nonomissible", we can perform the construction leading to $\mathcal{A}_0$ with the same modification and obtain a model $\mathcal{A}'_0$ which is prime instead of just $\mathcal{T}$-prime. The second point is that by the same token, the proof of Theorem 4 shows that $\mathcal{A}'_0$ is pseudo-minimal. The final point is that if $\mathcal{B}$ is another prime model of T, it is em-

beddable in $\mathcal{A}'_0$, which in turn is embeddable in $\mathcal{B}$. The isomorphism of $\mathcal{A}'_0$ and $\mathcal{L}$ follows as before. *

5. *Finite-dimensional extensions*

Starting with a finite-dimensional model $\mathcal{A}_0$, it is easy to construct nonisomorphic elementary extensions of $\mathcal{A}_0$. Letting $\mathcal{A}$ be any proper elementary extension of $\mathcal{A}_0$, u an element of $A - A_0$ of type p, X a basis of $p(\mathcal{A}_0)$, consider $T' = \mathrm{Th}[\mathcal{A}, X \cup \{u\}]$: T' has either a finite-dimensional model or a prime model $\mathcal{B}'$. On the other hand, $\mathcal{A}_0$ has a manimal elementary extension, *viz.* $\mathcal{B} = \mathcal{A} \restriction \mathrm{cl}^{\mathcal{A}}(A_0 \cup \{u\})$. However, nothing seems to prevent $\mathcal{B}$ and $\mathcal{B}'$ from being infinite-dimensional, in which case they must be isomorphic; the point is that *relative* type-dimensions could conceivably be larger in the smaller model $\mathcal{A}_0$ than in the larger model $\mathcal{B}$. We might mention the following very special case:

PROPOSITION 6 — *If* T *is irreducible,* $\aleph_0 < |T| < 2^{\aleph_0}$ *and* $|T|$ *is regular, then* T *has at least* $\aleph_0$ *nonisomorphic finite-dimensional models of power* $|T|$.

* If T had no finite-dimensional model of power $|T|$, every model in $\mathcal{T}$ would be $\aleph_0$-saturated, thus would realize every type of T in any number of variables. Then by [K, Theorem B], under the hypotheses on $|T|$, T would be reducible. So T has a finite-dimensional model of power $|T|$; call it $\mathcal{A}$. For the same reason, if $\mathcal{B}$ is a proper elementary extension of $\mathcal{A}$ and $u \in B - A$ realizes the type p of T, and X is a basis of $p(\mathcal{A})$ in $\mathcal{A}$, then $\mathrm{Th}[\mathcal{B}, X \cup \{u\}]$ has a finite-dimensional model whose contraction to $\mathcal{L}$, the language of T, is finite-dimensional and not isomorphic to $\mathcal{A}$. Iterating this construction, we obtain the desired models. *

A characteristic feature of finite-dimensional models is total homogeneity (a structure $\mathcal{A}$ is *totally homogeneous* if *any* partial automorphism of $\mathcal{A}$ can be extended to an automorphism of $\mathcal{A}$). Infinite-dimensional models are either not homogeneous (if their type-dimension is less than their power) or (in the other case) homogeneous and not totally homogeneous ([A 2, 3.15]):

PROPOSITION 7 — *The finite-dimensional models of* T *are totally homogeneous.*

* Let $\mathcal{A}$ be a finite-dimensional model of T, X a subset of A, f an $(\mathcal{A}, \mathcal{A})$-elementary map of domain X. Let $\langle p_\alpha ;\ \alpha < \lambda_0 \rangle$ be a well-ordering of the complete $\mathcal{P}$-types of T. We define by induction an increasing sequence f_α of partial automorphisms of $\mathcal{A}$ extending f such that $p_\alpha(\mathcal{A}) \subseteq$ domain $f_{\alpha+1} \cap$ range $f_{\alpha+1}$. Assuming f_α has been defined, with domain $f_\alpha = X_\alpha$, we consider $cl^{\mathcal{A}} X_\alpha \cap p_\alpha(\mathcal{A})$, of dimension say m_α. Let n_α be the dimension of $p_\alpha(\mathcal{A})$. If $m_\alpha = n_\alpha$, then $p_\alpha(\mathcal{A}) \subseteq cl^{\mathcal{A}} X_\alpha$, or else there would be $n_\alpha + 1$ independent elements in $p_\alpha(\mathcal{A})$; in that case we set $f_{\alpha+1}$ to be an elementary map extending f to $cl^{\mathcal{A}} X_\alpha$. If $m_\alpha < n_\alpha$, let x be an element in $p_\alpha(\mathcal{A}) - cl^{\mathcal{A}} X_\alpha$. By induction hypothesis, $\dim^{\mathcal{A}}(cl^{\mathcal{A}}(f_\alpha X_\alpha) \cap p_\alpha(\mathcal{A})) = \dim^{\mathcal{A}}(cl^{\mathcal{A}} X_\alpha \cap p_\alpha(\mathcal{A})) = m_\alpha$, so there is an element y in $p_\alpha(\mathcal{A}) - cl^{\mathcal{A}}(f_\alpha X_\alpha)$. We set

$f^1_\alpha x = y$ and $f^1_\alpha \upharpoonright X = f_\alpha$. f^1_α is $(\mathcal{A}, \mathcal{A})$-elementary.

We subject f^1_α to the same treatment as f_α, and thus obtain, after finitely many steps (at most $n_\alpha - m_\alpha$ in fact) an $(\mathcal{A},\mathcal{A})$-elementary map f^k_α extending f_α and whose domain and range each generate $p_\alpha(\mathcal{A})$. We let $f_{\alpha+1}$ be some extension of f^k_α to the closure of its domain. At a limit ordinal δ, we take $f_\delta = \bigcup_{\beta<\alpha} f_\beta$. f_{λ_0} is the desired isomorphism. *

6. *The n-reduction property*

Let T be a semi-minimal theory, $\mathcal{A}$ a model of T. As the algebraic closure has finite character, and as a set X free in $\mathcal{A}$ of elements realizing in $\mathcal{A}$ the same 1-type of T is indiscernible in $\mathcal{A}$, the following is clear: given two complete 1-types p, q of T, there is a number $N = N^{\mathcal{A}}_{p,q}$ such that if some elements of $p(\mathcal{A})$ generate in $\mathcal{A}$ a free element of $q(\mathcal{A})$, then any set free in $\mathcal{A}$ of N elements of $p(\mathcal{A})$ generates in $\mathcal{A}$ a free element of $q(\mathcal{A})$. Of course, $N^{\mathcal{A}}_{p,q}$ is actually independent of $\mathcal{A}$. We shall say that T *has the n-reduction property* if for any complete inessential extension T' of T, any model $\mathcal{A}'$ of T', any two complete 1-types p and q of T', $N^{\mathcal{A}'}_{p,q} \leqslant n$. All 'non-trivial' uncountable categorical theories known have the 1-reduction property.

Now let $\mathcal{D}_n$ be the class of models of T which for some complete 1-type of T have p-dimension at least n. Categoricity of T, which was not required in the presentation of n-reduction, is again assumed.

LEMMA 8 — *If* T *verifies n-reduction for some* $n \geqslant 1$, *and* $\mathcal{A}$ *is a model in* $\mathcal{D}_n$, *then* $\mathcal{A}$ *omits no 1-type of* T.

* Let p be a complete 1-type of T such that $\dim^{\mathcal{A}} p \geqslant n$, and let X be a basis of $p(\mathcal{A})$, completed by Z to a basis of A in $\mathcal{A}$. Let $\mathcal{C}$ be an $\aleph_0$-saturated elementary extension of $\mathcal{A}$, and X' extend X as a basis of $p(\mathcal{C})$. By [A 2, 3.1.a], $\mathcal{B} = \mathcal{C} \restriction \mathrm{cl}^{\mathcal{C}}(X' \cup Z) \preccurlyeq \mathcal{C}$. As $\mathcal{B}$ is an infinite-dimensional model of T, $\mathcal{B}$ is $\aleph_0$-saturated; therefore, given a complete 1-type q of T, there exists $z \in B$ realizing q in $\mathcal{B}$. z is generated in $[\mathcal{B}, Z]$ by some elements of X'. If $z \in A$, we are done. If not, z is free over Z in $[\mathcal{B}, Z]$. Therefore, by n-reduction, any n elements of X' generate in $[\mathcal{B}, Z]$ an element realizing q in $[\mathcal{B}, Z]$. In particular, pick n elements of X: they generate in $[\mathcal{B}, Z]$ an element z' realizing q. But $z' \in A$ and $[\mathcal{A}, Z] \preccurlyeq [\mathcal{C}, Z]$. Therefore z' realizes q in $\mathcal{A}$. *

LEMMA 9 — *Suppose* T *verifies n-reduction for some* $n \geqslant 1$, *and let* N *be any number. If* $\mathcal{A}$ *is a model in* $\mathcal{D}_{n+N}$, *then* $\mathcal{A}$ *omits no* (N + 1)*-type of* T.

* We show by induction on $k \leqslant N$ that $\mathcal{A}$ omits no (k + 1)-type of T. The case k = 0 follows from Lemma 8. Assuming the conclusion for k, let σ be a (k + 1)-type of T. Let $\mathcal{B}$ be an elementary extension of $\mathcal{A}$ with a (k + 1)-tuple $\langle b_0, \ldots, b_k \rangle$ realizing σ, and let $\sigma' = \tau(\langle b_1, \ldots, b_k \rangle, \mathcal{B})$. By the induction hypothesis, some sequence $a = \langle a_1, \ldots, a_k \rangle$ realizes σ' in $\mathcal{A}$. Let $T' = \mathrm{Th}(\mathcal{A}, a)$. T' verifies n-reduction, and for some complete 1-type p of T, $\dim^{(\mathcal{A}, a)} p \geqslant N + n - k \geqslant n$, as $\mathcal{A} \in \mathcal{D}_{n+N}$ and by the dimension theorem. Therefore $(\mathcal{A}, a)$ realizes every 1-type of T', by Lemma 8, in particular some element a_0 realizes in $(\mathcal{A}, a)$ the type $\{\varphi(v_0, \bar{a}_1, \ldots, \bar{a}_k) \mid \mathcal{B} \models \varphi[b_0, \ldots, b_k]\}$, so $a_0 a$ realizes σ in $\mathcal{A}$. *

Let us now substitute $\mathcal{D}_n$ for $\mathcal{T}$ in the construction of § 3; we obtain a model $\mathcal{B}_0^n$ of T. Theorem 3 can then be restated as follows, by Lemma 8:

LEMMA 10 — *Let* T *verify* n*-reduction for some* $n \geqslant 1$. *There exists a* $\mathcal{D}_n$*-prime model* $\mathcal{B}^n_0$ *of* T. *If* $\mathcal{B}^n_0$ *belongs to* $\mathcal{D}_n$, *then it is unique up to isomorphism.*

The main improvement due to n-reduction is the following:

THEOREM 11 — $\mathcal{B}^n_0$ *is finite-dimensional. In fact there is a nonalgebraic complete 1-type* p_0 *of T such that* $\mathcal{B}^n_0$ *has* p_0 *— dimension at most* n.

* (The notations are carried over from § 3). B^n_0 is determined as $cl^{\mathcal{A}} S$. $\langle p_\alpha ; \alpha < \lambda_0 \rangle$ is the well-ordering of the nonalgebraic $\mathcal{P}$-types used to construct S. Suppose for contradiction that $\dim^{\mathcal{B}^n_0} p_0$ is greater than n. For $1 \leqslant i \leqslant n + 1 - \ell(s_0)$ (note that $\ell(s_0) \leqslant n$ by n-reduction), let σ_i be a minimal (finite) sequence in S generating over s_0 an element x_i realizing p_0 and free over $s_0 x_1 \ldots x_{i-1}$. Let $\sigma = \bigcup_{1 \leqslant i \leqslant n} \sigma_i$ $(\sigma \neq \emptyset)$ and $z \in \sigma_i$ be an element whose type has highest rank α in the hierarchy $\langle p_\alpha ; \alpha < \lambda_0 \rangle$ among elements of σ. As σ_i is minimal, $x_i \in cl_{s_0}^{\mathcal{B}^n_0} \sigma_i - cl_{s_0}^{\mathcal{B}^n_0} \sigma_i^{z}$, so by exchange: $z \in cl_{s_0}^{\mathcal{B}^n_0} \sigma_i^{z} x_i$. Pick n elements $y_1, \ldots, y_n$ among $s_0 x_1 \ldots x_n^{x_i}$. By definition $\{x_i, y_1, \ldots, y_n\}$ is a free set.

Case A: p_α is nonisolated. Let $C \subseteq S_\alpha$ complete $x_i y_1 \ldots y_n$ to a basis of $S_{\alpha+1}$. As $x_i y_1 \ldots y_n$ generates over C an element z free over C and realizing p_α, by n-reduction $y_1 \ldots y_n$ generates over C an element z' realizing p_α. As p_α is nonisolated,

$Cy_1 \ldots y_n \subseteq S_\alpha$ generates no element realizing p_α, or else z would not have been thrown in: contradiction.

Case B: p_α is isolated by a strongly minimal formula ϑ. Let $C \subseteq S^z_{\alpha+1}$ complete $x_i y_1 \ldots y_n$ to a basis of $S_{\alpha+1}$. Let m be the length of a powerful sequence of ϑ over C. $m \geqslant 1$ or else z would not have been thrown into s_α. Now $x_i y_1 \ldots y_n$ generates, freely over C, an element z satisfying ϑ free over C. So $y_1 \ldots y_n$ alone generates over C an element z' satisfying ϑ free over C. z satisfies, for each N and some formula π_N of L(C) and some finite $N' \geqslant N$, the 1-formula $\exists v_1 \ldots \exists v_{m-1} (\bigwedge_{1 \leqslant i \leqslant m-1} \vartheta(v_i) \wedge \exists^{N'} v_m [\pi_N(v_0, v_1, \ldots, v_m) \wedge \vartheta(v_m)])$. Therefore z' satisfies the same formulas. On the other hand, as $S^z_{\alpha+1}$ contains a powerful sequence for φ over Cz, $S^z_{\alpha+1}$ contains $m-1$ solutions to ϑ free over C. The formulas written above, which z' satisfies, imply that $m-1$ solutions to ϑ free over C generate over C a number of solutions to ϑ which, being unbounded, is infinite (for N large enough, only free $v_1 \ldots v_{m-1}$ will do, so any free $v_1 \ldots v_{m-1}$ will do). Altogether, $S^z_{\alpha+1}$ generates infinitely many solutions to ϑ: this contradicts the definition of $z \in s_\alpha$. *

In order to obtain other finite-dimensional models of T, one simply starts with $\mathcal{B}^n_0$ and constructs successive finite-dimensional models $\mathcal{B}^n_i$ in the following way: let $\mathcal{C}$ be an $\aleph_0$-saturated elementary extension of $\mathcal{B}^n_i$, X a basis of $p_0(\mathcal{B}^n_i)$, u an element of $p_0(\mathcal{C}) - B^n_i$, and $T' = Th[\mathcal{C}, X \cup \{u\}]$. As T' has the same properties as T, including n-reduction, one can take a $\mathcal{D}'_n$-prime model $\mathcal{B}'$ of T' ($\mathcal{D}'_n$ being the class of models of T' which for some complete 1-type p of T have p-dimension at least n). $\mathcal{B}'$ is finite-dimensional, therefore its contraction to the language L of T is a finite-dimensional model $\mathcal{B}^n_{i+1}$ of T of strictly greater p_0-dimension that $\mathcal{B}^n_i$. Therefore

COROLLARY 12 — *If* T *verifies n-reduction for some* $n \geqslant 1$, T *has at least* $\aleph_0$ *nonisomorphic finite-dimensional models. If in addition, for some* N, T *has* $|T|$ *complete (1 + N)-types (in particular if* T *is irreducible and* $\mathrm{cf}|T| > \omega$), *then these models can be chosen to have power* $|T|$.

* The last part follows from Lemma 9: $\mathcal{B}^n_{n+N}$ has finite p_0-dimension at least $n + N$, therefore $\mathcal{B}^n_k$, for $k \geqslant n + N$, realizes every complete $(1 + N)$-type of T. *

LEMMA 13 — *Let* T *verify n-reduction for some* $n \geqslant 1$, *and* $\mathcal{A} \leqslant \mathcal{B}$ *be models of* T. *If there is a complete 1-type q of* T *such that* $\dim_A^{\mathcal{B}} q \geqslant n$, *then for any nonalgebraic 1-type* p *of* T, $p(\mathcal{A}) \neq p(\mathcal{B})$.

* In the case where $\mathcal{A}$ is infinite-dimensional, this was shown in the proof of [A 2, 3.16] (n-reduction superfluous). Now suppose $\mathcal{A}$ is finite-dimensional. Let $\mathcal{D}$ be an infinite-dimensional elementary extension of $\mathcal{B}$ and X a basis of $q(\mathcal{D})$ over A. Finally let $\mathcal{C} = \mathcal{D} \upharpoonright \mathrm{cl}^{\mathcal{D}}(A \cup X)$. As usual, $\mathcal{C} \leqslant \mathcal{D}$, and because $\mathcal{A}$ is finite-dimensional, $p(\mathcal{C})$ contains an element u not in A, therefore free over A in $\mathcal{C}$. In $\mathcal{D}$, u is generated over A by X, therefore by n-reduction any set free over A of n elements realizing q generates an element free over A realizing p. This is also true in $\mathcal{B} \leqslant \mathcal{D}$ and $\mathcal{B}$ contains n elements of the right sort; in $\mathcal{B}$ they generate over A an element u' free over A and realizing p: $u' \in p(\mathcal{B}) - p(\mathcal{A})$. *

Thus finite-dimensional models show some signs of good behavior under n-reduction: on one hand there are infinitely many of them, on the other no large "distortions" are possible when one extends the other, a limiting factor for possible isomorphism types. Yet only under the hypothesis of 1-reduction are we able to settle the matter completely.

7. 1-reduction and the elementary chain

As observed earlier, 1 reduction is verified by known examples of nontrivial uncountable categorical theories. It might not therefore be as preposterous a postulate as would seem at first.

Let T be a categorical semi-minimal theory with 1-reduction, and $\mathcal{B}_\omega$ an $\aleph_0$-saturated model of T. We first observe that the class $\mathcal{D}_1$ augmented, if it is an elementary substructure of $\mathcal{B}_\omega$, by $\mathcal{B}_0 = \mathcal{B}_\omega \restriction \mathrm{cl}^{\mathcal{B}_\omega} \varnothing$, constitutes up to isomorphism the class of all models of T. In other words, either $\mathcal{B}_0$ is a model of T, and then of course $\mathcal{B}_0$ is a prime model of T, or $\mathcal{B}_0$ is not a model of T and then the $\mathcal{D}_1$-prime model of T is a prime model of T. Next we show:

THEOREM 14 — *If* T *verifies 1-reduction, any model of* T *has a minimal prime extension.*

* Let $\mathcal{A}$ be a model of T. If $\mathcal{A}$ is infinite-dimensional, the claim has been established in [A 2]. If not, let X be a basis of A and, assuming without loss that $\mathcal{A} \leqslant \mathcal{B}_\omega$, let u be any element of B_ω — A: u realizes a nonalgebraic complete 1-type p. Let $\mathcal{B} = \mathcal{B}_\omega \restriction \mathrm{cl}^{\mathcal{B}_\omega}(A \cup \{u\})$. Let $\mathcal{A}'$ be any proper elementary extension of $\mathcal{A}$: there exists an element t in $q(\mathcal{A}')$ — $q(\mathcal{A})$ for some complete 1-type q. By 1-reduction and Lemma

13, $p(\mathcal{A}') - p(\mathcal{A})$ contains an element u'. Then clearly the identical map of X extended by sending u onto u' is elementary, and thus extends to an elementary embedding of $\mathcal{B}$ onto $\mathcal{A}'$ over A. Thus $\mathcal{B}$ is a prime extension of $\mathcal{A}$. It is obvious that $\mathcal{B}$ is also a minimal elementary extension of $\mathcal{A}$. *

We shall now construct an elementary tower of finite-dimensional models. Let $\mathcal{A}_0$ be defined as $\mathcal{B}_\omega \restriction \mathrm{cl}^{\mathcal{B}_\omega} \varnothing$ only if it is a model of T. If $\mathcal{A}_0$ is not defined, let $\mathcal{A}_1$ be a $\mathcal{D}_1$-prime model in $\mathcal{B}_\omega$. If $\mathcal{A}_0$ is defined, let $\mathcal{A}_1$ be a prime extension of $\mathcal{A}_0$ in $\mathcal{B}_\omega$ ($\mathcal{A}_1$ may or may not be $\mathcal{D}_1$-prime). By induction on $n \geqslant 1$, we let $\mathcal{A}_{n+1}$ be a prime extension of $\mathcal{A}_n$ in $\mathcal{B}_\omega$. As for each n there is clearly an N such that $\mathcal{A}_n$ is elementarily embeddable in the model $\mathcal{B}^1_N$ of the previous section, $\mathcal{A}_n$ is finite-dimensional. But let us see more precisely that $\dim^{\mathcal{A}_n} p_0 = n$. The proof of Theorem 11 shows that $\dim^{\mathcal{A}_1} p_0 = 1$, whether $\mathcal{A}_0$ or $\mathcal{A}_1$ be $\mathcal{D}_1$-prime. Therefore in $p_0(\mathcal{A}_1)$ there is a free element u_1 (which may or may not belong to s_0) such that s_0u_1 is a basis of $p_0(\mathcal{A}_1)$. As the proof of 14 shows, $\mathcal{A}_{n+1}$ can be seen as $\mathcal{B}_\omega \restriction \mathrm{cl}^{\mathcal{B}_\omega}(A_n \cup \{u_n\})$ for some $u_n \in p_0(\mathcal{B}_\omega) - p_0(\mathcal{A}_n)$. By induction, suppose that $s_0u_1...u_n$ generates $p(\mathcal{A}_n)$ in $\mathcal{A}_n$, and for contradiction that $s_0u_1...u_{n+1}$ does not generate $p(\mathcal{A}_{n+1})$ in $\mathcal{A}_{n+1}$. Let u be an element of $p(\mathcal{A}_{n+1})$ free over $s_0u_1...u_{n+1}$. By exchange, as Su_1 generates A_1 in $\mathcal{A}_1$, it is easily seen that for some $z \in S$ free over $S^z u_1...u_n = V$, $z \in \mathrm{cl}^{\mathcal{A}_{n+1}} Vu_{n+1}u$. By 1-reduction, u generates over V an element z' of same type as z free over V. Exchanging again: $u \in \mathrm{cl}^{\mathcal{A}_{n+1}} Vz'$. As z and z' are indiscernible over V, z therefore generates an element u' of type p_0 free over $s_0u_1...u_n$. But this means that $s_0u_1...u_n$ does not generate $p_0(\mathcal{A}_n)$, contrary to our induction hypothesis.

It is also clear that for any nonalgebraic complete type p' of T, $\dim^{\mathcal{A}_n} p' = n$. Indeed, the model $\mathcal{A}'_n$ obtained like $\mathcal{A}_n$ starting with p' instead of p is isomorphic to $\mathcal{A}_n$.

Finally, $\mathcal{A}_\omega = \bigcup_{n\in\omega} \mathcal{A}_n$ is the (essentially unique) model of type-dimension $\aleph_0$. Now let $\mathcal{A}$ be any finite-dimensional model; there is an embedding of $\mathcal{A}_0$ into $\mathcal{A}$ which, provided it is not onto, can be extended to an elementary embedding of $\mathcal{A}_1$ into $\mathcal{A}$, and so forth. The process ends after finitely many steps, or else $\mathcal{A}$ would be an elementary extension of an isomorphic image of $\mathcal{A}_\omega$, and therefore be infinite-dimensional. Thus for some $k \in \omega$, $\mathcal{A}$ is isomorphic to $\mathcal{A}_k$.

If T verifies 1-reduction, the notion of type-dimension, previously defined for infinite-dimensional models, clearly makes sense also for finite-dimensional models. Our conclusion may therefore be worded in the following way:

THEOREM 15 — *If* T *verifies 1-reduction,* T *has an elementary chain of models* $<\mathcal{A}_\beta; \varepsilon \leqslant \beta>$ $(\varepsilon = 0$ or $\varepsilon = 1)$ *such that* $\mathcal{A}_\varepsilon$ *is a prime model,* $\mathcal{A}_{\beta+1}$ *is a minimal prime extension of* $\mathcal{A}_\beta$, $\mathcal{A}_\delta = \bigcup_{\beta<\delta} \mathcal{A}_\beta$ *for a limit ordinal* δ, *and any model* $\mathcal{A}$ *of* T *is isomorphic to any* $\mathcal{A}_\beta$ *such that* $|\beta|$ *equals the type-dimension of* $\mathcal{A}$. *For* $\beta \geqslant \aleph_0$, $|A_\beta| = |\beta| + |T|$. *For* $\beta < \aleph_0$, $|A_\beta| \leqslant T$. *Therefore* T *has* $|\alpha| + \aleph_0$ *isomorphism types of models of power at most* $|T| = \aleph_\alpha$. *If in addition for some* $N \geqslant 1$, T *has* $|T|$ *complete N-types, there is an* $m \leqslant N + 1$

such that $\mathcal{A}_m$ belongs to $\mathcal{T}$ and is $\mathcal{T}$-prime; then in power $|T|$, T *has* $\aleph_0$ *finite-dimensional models.*

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The significance of the n-reduction property, in particular when n = 1, for categorical theories, remains to be ascertained; therefore the question of how many finite-dimensional models, if any, a categorical semi-minimal theory can have, remains open.

[A] D. ANDLER, *Models of uncountable theories categorical in power*, doct. diss., Univ. of California, Berkeley, 1973.

[A1] D. ANDLER, *Notices of the A.M.S.* 19 (Nov. 1972), 72T-E109.

[A2] D. ANDLER, Semi-minimal theories and categoricity, *J. of Symbolic* Logic 40 (1975), 419-438.

[BL] J.T. BALDWIN and A.H. LACHLAN, On strongly minimal sets, *J. of Symbolic Logic* 36 (1971), 79-96.

[K] H.J. KEISLER, On theories categorical in their own power, *J. of Symbolic Logic* 36 (1971), 240-244.

[M1] M.D. MORLEY, Categoricity in power, *Trans. A.M.S.* 114 (1965), 514-538.

[M2] M.D. MORLEY, Countable models of $\aleph_1$-categorical theories, *Israel J. of Math.* 5 (1967), 65-72.

[Ma] W.E. MARSH, *On ω_1-categorical but not ω-categorical theories*, doct. diss., Dartmouth College, 1966.

[S1] S. SHELAH, Stability, the f.c.p. and superstability, *Annals of Math. Logic* 3 (1971), 271-362.

[S2] S. SHELAH, Categoricity of uncountable theories, in *Proc. of the Tarski Symp.*, Berkeley 1971, L. Henkin ed., Proc. of Symp. in Pure Math. XXV, A.M.S. 1974.

[V] R.L. VAUGHT, Denumerable models of complete theories, in *Infinitistic methods*. Pergamon Press, New-York & Krakow, 1961, 303-321.

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