COUNTING COUNTABLE E.C. STRUCTURES

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These lectures are concerned with the following central theme:

> Given a countable first order theory T, to describe the spectrum of the countable members of E_T with particular reference to the subclasses F_T, G_T.

Thē lectures are divided into 14 sections, and these sections are grouped into four chapters. We give here a brief description of the content of these chapters.

Chapter A contains the required background material on e.c. structures, etc. Most of this material is now well known, however, if required, the reader can find full details in [13].

Chapter B is an $\exists_1$-version of the classical results concerning the countable models of a complete theory (i.e. the results of [1, pp 93-106]). This chapter is based on [12] which itself is based on [5], [6], [7].

Chapters C, D are based on [4] however some of the proofs given here are quite different to those in [4]. Both chapters are concerned with counting e.c. structures up to elementary equivalence. Chapters C contains various cardinality results and chapter D shows there is an underlying topology which partly controls the behaviour of the generic structures of a theory.

CHAPTER A

Required background material

§ 1. *Notation, terminology, etc.*

The subject matter of these lectures is a part of first order model theory with the occasional inessential use of $L_{\infty,\omega}$-languages. Our notation and terminology is standard except for a few minor differences and additons. The main differences from the standard notation is that, for typographical reasons, we use $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, ... for structures (instead of the usual gothic letters) and we use *K, E, F, G,* ... for classes of structures (instead of the usual script letters).

Throughout L is some fixed countable first order language. All our model theoretic concepts (e.g. formula, structure, theory, etc.) are relative to L. The countability of L is important.

For each set of formulas Γ, fv(Γ) is the set of variables occuring free in members of Γ. A type is such a set Γ with fv(Γ) finite. Sometimes we write $\Gamma(\mathbf{v})$ for Γ to indicate that $\mathbf{v}$ is the finite sequences of free variables of Γ. This notation is most frequently used when Γ is a singleton $\{\varphi\}$, when we write fv(φ) for fv($\{\varphi\}$) and $\varphi(\mathbf{v})$ for $\{\varphi\}(\mathbf{v})$. A sentence is, of course, a formula φ with fv(φ) $= \varnothing$.

For each $n \in \omega$ the sets of formulas $\forall_n$, $\exists_n$ are defined in the usual way. A type Γ is an $\forall_n$-type or an $\exists_n$-type if $\Gamma \subseteq \forall_n$ or $\Gamma \subseteq \exists_n$.

Each type Γ gives us two $L_{\infty,\omega}$-formulas, i.e. the conjunction $\bigwedge\Gamma$ and the disjunction $\bigvee\Gamma$ of Γ.

A theory T is just a deductively closed (consistent) set of sentences. At various places we tacitly assume that each occuring theory has no finite models. For each theory T we write *Md*(T) for the class of models of T and *Sb*(T) for the class of submodels of T (i.e. the class of all structures $\mathcal{A}$ such that there is some $\mathcal{B} \models T$ with $\mathcal{A} \subseteq \mathcal{B}$). Two theories T, T' are cotheories if $Sb(T) = Sb(T')$ i.e. if $T \cap \forall_1 = T' \cap \forall_1$.

For a theory T and formula φ we write $T \models \varphi$ to indicate that each model of T satisfies the universal closure of φ. This notation is particularly used when φ is an $L_{\infty,\omega}$-formula.

Let $\mathcal{A}$ be a structure. Quite often we will be concerned with a finite sequence **a** of elements of $\mathcal{A}$. For convenience we call such a sequence a point of $\mathcal{A}$. Very often such a point **a** will occur in conjunction with a formula $\varphi(\mathbf{v})$, when it will be assumed that the point **a** exactly matches the sequence **v**. Thus we may write $\mathcal{A} \models \varphi(\mathbf{a})$.

Given a point **a** of a structure $\mathcal{A}$ we can form a new structure $(\mathcal{A}, \mathbf{a})$ by enriching $\mathcal{A}$ with **a**. Of course this new structure is a structure for a larger language.

The sets of formulas $\forall_n$, $\exists_n$ induce several relations between structures. First there are the well known substructure relations $\mathcal{A} <_n \mathcal{B}$ and the associated n-embeddings $\mathcal{A} \to \mathcal{B}$. Then there are the relations.

$$\mathcal{A} \Rrightarrow (\exists_n)\, \mathcal{B} \quad , \quad \mathcal{A} \Rrightarrow (\forall_n)\, \mathcal{B} \quad , \quad \mathcal{A} \equiv_n \mathcal{B}.$$

The first of these means that each $\exists_n$-sentence which holds in $\mathcal{A}$ also holds in $\mathcal{B}$. The second is just $\mathcal{B} \Rrightarrow (\exists_n)\, \mathcal{A}$, and the third is the conjunction of the first two. The following theorems give a useful characterization of $\Rrightarrow (\exists_{n+1})$.

1.1. *THEOREM. For each two structures* $\mathcal{A}$, $\mathcal{B}$ *and* $n \in \omega$ *the following are equivalent.*

(i) $\mathcal{A} \Rrightarrow (\exists_{n+1})\, \mathcal{B}$.

(ii) *There is a structure* $\mathcal{C}$ *together with an* n-*embedding* $\mathcal{A} \to \mathcal{C}$ *and an elementary embedding* $\mathcal{B} \to \mathcal{C}$.

(iii) *There is a structure* $\mathcal{C}$ *together with an* n-*embedding* $\mathcal{A} \to \mathcal{C}$ *and an* n+1-*embedding* $\mathcal{B} \to \mathcal{C}$.

The relation $\Rrightarrow (\exists_1)$ will be used quite often when we are dealing with theories T with JEP. A characterization of this property (JEP) is given in 2-18.

Let $\mathcal{A}$, $\mathcal{B}$ be a pair of structures. A partial isomorphism for $\mathcal{A}$, $\mathcal{B}$ is a pair $(\mathbf{a}\,;\mathbf{b})$ of points $\mathbf{a}$ of $\mathcal{A}$ and $\mathbf{b}$ of $\mathcal{B}$ such that $(\mathcal{A}, \mathbf{a}) \equiv_0 (\mathcal{B}, \mathbf{b})$. A *back and forth system* (or p-system) for $\mathcal{A}$, $\mathcal{B}$ is a non-empty set I of partial isomorphisms for $\mathcal{A}$, $\mathcal{B}$ such that for each $(\mathbf{a},\mathbf{b}) \in I$ the following hold.

(Forth). For each $x \in \mathcal{A}$ there is some $y \in \mathcal{B}$ such that $(\mathbf{a},x\,;\mathbf{b},y) \in I$.

(Back). For each $y \in \mathcal{B}$ there is some $x \in \mathcal{A}$ such that $(\mathbf{a},x\,;\mathbf{b},y) \in I$.

We write $\mathcal{A} \equiv_p \mathcal{B}$ if there is a p-system for $\mathcal{A}$, $\mathcal{B}$ (so $\mathcal{A} \equiv_p \mathcal{B}$ hold if and only if $\mathcal{A}$, $\mathcal{B}$ satisfy exactly the same $L_{\infty,\omega}$-sentences). If $\mathcal{A}$, $\mathcal{B}$ are both countable then $\mathcal{A} \equiv_p \mathcal{B}$ holds exactly when $\mathcal{A} \cong \mathcal{B}$.

§ 2. *E.C. structures and companion theories*

This section contains all the required results concerning the various kinds of e.c. structures. We do not give any proofs and, in fact, the order in which we give the results is not the order in which they are most conveniently proved.

We start, at the beginning, with a definition.

2.1. *DEFINITION.* Let T be a theory. A structure $\mathcal{A}$ is *e.c.* for T if $\mathcal{A} \in Sb(T)$ and for each model $\mathcal{B}$ of T,

$$\mathcal{A} \subseteq \mathcal{B} \Rightarrow \mathcal{A} \prec_1 \mathcal{B}.$$

We let E_T be the class of structures e.c. for T and put $T^e = Th(E_T)$. We call T^e the *e-companion* of T.

There is a useful syntactical characterization of e.c. structures. For each theory T and formula φ let $\Omega(T,\varphi)$ be the type

$\{\varphi\} \cup \{\neg\vartheta : \vartheta$ is an $\exists_1$-formula, consistent with T, such that $fv(\vartheta) \subset fv(\varphi)$ and $T \vdash \vartheta \rightarrow \varphi\}$.

Notice that if φ is an $\forall_1$-formula then $\Omega(T, \varphi)$ is an $\forall_1$-type.

2.2. *THEOREM.* *Let* T *be a theory and* $\mathcal{A}$ *a submodel of* T. *The following are equivalent.*

(i) $\mathcal{A}$ *is e.c. for* T.
(ii) *For each* $\forall_1$*-formula* φ, $\mathcal{A}$ *omits the type* $\Omega(T,\varphi)$.

The class E_T itself can be implicitly characterized. Remember that we say a class E is cofinal in a class S if $E \subseteq S$ and for each $\mathcal{A} \in S$ there is some $\mathcal{B} \in E$ with $\mathcal{A} \subseteq \mathcal{B}$. .

2.3. *THEOREM.* *Let* T *be a theory. Then* E_T *is the unique class such that the following hold.*

(i) E_T *is cofinal in* $Sb(T)$.
(ii) *For each pair of members* $\mathcal{A}$, $\mathcal{B}$ *of* E_T, *if* $\mathcal{A} \subseteq \mathcal{B}$ *then* $\mathcal{A} <_1 \mathcal{B}$.
(iii) *For each pair of structures* $\mathcal{A}$, $\mathcal{B}$, *if* $\mathcal{A} <_1 \mathcal{B} \in E_T$ *then* $\mathcal{A} \in E_T$.

There are two particularly nice kinds of e.c. structures, namely the generic structures. First we look at the f-generic structures.

2.4. *DEFINITION.* Let T be a theory

(i) T is *f-complete* if for each formula φ, consistent with T, there is an $\exists_1$-formula ϑ, consistent with T, such that $fv(\vartheta) \subseteq fv(\varphi)$ and $T \vdash \vartheta \rightarrow \varphi$.
(ii) A model $\mathcal{A}$ of T is a *completing model* of T if for $\mathcal{B} \models T$, if $\mathcal{A} \subseteq \mathcal{B}$ then $\mathcal{A} < \mathcal{B}$.

These two concepts are related for we have the following.

2.5. *THEOREM.* *A theory is f-complete if and only if it is the theory of its class of completing models.*

In general a theory does not have a completing model, however each theory does have a companion with a controlling class of completing models.

2.6. *THEOREM.* *Each theory has a unique f-complete cotheory.*

2.7. *DEFINITION.* Let T be a theory. We write T^f for the unique f-complete cotheory of T and call T^f the *f-companion* of T. We let F_T be the class of completing models of T^f (so that $T^f = \mathrm{Th}(F_T)$). The members of F_T are called the structures *f-generic* for T.

We easily check that $F_T \subseteq E_T$ (so that $T^e \subseteq T^f$) and that

$$\mathcal{A} <_1 \mathcal{B} \in F_T \Rightarrow \mathcal{A} \in F_T.$$

Note however that, in general, F_T is not cofinal in $Sb(T)$.

There is a characterization of f-generic structures along the lines of 2.2. Notice that in this characterization we use the types $\Omega(T^f,\varphi)$, and not the types $\Omega(T,\varphi)$.

2.8. *THEOREM.* *Let* T *be a theory and* $\mathcal{A}$ *a submodel of* T. *The following are equivalent.*

(i) $\mathcal{A}$ *is f-generic for* T.
(ii) *For each formula* φ, $\mathcal{A}$ *omits the type* $\Omega(T^f,\varphi)$.

The second kind of generic structure can be approached via a characterization similar to 2.3.

2.9. *THEOREM.* *Let* T *be a theory. There is a unique class* G_T *such that the following hold.*

(i) G_T *is cofinal in* $Sb(T)$.
(ii) *For each pair members* $\mathcal{A}$, $\mathcal{B}$ *of* G_T, *if* $\mathcal{A} \subseteq \mathcal{B}$ *then* $\mathcal{A} < \mathcal{B}$.
(iii) *For each pair of structures* $\mathcal{A}$, $\mathcal{B}$, *if* $\mathcal{A} < \mathcal{B} \in G_T$ *then* $\mathcal{A} \in G_T$.

2.10. *DEFINITION.* Let T be a theory. The members of G_T (characterized is 2.9) are called the structures *g-generic* for T. We put $T^g = \mathrm{Th}(G_T)$ and call T^g the *g-companion* of T.

Again we easily see that $G_T \subseteq E_T$ so that $T^e \subseteq T^g$. In general there appears to be very little connection between F_T and G_T.

These g-generic structures are intimately connected with the $\aleph_0 - \exists_1$-saturated structures which we consider in section 6. In fact G_T is most conveniently constructed using the following (where U_T is the class defined in 6.1.)

2.11. *THEOREM. Let* T *be a theory. For each structure* $\mathcal{A}$ *the following are equivalent.*

(i) $\mathcal{A}$ *is g-generic for* T.
(ii) *There is some* $\mathcal{B} \in U_T$ *such that* $\mathcal{A} < \mathcal{B}$.

So far we have seen three companions T^e, T^f, T^g of a theory T. It is easy to verify that these are cotheories of T, depend only on $T \cap \forall_1$ and satisfy

$$T \cap \forall_2 \subseteq T^e \cap \forall_2 = T^f \cap \forall_2 = T^g \cap \forall_2.$$

There is a fourth companion of T which will play a crucial role in chapter B.

2.12. *DEFINITION.* Let T be a theory. T is *0-complete* if for each $\forall_1$-formula φ, consistent with T, there is an $\exists_1$-formula ϑ, consistent with T, such that $fv(\vartheta) \subseteq fv(\varphi)$ and $T \vdash \vartheta \rightarrow \varphi$.

2.13. *THEOREM. Let* T *be a theory. There is a unique* $\forall_2$-*axiomatizable, 0-complete cotheory* T^o *of* T. *This theory* T^o *is axiomatized by* $T^e \cap \forall_2 = T^f \cap \forall_2 = T^g \cap \forall_2$ *and, for each* $\forall_2$-*axiomatizable cotheory* T' *of* T, $T' \subseteq T^o$.

We call this theory T^o the *0-companion* of T. Notice that unlike the other three companions of T, this companion T^o is not associated with a class of submodels of T (except, of course, $Md(T^o)$).

Broadly speaking the above concepts can be regarded as tools for studying the existence or non-existence of model companions. We give the relevant results.

2.14. *DEFINITION.* Let T be a theory. A *model companion* of T is a model complete cotheory T^m of T.

2.15. *THEOREM. Each theory has at most one model companion, but may have none.*

2.16. *THEOREM. Let* T *be a theory. The following are equivalent.*

(i) T *has a model companion* T^m.
(ii) E_T *is an elementary class.*
(iii) F_T *is an elementary class.*
(iv) G_T *is an elementary class.*
(v) T^o *is model complete.*

Moreover if T^m *exists then* $E_T = F_T = G_T = Md(T^m)$ *and* $T^e = T^f = T^g = T^o = T^m$.

Finally in this section we give two results concerned with JEP.

2.17. *THEOREM. Let* T *be a theory, let* $\mathcal{A}$, $\mathcal{B}$ *be two structures both e.c. for* T, *and let* **a** *be a point of* $\mathcal{A}$ *and* **b** *a point of* $\mathcal{B}$ *such that*

$$(\mathcal{A},\mathbf{a}) \Rrightarrow(\exists_1)\ (\mathcal{B},\mathbf{b}).$$

Then $(\mathcal{A},\mathbf{a}) \equiv_2 (\mathcal{B},\mathbf{b})$. *Moreover if* $\mathcal{A}$, $\mathcal{B}$ *are both f-generic for* T *or both g-generic for T then* $(\mathcal{A},\mathbf{a}) \equiv (\mathcal{B},\mathbf{b})$.

2.18. *THEOREM. Let* T *be a theory. The following are equivalent.*

(i) T *has JEP.*
(ii) T^f *is complete.*
(iii) T^g *is complete.*
(iv) *For each pair* α,β *of* $\forall_1$*-sentences,* if $T \vdash \alpha \vee \beta$ *then either* $T \vdash \alpha$ *or* $T \vdash \beta$.

§ 3. *An omitting types result*

Remember that a type Γ is principal over a theory T if there is some formula ψ, consistent with T, such that $fv(\psi) \subseteq$

fv(Γ) and $T \vdash \psi \rightarrow \bigwedge \Gamma$. The following is the classical omitting types theorem.

3.1. *THEOREM. Let* T *be a theory and* Γ *a countable collection of types each non-principal over* T. *Then there is some countable model of* T *which omits each member of* Γ.

This result is one of crucial tools needed to study the countable models of a theory. We will be concerned with the countable e.c. structures of a theory and consequently we will need a corresponding omitting types theorem. This section is devoted to a discussion of the appropriate result.

3.2. *DEFINITION.* Let T be a theory. A type Π is $\exists_1$-*principal over* T if there is some $\exists_1$-formula ψ consistent with T, such that $fv(\psi) \subseteq fv(\Pi)$ and $T \models \psi \rightarrow \bigwedge \Pi$.

This concept is used almost exclusively with $\forall_1$-types. The following is the result corresponding to 3.1.

3.3. *THEOREM. Let* T *be a theory and* Π *a countable collection of* $\forall_1$-*types each non-*$\exists_1$-*principal over* T. *Then there is some countable member* $\mathcal{A}$ *of* F_T *which omits each member of* Π.

PROOF. Notice that each member of Π is non-$\exists_1$-principal over T^f and so (using the f-completeness of T^f) each member of Π is non-principal over T^f. In a similar way, for each formula φ the type $\Omega(T^f, \varphi)$ (of 2.8) is non-principal over T^f. The required result now follows by applying 3.1 to the countable collection

$$\Pi \cup \{\Omega(T^f, \varphi) : \varphi \text{ a formula}\}$$

and then using 2.8.

This result is, in fact, equivalent to 3.1. To see this we remember that there is a definitional extension T' of T which is model complete and there is a 1.1 correspondence between the models of T and the models of T'. Of course T' is a theory in a larger language but this language is still countable. The models of T are exactly the reducts of the models of T'.

Now each member of Γ (of 3.1) is equivalent modulo T' to an

$\forall_1$-type, and these types are non-principal over T'. Thus, applying 3.3 we obtain some countable $\mathcal{A}' \in F_{T'} = Md(T)$ which omits Γ. But then the reduct $\mathcal{A}$ of $\mathcal{A}'$ is a model of T and omits Γ, as required.

CHAPTER B

Large and small e.c. structures

§ 4. $\exists_1$-*atomic structures and theories*

The study of classical atomic structures and theories uses the tool of complete formulas. In the same way the study of $\exists_1$-atomic structures and theories uses an analogous tool of $\exists_1$-complete formulas. These formulas are isolated in the following theorem.

4.1. *THEOREM. Let* T *be a theory and let* ϑ *be an* $\exists_1$-*formula consistent with* T. *The following are equivalent.*

(i) *For each* $\forall_1$-*formula* φ *with* $\mathrm{fv}(\varphi) \subseteq \mathrm{fv}(\vartheta)$, *if* $\vartheta \wedge \varphi$ *is consistent with* T° *then* $T \vdash \vartheta \rightarrow \varphi$.

(ii) *For each* $\exists_1$-*formula* ψ *with* $\mathrm{fv}(\psi) \subseteq \mathrm{fv}(\vartheta)$, *if* $\vartheta \wedge \psi$ *is consistent with* T *then* $T^\circ \vdash \vartheta \rightarrow \psi$.

(iii) *For each two* $\exists_1$-*formulas* ψ_1, ψ_2 *with* $\mathrm{fv}(\psi_1, \psi_2) \subseteq \mathrm{fv}(\vartheta)$, *if both* $\vartheta \wedge \psi_1$ *and* $\vartheta \wedge \psi_2$ *are consistent with* T *then* $\psi_1 \wedge \psi_2$ *is consistent with* T.

PROOF. (i) $\Rightarrow$ (ii). Suppose (i) holds and let ψ be an $\exists_1$-formula such that $\mathrm{fv}(\psi) \subseteq \mathrm{fv}(\vartheta)$ and $\vartheta \wedge \psi$ is consistent with T. If $T^\circ \nvdash \vartheta \rightarrow \psi$ then $\vartheta \wedge \neg\psi$ is consistent with T° so (since $\neg\psi$ is an $\forall_1$-formula) (i) gives $T \vdash \vartheta \rightarrow \neg\psi$. This contradicts the consistency of $\vartheta \wedge \psi$ with T, and so we have (ii).

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i). Suppose (iii) holds and let φ be an $\forall_1$-formula such that $fv(\varphi) \subseteq fv(\vartheta)$ and $\vartheta \wedge \varphi$ is consistent with T^o. There is some $\mathcal{A} \in E_T$ and point $\mathbf{a}$ of $\mathcal{A}$ such that $\mathcal{A} \models \vartheta(\mathbf{a}) \wedge \varphi(\mathbf{a})$. (For otherwise $T^e \vdash \neg\vartheta \vee \neg\varphi$ so that $T^o \vdash \neg\vartheta \vee \neg\varphi$, which contradicts the consistency of $\vartheta \wedge \varphi$ with T^o.) Thus there is some $\exists_1$-formula ψ such that

$$\mathcal{A} \models \psi(\mathbf{a}), \qquad T \vdash \psi \rightarrow \varphi.$$

The first of these shows that $\vartheta \wedge \psi$ is consistent with T, and the second asserts that $\psi \wedge \neg\varphi$ is inconsistent with T. Thus, by (iii), we see that $\vartheta \wedge \neg\varphi$ is inconsistent with T, i.e. $T \vdash \vartheta \rightarrow \varphi$, as required.

4.2. *DEFINITION.* Let T be a theory. An $\exists_1$-formula ϑ is $\exists_1$-*complete over* T if ϑ is inconsistent with T and satisfies the equivalent conditions of 4.1.

There are several different ways of characterizing the $\exists_1$-atomic structures. No one of there ways is more useful than the others so we will first prove their equivalence before formally defining the $\exists_1$-atomic structures.

4.3. *THEOREM. Let* T *be a theory. For each structure* $\mathcal{A}$ *the following are equivalent.*

(i). $\mathcal{A} \in Sb(T)$ *and for each point* $\mathbf{a}$ *of* $\mathcal{A}$ *there is an* $\exists_1$-*formula* ϑ *such that*

$$\mathcal{A} \models \vartheta(\mathbf{a}), \qquad T \models \vartheta \rightarrow \bigwedge \Pi$$

where Π *is the* $\forall_1$-*type of* $\mathbf{a}$ *in* $\mathcal{A}$.

(ii). $\mathcal{A} \models T^o$ *and each point of* $\mathcal{A}$ *realizes a formula which is* $\exists_1$-*complete over* T.

(iii). $\mathcal{A} \in E_T$ *and for each point* $\mathbf{a}$ *of* $\mathcal{A}$ *there is an* $\exists_1$-*formula* ϑ *such that*

$$\mathcal{A} \models \vartheta(\mathbf{a}), \qquad T^o \models \vartheta \rightarrow \bigwedge \Sigma$$

where Σ *is the* $\exists_1$-*type of* a *in* $\mathcal{A}$.

PROOF. (i) $\Rightarrow$ (ii). Suppose $\mathcal{A}$ satisfies (i). Clearly $\mathcal{A} \in E_T$ so that $\mathcal{A} \models T^o$. Let **a** be a point of $\mathcal{A}$ and let ϑ be the $\exists_1$-formula given by (i). We show that ϑ is $\exists_1$-complete over T, and hence verify (ii). To do this we verify 4.1(i).

Let φ be any $\forall_1$-formula such that $fv(\varphi) \subseteq fv(\vartheta)$ and $\vartheta \wedge \varphi$ is consistent with T^o. As in the proof of 4.1 (iii) $\Rightarrow$ (i) there is some $\mathcal{B} \in E_T$ and point **b** of $\mathcal{B}$ such that $\mathcal{B} \models \vartheta(\mathbf{b}) \wedge \varphi(\mathbf{b})$. The assumed property of ϑ, (i), gives

$$(\mathcal{A}, \mathbf{a}) \quad \Rrightarrow(\forall_1) \quad (\mathcal{B}, \mathbf{b})$$

so that (since $\mathcal{B} \in E_T$)

$$(\mathcal{A}, \mathbf{a}) \quad \equiv_1 \quad (\mathcal{B}, \mathbf{b}).$$

Thus we have $\mathcal{A} \models \varphi(\mathbf{a})$, i.e. $\varphi \in \Pi$, and hence (again using

(i)) $T \vdash \vartheta \rightarrow \varphi$, as required.

(ii) $\Rightarrow$ (iii). Suppose $\mathcal{A}$ satisfies (ii) and let **a** be a point of $\mathcal{A}$. Let ϑ be the $\exists_1$-formula given by (ii) (so that $\mathcal{A} \models \vartheta(\mathbf{a})$) and let Σ, Π be the $\exists_1$-type of **a** in $\mathcal{A}$. Then (since $\mathcal{A} \models T^o$) $\{\vartheta\} \cup \Sigma \cup \Pi$ is consistent with T^o so that (since ϑ is $\exists_1$-complete over T) 4.1 gives

$$T \models \vartheta \rightarrow \bigwedge \Pi, \qquad T^o \models \vartheta \rightarrow \bigwedge \Sigma.$$

The first of these shows that $\mathcal{A} \in E_T$ and the second completes the proof of (iii).

(iii) $\Rightarrow$ (i). Suppose $\mathcal{A}$ satisfies (iii), so that (trivially) $\mathcal{A} \in Sb(T)$. Let **a** be a point of $\mathcal{A}$ and let Σ, Π be the $\exists_1$-type and $\forall_1$-type of **a** in $\mathcal{A}$. Since $\mathcal{A} \in E_T$ we have

$$T \models \bigwedge \Sigma \rightarrow \bigwedge \Pi.$$

Now let ϑ be the $\exists_1$-formula given by (iii), i.e. $\mathcal{A} \models \vartheta(\mathbf{a})$ and

$$T^o \models \vartheta \rightarrow \bigwedge \Sigma.$$

The two displayed relations give $T \models \vartheta \rightarrow \bigwedge \Pi$, as required.

Notice that the formula ϑ in (i), (iii) is, in fact. $\exists_1$-complete over T.

4.4. *DEFINITION.* Let T be a theory. A structure $\mathcal{A}$ is $\exists_1$-*atomic* for T if $\mathcal{A}$ satisfies the equivalent conditions of 4.3. We let SE_T be the class of structures which are $\exists_1$-atomic for T.

Trivially $SE_T \subseteq E_T$, and we easily check that $\mathcal{A} <_1 \mathcal{B} \in SE_T \Rightarrow \mathcal{A} \in SE_T$.

(In [12] these $\exists_1$-atomic structures are called *strongly e.c. structures,* hence the notation SE_T.)

As can be expected only certain theories have $\exists_1$-atomic structures. These are just the $\exists_1$-atomic theories.

4.5. *DEFINITION.* A theory T is $\exists_1$-*atomic* if T has JEP and satisfies the following condition. For each $\exists_1$-formula ψ consistent with T, there is an $\exists_1$-formula ϑ which is $\exists_1$-complete over T such that $fv(\vartheta) \subseteq fv(\psi)$ and $T^\circ \vdash \vartheta \rightarrow \psi$.

4.6. *THEOREM. Let* T *be a theory with JEP. The following are equivalent.*

(i). *There is some (countable) structure which is* $\exists_1$-*atomic for* T.

(ii). T *is* $\exists_1$-*atomic.*

PROOF. (i) $\Rightarrow$ (ii). Suppose $\mathcal{A}$ is $\exists_1$-atomic for T and let ψ be an $\exists_1$-formula which is consistent with T. Since T has JEP ψ is realized in $\mathcal{A}$, i.e. there is some point $\mathbf{a}$ of $\mathcal{A}$ such that $\mathcal{A} \models \psi(\mathbf{a})$. By 4.3 there is some formula ϑ which is $\exists_1$-complete over T such that $\mathcal{A} \models \vartheta(\mathbf{a})$ and $T^\circ \vdash \vartheta \rightarrow \psi$. This verifies (ii).

(ii) $\Rightarrow$ (i). Suppose T is $\exists_1$-atomic. For each finite set V of variables let $\Pi(V)$ be the $\forall_1$-type

$\{\neg\vartheta : \vartheta$ is a formula such that $fv(\vartheta) \subseteq V$ and is $\exists_1$-complete over T$\}$.

Let Π be the countable set of all these $\forall_1$-types.

Suppose that, for some V, $\Pi(V)$ is $\exists_1$-principal over T. Then there is some $\exists_1$-formula ψ, consistent with T, such that $fv(\psi) \subseteq V$ and $T \models \psi \rightarrow \bigwedge \Pi(V)$. Since T is $\exists_1$-atomic we may assume that ψ is $\exists_1$-complete over T, so that $\neg\psi \in \Pi(V)$. But then $T \vdash \psi \rightarrow \neg\psi$, which contradicts the consistency of ψ with T.

This shows that no member of Π is $\exists_1$-principal over T. Thus, by 3.3, there is some countable $\mathcal{A} \in F_T$ which omits each member of Π. But then, 4.3 (ii), shows that $\mathcal{A}$ is $\exists_1$-atomic for T, which gives (i).

Superficially there is another possible definition of $\exists_1$-atomicity of theories. We may consider those theories T which have JEP and the following property. For each $\forall_1$-formula φ consistent with T^o, there is an $\exists_1$-formula ϑ which is $\exists_1$-complete over T such that $fv(\vartheta) \subseteq fv(\varphi)$ and $T \vdash \vartheta \rightarrow \varphi$. I have made no study of this property. Note, however, that it is weaker than $\exists_1$-atomicity.

§ 5. Further properties of $\exists_1$-atomicity

$\exists_1$-atomic structures (like their classical analogues) have very strong back-and-forth properties. These properties are derived from the following lemma.

5.1. *LEMMA. Let* T *be a theory,* $\mathcal{A}$ *a structure which is* $\exists_1$-*atomic for* T*, and* $\mathcal{B}$ *a model of* T^o. *Let* **a** *be a point of* $\mathcal{A}$ *and* **b** *a point of* $\mathcal{B}$ *such that*

$$(\mathcal{A}, \mathbf{a}) \quad \Rrightarrow(\exists_1) \quad (\mathcal{B}, \mathbf{b})$$

Then for each element x *of* $\mathcal{A}$ *there is some element* y *of* $\mathcal{B}$ *such that*

$$(\mathcal{A}, \mathbf{a}, x) \quad \Rrightarrow(\exists_1) \quad (\mathcal{B}, \mathbf{b}, y).$$

PROOF. Let x be an element of $\mathcal{A}$ and let $\Sigma(\mathbf{v}, v)$ be the $\exists_1$-type of **a**, x in $\mathcal{A}$. (So **v** is a sequence of variables matching

a and v is a single variable). By 4.3 (iii) there is an $\exists_1$-formula $\vartheta(\mathbf{v}, v)$ such that

$$\mathcal{A} \models \vartheta(\mathbf{a}, x), \qquad T^{\circ} \models \vartheta \rightarrow \bigwedge \Sigma.$$

In particular $\mathcal{A} \models (\exists\, v)\, \vartheta(\mathbf{a}, v)$ so that $\mathcal{B} \models (\exists\, v)\, \vartheta\, (\mathbf{b}, v)$ i.e. there is some element y of $\mathcal{B}$ such that $\mathcal{B} \models \vartheta\, (\mathbf{b}, y)$. Since $\mathcal{B} \models T^{\circ}$ this gives $\mathcal{B} \models \Sigma\, (\mathbf{b}, y)$, which is the required result.

Our first use of 5.1 is to prove a uniqueness result.

5.2. *THEOREM. Let* T *be a theory with JEP. Let* $\mathcal{A}$, $\mathcal{B}$ *be two structures both* $\exists_1$*-atomic for* T. *Then* $\mathcal{A} \equiv_p \mathcal{B}$, *in particular if* $\mathcal{A}$, $\mathcal{B}$ *are countable then* $\mathcal{A} \cong \mathcal{B}$.

PROOF. Let I be the set of all pairs $(\mathbf{a}, \mathbf{b})$ of points *a* of $\mathcal{A}$ and **b** of $\mathcal{B}$ such that

$$(\mathcal{A}, \mathbf{a}) \equiv_1 (\mathcal{B}, \mathbf{b}).$$

(Of course, since both $\mathcal{A}$ and $\mathcal{B}$ are e.c. for T, this relation can be replaced by $\Rrightarrow(\exists_1)$, $\Rrightarrow(\forall_1)$ or $\equiv_2$.) Notice that, since T has JEP, I is non-empty since $(\emptyset, \emptyset) \in I$.

Now 5.1 shows that I is a back-and-forth system, which gives the required result.

The next theorem is the $\exists_1$-analogue of the classical result that the countable atomic model of a complete theory is the prime model of the theory. (A prime model of a theory is a model which is elementarily embeddable in each model of the theory.)

5.3. *THEOREM. Let* T *be a theory with JEP. For each structure* $\mathcal{A}$ *the following are equivalent.*

(i) $\mathcal{A}$ *is coutable and* $\exists_1$*-atomic for* T.

(ii) $\mathcal{A} \in E_T$ *and is embeddable in each model of* T°.

(iii) $\mathcal{A}$ *is a prime model of* T^{f}.

(iv) $\mathcal{A} \in E_T$ *and is embeddable in each member of* F_T.

PROOF. (i) $\Rightarrow$ (ii). Suppose $\mathcal{A}$ satisfies (i) so that (since $SE_T \subseteq E_T$) $\mathcal{A} \in E_T$. Let $\mathcal{B}$ be any model of T^o. Since T has JEP we have $\mathcal{A} \Rrightarrow (\exists_1)\ \mathcal{B}$ so, since $\mathcal{A}$ is countable, repeated use of 5.1 produces an embedding of $\mathcal{A}$ in $\mathcal{B}$.

(ii) $\Rightarrow$ (iii). Suppose $\mathcal{A}$ satisfies (ii) and let $\mathcal{B}$ be a model of T^f and let $\mathcal{C}$ be a member of F_T (so $\mathcal{C}$ is a particular kind of model of T^f). Both $\mathcal{B}$, $\mathcal{C}$ are models of T^o and so there are embeddings f, g of $\mathcal{A}$ into $\mathcal{B}$, $\mathcal{C}$ respectively. Since $\mathcal{A} \in E_T$ both f, g are 1-embeddings so, using g, we see that $\mathcal{A} \in F_T$. I.e. $\mathcal{A}$ is a completing model of T^f, and so f is an elementary embedding. Thus we have (iii).

(iii) $\Rightarrow$ (iv) is trivial.

(iv) $\Rightarrow$ (i). Suppose $\mathcal{A}$ satisfies (iv) so that (since F_T has a countable member) $\mathcal{A}$ is a countable submodel of T.

Let **a** be a point of $\mathcal{A}$ and let Π be the $\forall_1$-type of a in $\mathcal{A}$. By (iv) we see that Π is realized in every member of F_T hence, by 3.3, Π is $\exists_1$-principal over T. Thus there is some $\exists_1$-formula ϑ, consistent with T, such that $fv(\vartheta) \subseteq fv(\Pi)$ and $T \models \vartheta \rightarrow \Lambda\Pi$. By 4.3(i) it is sufficient to show that $\mathcal{A} \models \vartheta(\mathbf{a})$.

If $\mathcal{A} \models \neg\vartheta(\mathbf{a})$ then $\neg\vartheta \in \Pi$ so that $T \vdash \vartheta \rightarrow \neg\vartheta$ which contradicts the consistency of ϑ with T. This completes the proof.

5.4. *COROLLARY. Let* T *be a theory with JEP. Then* $SE_T \subseteq F_T$.

PROOF. We have seen above that each countable member of SE_T is in F_T. But a structure is a member of F_T if and only if each of its countable elementary substructures is in F_T, which gives the required result.

Having seen 5.3 the following characterization of $\exists_1$-atomic structures is no surprise.

5.5. *THEOREM. Let* T *be a theory with* JEP. *For each structure* $\mathcal{A}$ *the following are equivalent.*

(i) $\mathcal{A}$ *is* $\exists_1$*-atomic for* T.
(ii) $\mathcal{A}$ *is an atomic model of* T^f.

PROOF. (i) $\Rightarrow$ (ii). Suppose $\mathcal{A} \in SE_T$ so (by 5.4) $\mathcal{A} \in F_T$. Let $\mathbf{a}$ be a point of $\mathcal{A}$ and let Σ, Γ be the $\exists_1$-type and full full type of $\mathbf{a}$ in $\mathcal{A}$. Since $\mathcal{A} \in F_T$ we have

$$T^f \models \Lambda\Sigma \rightarrow \Lambda\Gamma.$$

But $\mathcal{A} \in SE_T$ so there is some $\vartheta \in \Sigma$ such that $T^o \models \vartheta \rightarrow \Lambda\Sigma$. This gives $T^f \vdash \vartheta \rightarrow \Lambda\Gamma$, which shows that $\mathcal{A}$ is an atomic model of T^f.

(ii) $\Rightarrow$ (i). Suppose $\mathcal{A}$ satisfies (ii), so that $\mathcal{A} \in Sb(T)$. Let $\mathbf{a}$ be a point of $\mathcal{A}$ and let Π be the $\forall_1$-type of $\mathbf{a}$ in $\mathcal{A}$. Since $\mathcal{A}$ is atomic there is some formula φ such that $\mathcal{A} \models \varphi(\mathbf{a})$ and $T^f \models \varphi \rightarrow \Lambda\Pi$. The f-completeness of T^f now gives some $\exists_1$-formula, consistent with T^f, such that $fv(\vartheta) \subseteq fv(\varphi)$ and $T^f \models \vartheta \rightarrow \varphi$, and hence $T^f \models \vartheta \rightarrow \Lambda\Pi$. Thus (considering the quantifier complexity of $\vartheta \rightarrow \Lambda\Pi$) we have $T \vdash \vartheta \rightarrow \Lambda\Pi$, and so it is sufficient to show that $\mathcal{A} \models \vartheta(\mathbf{a})$. But if $\mathcal{A} \models \neg\vartheta(\mathbf{a})$ then $\neg\vartheta \in \Pi$ and hence $T \vdash \vartheta \rightarrow \neg\vartheta$, which contradicts the consistency of ϑ with T.

5.6. *COROLLARY. Let* T *be a theorem with JEP. Then* T *is* $\exists_1$*-atomic if and only if* T^f *is atomic.*

The results of this section show there is a strong connection between f-generic structures and $\exists_1$-atomic structure. This connection reinforces the feeling that, in some sense, f-generic structures are small.

To complete this section we state, without proof, a characterization of $\exists_1$-completeness.

5.7. *THEOREM. Let* T *be a theory and let* ϑ *be an* $\exists_1$*-formula consistent with* T. *The following are equivalent.*

(i) ϑ *is* $\exists_1$*-complete over* T.
(ii) ϑ *is complete over* T^f.
(iii) ϑ *is complete over* T^g.

§ 6. $\aleph_0$ - $\exists_1$ - saturated structures

We now come to the large e.c. structures i.e. those e.c. structures which are the $\exists_1$-analogue of saturated structures.

First we need a little terminology.

Let $\mathcal{M}$ be some fixed structure. We extend the underlying language L to a new language L_M by adding to L a name for each member of $\mathcal{M}$. These additional constant symbols are called the parameters of L_M. An $\aleph_0$ - $\exists_1$-type over $\mathcal{M}$ is an $\exists_1$-type Σ in the language L_M such that Σ contains no more than finitely many parameters.

The reader will probably recognize the following definition.

6.1. *DEFINITION.* Let T be a theory. A structure $\mathcal{M}$ is $\aleph_0$ - $\exists_1$-*saturated* for T if $\mathcal{M} \in Sb(\mathrm{T})$ and for each $\aleph_0$ - $\exists_1$-type Σ over $\mathcal{M}$, if $\mathcal{M}$ is realized in some extension of $\mathcal{M}$ which is a model of T the Σ is also realized in $\mathcal{M}$. We let U_{T} be the class of structures which are $\aleph_0$ - $\exists_1$-saturated for T.

These $\aleph_0$ - $\exists_1$-saturated structures are, of course, nothing more than the well known existentially universal structures.

We will sketch here some of the properties of U_{T} and its members. A full account can be found in [13] where the $\aleph_0$ - $\exists_1$- saturated structures are called $\aleph_0$-closed structures).

The reader should contrast the well known connection between these large e.c. structures and g-generic structures with the connection between the small e.c. structures and f-generic structures given in section.

Almost trivially we have $U_{\mathrm{T}} \subseteq E_{\mathrm{T}}$ and by a fairly routine argument we can show that U_{T} is cofinal in $Sb(\mathrm{T})$. The $\aleph_0$ - $\exists_1$-saturated structures have back-and-forth properties analogous to those of $\aleph_0$-saturated structures and, in some way, dual to those of $\exists_1$-atomic structures. The following two results are sufficient for our purpose.

6.2. *THEOREM. Let* T *be a theory with JEP and let* $\mathcal{A}$, $\mathcal{B}$ *be two structures both* $\aleph_0$ - $\exists_1$*saturated for* T. *Then* $\mathcal{A} \equiv_p \mathcal{B}$, *in particular if* $\mathcal{A}$, $\mathcal{B}$ *are countable then* $\mathcal{A} \cong \mathcal{B}$.

6.3. *THEOREM. Let* T *be a theory with JEP and let* $\mathcal{M}$ *be a structure* $\aleph_0$ - $\exists_1$*-saturated for* T. *Then each countable submodel of* T *is embeddable in* $\mathcal{M}$.

In general a theory will not have a countable $\aleph_0$ - $\exists_1$-saturated structure, simply because there will be too many $\exists_1$-types for any countable structure to deal with. To discuss the relevant properties of these $\exists_1$-types it is convenient to make the following definition.

6.4. *DEFINITION.* Let T be a theory. An $\exists_1$-type Σ is a *max-* $\exists_1$-type of T if Σ is consistent with T and for each $\exists_1$-formula ϑ such that $fv(\vartheta) \subseteq fv(\Sigma)$, if $\Sigma \cup \{\varphi\}$ is consistent with T then $\varphi \in \Sigma$. For each finite set V of variables we let m(T, V) be the number of max- $\exists_1$-types Σ of T such that $fv(\Sigma) \subseteq V$.

Clearly every $\exists_1$-type consistent with T can be extended to a max-$\exists_1$-type of T. These max-$\exists_1$-types are located in the following theorem.

6.5. *THEOREM. Let* T *be a theory. For each* $\exists_1$*-type* Σ *the following are equivalent.*

(i) Σ *is a max-*$\exists_1$*-type of* T.

(ii) *There is some* $\mathcal{A} \in E_T$ *and some point* **a** *of* $\mathcal{A}$ *such that* Σ *is the* $\exists_1$*-type of* **a** *in* $\mathcal{A}$.

PROOF. (i) $\Rightarrow$ (ii). Suppose Σ is a max-$\exists_1$-type of T. Since Σ is consistent with T, Σ is realized in some model $\mathcal{B}$ of T. Let $\mathcal{B} \subseteq \mathcal{A}$, where $\mathcal{A} \in E_T$, so that (since Σ is an $\exists_1$-type) Σ is also realized in $\mathcal{A}$. Let **a** be a point of $\mathcal{A}$ which realizes Σ, and let ϑ be any $\exists_1$-formula such that $\mathcal{A} \models \varphi(\mathbf{a})$. Then $\Sigma \cup \{\vartheta\}$ is consistent with T and so, by (i), $\vartheta \in \Sigma$. This shows that Σ is the $\exists_1$-type of **a** in $\mathcal{A}$.

(ii) $\Rightarrow$ (i). Suppose $\mathcal{A}$, **a**, Σ are as in (ii), so clearly Σ is consistent with T. Let ϑ be any $\exists_1$-formula such that $fv(\vartheta) \subseteq fv(\Sigma)$ and $\Sigma \cup \{\vartheta\}$ is consistent with T. To show (i) it is sufficient to show that $\mathcal{A} \models \vartheta(\mathbf{a})$.

Since $\Sigma \cup \{\vartheta\}$ is consistent with T there is some model $\mathcal{B}$ of T and point **b** of $\mathcal{B}$ such that

$$\mathcal{B} \models \Sigma(\mathbf{b}), \qquad \mathcal{B} \models \vartheta(\mathbf{b}).$$

The first of these gives

$$(\mathcal{A}, \mathbf{a}) \quad \Rrightarrow(\exists_1) \quad (\mathcal{B}, \mathbf{b})$$

so that, since $\mathcal{A} \in E_T$,

$$(\mathcal{A}, \mathbf{a}) \quad \equiv_1 (\mathcal{B}, \mathbf{b}).$$

Thus we have $\mathcal{A} \models \vartheta(\mathbf{a})$, as required.

The following theorem characterizes those theories which have a countable $\aleph_0$ - $\exists_1$-saturated structure.

6.6. *THEOREM. Let* T *be a theory with JEP. The following are equivalent.*

(i) *For each finite set* V *of variables,* $m(T, V) < 2^{\aleph_0}$
(ii) *For each finite set* V *of variables,* $m(T, V) \leqslant \aleph_0$
(iii) *There is a countable member of* U_T.
(iv) *There is some countable* $\mathcal{M} \in Sb(T)$ *such that each countable model of* T *is embeddable in* $\mathcal{M}$.

PROOF. (i) $\Rightarrow$ (ii). This uses a splitting argument similar to those in section 10 (later). We do not give the details.

(ii) $\Rightarrow$ (i) is trivial.

(ii) $\Rightarrow$ (iii). This follows from an analysis of the construction of $\aleph_0$ - $\exists_1$-saturated structures. Such structures are constructed as the union of an ascending chain of submodels of T. Condition (ii) enables us to keep down the length of this chain and the cardinality of its components.

(iii) $\Rightarrow$ (iv) follows from 6.3.

(iv) $\Rightarrow$ (ii) is trivial.

Let $i(E_T)$ be the number of countable members of E_T, up to isomorphism. The following corollary holds by 6.5 and 6.6 (i).

6.7. *COROLLARY. Let* T *be a theory with JEP. If* $i(E_T) < 2^{\aleph_0}$ *then there is a countable member of* U_T.

The final theorem of this section is the $\exists_1$-analogue of [1, Theorem 2.3.14].

6.8. *THEOREM. Let* T *be a theory with JEP. If* T *has a countable* $\aleph_0$-$\exists_1$*-saturated structure then* T *has a (countable)* $\exists_1$*-atomic structure.*

PROOF. Suppose T does not have an $\exists_1$-atomic structure so, by 6.6. there is some $\exists_1$-formula ϑ which is consistent with T but is not $\exists_1$-completable over T. We use this formula ϑ to construct a binary tree of essentially different $\exists_1$-formulas, each consistent with T but not $\exists_1$-completable over T. Hence we get $m(T, fv(\vartheta)) = 2^{\aleph_0}$.

The following is the crucial lemma.

6.9. *LEMMA. Let* T *be a theory with JEP and let* ϑ *be an* $\exists_1$*-formula consistent with* T *but not* $\exists_1$*-completable over* T. *Then there are* $\exists_1$*-formulas* ϑ_0, ϑ_1, *both consistent with* T *but not* $\exists_1$*-completable over* T, *such that the following hold.*

(i) $fv(\vartheta_0, \vartheta_1) \subseteq fv(\vartheta)$.
(ii) $T^{\circ} \vdash \vartheta_i \rightarrow \vartheta$, *where* $i = 0, 1$.
(iii) $T \vdash \neg(\vartheta_0 \wedge \vartheta_1)$.

PROOF. Since ϑ is not $\exists_1$-complete over T there are, by 4.1. two $\exists_1$-formulas ψ_0, ψ_1 such that $fv(\psi_0, \psi_1) \subseteq fv(\vartheta)$, both $\vartheta \wedge \psi_0$, $\vartheta \wedge \psi_1$ are consistent with T but $\psi_0 \wedge \psi_1$ is not consistent with T, i.e. $T \vdash \neg(\psi_0 \wedge \psi_1)$. We simply put $\vartheta_i = \vartheta \wedge \psi_i$.

Finally in this section we consider the $\exists_1$-analogue of Vaught's amazing result [1, Theorem 2.3.15]. I state this in the form of an exercise (since I have not verified all the details of the proof).

6.10. *EXERCISE.* *Let* T *be a theory with JEP. Then* $i(E_T) \neq 2$.

§ 7. Categoricity results

This section contains three results concerning $\aleph_0$-categoricity. The first two of these are both, in some ways, $\exists_1$-analogues of the Engeler — Ryll — Nardzewski — Svenonius characterization of $\aleph_0$-categorical complete theories.

7.1. *THEOREM.* *Let* T *be a theory with JEP. The following are equivalent.*

(i) $i(E_T) = 1$.
(ii) $SE_T \cap U_T \neq \varnothing$.
(iii) *Each max-* $\exists_1$*-type of* T *contains a formula which is* $\exists_1$*-complete over* T.
(iv) $SE_T = E_T$.

PROOF. (i) $\Rightarrow$ (ii). Suppose (i) holds so that, by 6.7, there is some countable member $\mathcal{M}$ of U_T. But then, by 6.8, there is some countable member $\mathcal{A}$ of SE_T. Now (i) gives $\mathcal{M} \cong \mathcal{A}$, which gives (ii).

(ii) $\Rightarrow$ (iii). Let $\mathcal{M}$ be any member of $SE_T \cap U_T$, and let Σ be a max- $\exists_1$-type of T. Since $\mathcal{M} \in U_T$ and T has JEP, Σ is realized in $\mathcal{M}$ by some point $\mathbf{a}$ of $\mathcal{M}$. The maximality of Σ implies that Σ is the $\exists_1$-type of $\mathbf{a}$ in $\mathcal{M}$. But $\mathcal{M} \in SE_T$ so 4.3 (ii) gives (iii).

(iii) $\Rightarrow$ (iv) follows from 6.5 and 4.3 (ii).

(iv) $\Rightarrow$ (i) follows by 5.2.

7.2. *THEOREM.* *Let* T *be a theory with JEP. The following are equivalent.*

(i) T^o *is* $\aleph_0$*-categorical.*
(ii) *For each* $\forall_1$*-type* Π *consistent with* T^o *there is a formula* ϑ, $\exists_1$*-complete over* T, *such that* $fv(\vartheta) \subseteq fv(\Pi)$ *and* $T \models \vartheta \rightarrow \Lambda\Pi$.

(iii). *For each finite set* V *of variables there are formulas* $\vartheta_1, \ldots, \vartheta_n$ *(where* n *depends on* V*) each one* $\exists_1$*-complete over* T *such that* $fv(\vartheta_1, \ldots, \vartheta_n) \subseteq V$ *and* $T^o \vdash \vartheta_1 \vee \ldots \vee \vartheta_n$.

(iv). *For each finite set* V *of variables,* $m(T, V) < \aleph_0$.

(v) *For each finite set* V *of variables there are, up to* T^o*-equivalence, only finitely many* $\exists_1$*-formulas* ϑ *with* $fv(\vartheta) \subseteq V$.

(vi) T *has an* $\aleph_0$*-categorical model companion.*

PROOF. (i) $\Rightarrow$ (ii). Suppose T^o is $\aleph_0$-categorical and let $\mathcal{A}$ be the unique countable model of T^o. Notice that $\mathcal{A}$ is $\exists_1$-atomic for T. Let Π be an $\forall_1$-type consistent with T^o, so that Π is realized in $\mathcal{A}$. The required result now follows by 4.3.

(ii) $\Rightarrow$ (iii). Suppose (ii) holds and let V be a finite set of variables. Consider the $\forall_1$-type Π

$$\{\neg\vartheta : fv(\vartheta) \subseteq V \text{ and } \vartheta \text{ is } \exists_1\text{-complete over } T\}.$$

Condition (ii) implies that Π is not consistent with T^o, and so we get (iii).

(iii) $\Rightarrow$ (iv). Suppose (ii) holds and let V be a finite set of variables. Let $\vartheta_1, \ldots, \vartheta_n$ be the $\exists_1$-formulas given by (iii). Let Σ be a max- $\exists_1$-type of T such that $fv(\Sigma) \subseteq V$. The maximality of Σ gives us some ϑ_i such that $\vartheta_i \in \Sigma$. But ϑ_i is $\exists_1$-complete over T and so Σ is determined by ϑ_i. Thus we have $m(T, V) \leqslant n$, which gives (iv).

(iv) $\Rightarrow$ (v) follows since two $\exists_1$-formulas are T^o equivalent if and only if they are equivalent in each member of E_T, and hence if and only if they are members of exactly the same max - $\exists_1$-types of T.

(v) $\Rightarrow$ (vi). Suppose (v) holds. We will first show that T^o is model complete (and so T^o is the model companion of T) and then show that T^o is $\aleph_0$ - categorical.

Let φ be any $\forall_1$-formula consistent with T^o and consider the $\exists_1$-type Θ of all $\exists_1$-formulas ϑ such that ϑ is consistent with T, $fv(\vartheta) \subseteq fv(\varphi)$, and $T^o \vdash \vartheta \rightarrow \varphi$. We know that $\Theta \neq \varnothing$ and (v) gives us some $\vartheta \in \Theta$ such that $T^o \models \vartheta \leftrightarrow \bigvee \Theta$.

If $T^\circ \vdash \varphi \rightarrow \vartheta$ then $\varphi \wedge \neg\vartheta$ is consistent with T° so (by 0-completeness) there is some $\exists_1$-formula ψ consistent with T° such that $fv(\psi) \subseteq fv(\varphi \wedge \neg\vartheta) \subseteq fv(\varphi)$ and $T^\circ \vdash \psi \rightarrow (\varphi \wedge \neg\vartheta)$. In particular $T^\circ \vdash \psi \rightarrow \varphi$ so that $\psi \in \Theta$ and hence $T^\circ \vdash \psi \rightarrow \vartheta$. But also $T^\circ \vdash \psi \rightarrow \neg\vartheta$ so we have a contradiction (since ψ is consistent with T°). This shows that $T^\circ \vdash \varphi \rightarrow \vartheta$ so that $T^\circ \vdash \varphi \leftrightarrow \vartheta$ and hence (by a well known characterization) T° is model complete.

Consider now any model $\mathcal{A}$ of T°, and let **a** be a point of $\mathcal{A}$ and Σ the $\exists_1$-type of **a** in $\mathcal{A}$. Since T° is model complete we have $\mathcal{A} \in E_T$. Also (v) gives us some $\exists_1$-formula ϑ such $T^\circ \vdash \vartheta \leftrightarrow \Lambda\Sigma$. Thus 4.3 (iii) shows that $\mathcal{A} \in SE_T$.

The $\aleph_0$-categoricity of T° now follows by 5.2.

(vi) $\Rightarrow$ (i) holds since if T has a model companion then T° is this companion.

Finally in this section we derive the following result of Saracino (see [10]).

7.3. *COROLLARY*. Each $\aleph_0$-categorical theory has an $\aleph_0$-categorical model companion.

PROOF. Let T be an $\aleph_0$-categorical theory (so that T has JEP). The classical Engeler — Ryll — Nardzewski — Svenonius theorem shows that for each finite set V of variables there are, up to T-equivalence, only finitely many $\exists_1$-formulas. We now easily check that 7.2 (v) holds, and so we get the required result.

§ 8. Some further remarks

This chapter has been on the whole, concerned with theories T which have JEP. We have isolated 7 classes of such theories using properties (i) — (vii) (below). These classes form an increasing chain i.e. (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), etc.

(i) T *is* $\aleph_0$-*categorical.*
(ii) T *has an* $\aleph_0$-*categorical model companion.*

(iii) $i(E_T) = 1$

(iv) $i(E_T) < 2^{\aleph_0}$

(v) T *has a countable* $\aleph_0$-$\exists_1$-*saturated structure.*

(vi) T *has an* $\exists_1$-*atomic structure.*

(vii) T *has the property mentioned at the end of section 4.*

A further analysis of these and related properties ought to be carried out, and would (I believe) bring to light more interesting results. The first job is, of course, to find examples which show that the seven properties are distinct.

Several of these results of this chapter are $\exists_1$-analogues of classical results. In fact these $\exists_1$-results imply the classical results. To see this we note that $\exists_1$-results immediately gives us the classical results restricted to model complete theories. But (as in section 3) using suitable definitional extensions it is sufficient to prove the classical results for model complete theories.

CHAPTER C

Counting the $\equiv$ - blocks of e.c. structures

§ 9. Minor results

Associated with each theory T there are three cardinalities $i(E_T)$, $i(F_T)$, $i(G_T)$. These are just the number of countable members of E_T, F_T, G_T respectively, these members being counted up to isomorphism. Very little is known about these cardinalities. (In fact I include somewhere or other in these notes all the non-trivial general results that I know). The major open problems are the three generalizations of Vaught's conjecture i.e. if K is one of E_T, F_T, or G_T then

$$\aleph_0 < i(K) \Rightarrow i(K) = 2^{\aleph_0}$$

In this chapter we look at a different set $j(E_T)$, $j(F_T)$, $j(G_T)$

of cardinalities associated with T. These are much easier to handle and, consequently, we can prove rather more about them.

9.1. *DEFINITION.* Let T be a theory and let K be any of E_T, F_T, G_T. The cardinal number $j(K)$ is the number of members of K, these members being counted up to elementary equivalence.

For each of these classes K we have

$$\mathcal{A} < \mathcal{B} \in K \;\Rightarrow\; \mathcal{A} \in K$$

so that $j(K)$ is determined by the countable members of K. In particular $j(K) \leqslant i(K)$. Much of this chapter is concerned with methods of making $j(K)$ large and so, indirectly, with methods of making $i(K)$ large.

Our first result gives us the relative strengths of these cardinalities.

9.2. *THEOREM.* *For each theory* T,

$$1 \leqslant j(F_T) \leqslant j(G_T) \leqslant j(E_T) \leqslant 2^{\aleph_0}$$

PROOF. The two inequalities $1 \leqslant j(F_T)$, $j(E_T) \leqslant 2^{\aleph_0}$ hold since the underlying language is countable, and $j(G_T) \leqslant j(E_T)$ holds since $G_T \subseteq E_T$. Thus it remains to prove $j(F_T) \leqslant j(G_T)$.

Let $\{\mathcal{A}_j : j \in J\}$ be a set of $\equiv$ - representatives of F_T (i.e. for each $\mathcal{B} \in F_T$ there is some $j \in J$ such that $\mathcal{A}_j \equiv \mathcal{B}$, and for each $j_1, j_2 \in J$ if $j_1 \neq j_2$ then $\mathcal{A}_{j_1} \not\equiv \mathcal{A}_{j_2}$). Thus $j(F_T) = |J|$. For each $j \in J$ let $\mathcal{B}_j$ be a member of G_T such that $\mathcal{A}_j \subseteq \mathcal{B}_j$

Notice that, in fact, $\mathcal{A}_j < \mathcal{B}_j$ so, for each $j_1, j_2 \in J$

$$\mathcal{B}_{j_1} \equiv \mathcal{B}_{j_2} \Rightarrow \mathcal{A}_{j_1} \equiv_1 \mathcal{A}_{j_2} \Rightarrow \mathcal{A}_{j_1} \equiv \mathcal{A}_{j_2}$$

the second implication following by 2.17.

This shows that $j(G_T) \geqslant |J|$, as required.

Let Σ be a set of sentences consistent with the theory T. We write $T + \Sigma$ for the theory axiomatized by $T \cup \Sigma$. If Σ is a singleton $\{\sigma\}$ then we write $T + \sigma$ for $T + \{\sigma\}$. Notice that the notation $T + \Sigma$ is used only when Σ is consistent with T.

A *finite extension* of T is simply an extension of T of the form $T + \Sigma$ where Σ is finite. Similarly an $\exists_1$-*extension* of T is an extension $T + \Sigma$ where Σ is a set of $\exists_1$-sentences.

We easily verify that if Σ is a set of $\exists_1$-sentences consistent with T then Σ is also consistent with any cotheory of T. In particular we have cotheories $T + \Sigma$, $T^e + \Sigma$, $T^f + \Sigma$, $T^g + \Sigma$.

9.3. *THEOREM. Let* T *be a theory and* σ *an* $\exists_1$-*sentence consistent with* T. *Then the following hold.*

(e). $(T + \sigma)^e = T^e + \sigma$, $E_{T+\sigma} = E_T \cap Md(\sigma)$.

(f). $(T + \sigma)^f = T^f + \sigma$, $F_{T+\sigma} = F_T \cap Md(\sigma)$.

(g). $(T + \sigma)^g = T^g + \sigma$, $G_{T+\sigma} = G_T \cap Md(\sigma)$.

PROOF. (e). Let $K = E_T \cap Md(\sigma)$. Clearly, using E_T, each embedding between members of K is a 1-embedding, and since σ is an $\exists_1$-sentence

$$\mathcal{A} <_1 \mathcal{B} \in K \Rightarrow \mathcal{A} \in K.$$

Thus to show that $K = E_{T+\sigma}$ it is sufficient to show that

K is cofinal in $Sb(T + \sigma)$.

Since σ is an $\exists_1$-sentence we easily check that $Sb(T + \sigma) = Sb(T) \cap Md(\sigma)$, so that $K \subseteq Sb(T + \sigma)$. Now consider any $\mathcal{A} \in Sb(T + \sigma)$. Since $\mathcal{A} \in Sb(T)$ there is some $\mathcal{B} \in E_T$ such that $\mathcal{A} \subseteq \mathcal{B}$. But $\mathcal{A} \models \sigma$ so (again since σ is an $\exists_1$-sentence) $\mathcal{B} \models \sigma$ and hence $\mathcal{B} \in K$. Thus K is cofinal in $Sb(T + \sigma)$, as required.

Finally we note that

$$\begin{aligned}(\mathrm{T} + \sigma)^{e} &= \mathrm{Th}(E_{\mathrm{T}+\sigma}) \\ &= \mathrm{Th}(E_{\mathrm{T}} \cap Md(\sigma)) \\ &= \mathrm{Th}(E_{\mathrm{T}}) + \sigma \\ &= \mathrm{T}^{e} + \sigma\end{aligned}$$

which completes the proof of (e).

(f). Since σ is an $\exists_1$-sentence we see that $\mathrm{T} + \sigma$ and $\mathrm{T}^f + \sigma$ are cotheories. Now consider and formula φ consistent with $\mathrm{T}^f + \sigma$. Thus $\sigma \wedge \varphi$ is consistent with T^f and so there is an $\exists_1$-formula ϑ consistent with T^f, such that $\mathrm{fv}(\vartheta) \subseteq \mathrm{fv}(\sigma \wedge \varphi) = \mathrm{fv}(\varphi)$ and $\mathrm{T}^f \vdash \vartheta \rightarrow \sigma \wedge \varphi$.

This shows that ϑ is consistent with $\mathrm{T}^f + \sigma$ and $\mathrm{T}^f + \sigma \vdash \vartheta \rightarrow \varphi$. Hence $\mathrm{T}^f + \sigma$ is f-complete so that $(\mathrm{T} + \sigma)^f = \mathrm{T}^f + \sigma$.

To prove (f) it is now sufficient to show that $K = F_{\mathrm{T}} \cap Md(\sigma)$ is the class of completing models of $\mathrm{T}^f + \sigma$.

Let $\mathcal{A} \in K$, so that $\mathcal{A}$ is a model of $\mathrm{T}^f + \sigma$, and consider any model $\mathcal{B}$ of $\mathrm{T}^f + \sigma$ such that $\mathcal{A} \subseteq \mathcal{B}$. Then $\mathcal{A} \in F_{\mathrm{T}}$ and $\mathcal{A} \subseteq \mathcal{B} \models \mathrm{T}^f$ so that $\mathcal{A} < \mathcal{B}$. Thus $\mathcal{A}$ is a completing model of $\mathrm{T}^f + \sigma$. Conversely let $\mathcal{A}$ be a completing model of $\mathrm{T}^f + \sigma$, in particular $\mathcal{A} \in Md(\sigma)$.
Suppose $\mathcal{A} \subseteq \mathcal{B} \models \mathrm{T}^f$. Since σ is an $\exists_1$-sentnce and $\mathcal{A} \models \sigma$ we have $\mathcal{B} \models \sigma$ i.e. $\mathcal{B}$ is a model of $\mathrm{T}^f + \sigma$. Thus $\mathcal{A} < \mathcal{B}$, which shows that $\mathcal{A}$ is a completing model of T^f i.e. $\mathcal{A} \in F_{\mathrm{T}}$. Hence $\mathcal{A} \in K$ as required.

(g). This is proved in the same way as (e).

The final result of this section generalizes a part of 2.18, which can be stated as

$$\mathrm{j}(F_{\mathrm{T}}) = 1 \quad \Leftrightarrow \quad \mathrm{j}(G_{\mathrm{T}}) = 1.$$

9.4. *THEOREM. Let* T *be a theory. If either of* $\mathrm{j}(F_{\mathrm{T}})$, $\mathrm{j}(G_{\mathrm{T}})$ *is finite then both are and* $\mathrm{j}(F_{\mathrm{T}}) = \mathrm{j}(G_{\mathrm{T}})$.

PROOF. Clearly, by 9.2. if $\mathrm{j}(G_{\mathrm{T}})$ is finite then so is $\mathrm{j}(F_{\mathrm{T}})$. So we may assume that $\mathrm{j}(F_{\mathrm{T}})$ is finite.

Suppose there is some $\mathcal{A} \in F_T$ such that Th($\mathcal{A}$) is not a finite extension of T^f. Then there is a sequence $(\varrho_r : r < \omega)$ of sentences such that

$$\varrho_0 \in \text{Th}(\mathcal{A}) - T^f$$

and, for each $r < \omega$,

$$\varrho_{r+1} \in \text{Th}(\mathcal{A}) - (T^f + \varrho_0 \wedge \ldots \wedge \varrho_r).$$

Consider the sequence $(\tau_r : r < \omega)$ defined by

$$\tau_0 = \tau \wedge \neg \varrho_0, \qquad \varrho_{r+1} = \varrho_0 \wedge \ldots \wedge \varrho_r \wedge \neg \varrho_{r+1}.$$

Each of these sentences is consistent with T^f (and so holds in some member of F_T) and they are pairwise inconsistent with T^f. Thus we have $j(F_T) \geqslant \aleph_0$, which contradicts the finiteness of j (F_T).

This shows that for each $\mathcal{A} \in F_T$ there is some sentence σ such that Th($\mathcal{A}$) $= T^f + \sigma$. Notice that (since $\mathcal{A}$ is a completing model of T^f) we can assume that σ is an $\exists_1$-sentence. With this $\exists_1$-sentence 9.3(f) shows that $(T + \sigma)^f$ is complete so, by 2.18, $T + \sigma$ has JEP.

Suppose $j(F_T) = n$. The above argument gives us $\exists_1$-sentences $\sigma_1, \ldots \sigma_n$ such that the following hold.

(i) Each $T + \sigma_1$ has JEP.
(ii) $T^o \vdash \sigma_1 \vee \ldots \vee \sigma_n$.
(iii) The sentence $\sigma_1, \ldots, \sigma_n$ are pairwise inconsistent with T.

Now, for each $1 \leqslant i \leqslant n$, let

$$G_i = G_T \cap Md(\sigma_i) = G_{T+\sigma_i}.$$

By (i) we have $j(G_i) = 1$ and (ii) gives

$$G_T = G_1 \cup \ldots \cup G_n.$$

But (iii) shows that this is a partition of G_T, so that $j(G_T) = n$, as required.

§ 10. Major results

This section contains the four major results concerning the three j-cardinalities, and some indications of one method of obtaining these results. We begin, however, we a result concerning $i(F_T)$.

10.1 *THEOREM. Let* T *be a theory with JEP. If* $i(F_T) < 2^{\aleph_0}$ *then* T^f *is atomic.*

This result is proved in [2]. In fact it can be obtained as a corollary of the following theorem. The details of this can be found in [11].

10.2. *THEOREM. Let* T *be a theory such that no finite* $\exists_1$-*extension of* T *has JEP.* Then $j(F_T) = 2^{\aleph_0}$.

The other three results are simply the analogues of Vaught's conjecture stated in terms of j.

10.3. *THEOREM. For each theory* T *the following hold.*

(e). $\aleph_0 < j(E_T) \Rightarrow j(E_T) = 2^{\aleph_0}$

(f). $\aleph_0 < j(F_T) \Rightarrow j(F_T) = 2^{\aleph_0}$

(g). $\aleph_0 < j(G_T) \Rightarrow j(G_T) = 2^{\aleph_0}$

The proofs (that we discuss in this section) of 10.2 and the three parts of 10.3 are based on splitting arguments. There are similarities and some essential difference between these proofs. The details of 10.2 can be found in [11]. Here we will discuss the details of 10.3(g) (which, in fact, is the easiest to prove) and then sketch the modification required to prove 10.3(e) and 10.3(f).

For the rest of this section let T be a fixed theory.

For each set X of sentences let

$$G(X) = G_T \cap Md(X).$$

The splitting construction is achieved using the following lemma.

10.4. *LEMMA. Let* X *be a finite set of sentences such that* $j(G(X)) > \aleph_0$. *Then there are* $\exists_1$*-senteces* σ_0, σ_1 *such that*

(i) $j(G(X \cup \{\sigma_0\})) > \aleph_0$ and $j(G(X \cup \{\sigma_1\})) > \aleph_0$,
(ii) $T^g \vdash \neg \sigma_0 \vee \neg \sigma_1$.

PROOF. Let Σ be the set of $\exists_1$-sentences σ such that $j(G(X \cup \{\sigma\})) \leqslant \aleph_0$, and put
$G'(X) = \cup \{G(X \cup \{\sigma\}) : \sigma \in \Sigma\}$.

Thus, since $G'(X)$ is essentially a countable union of countable sets, we have $j(G'(X)) \leqslant \aleph_0$
In particular

$$j(G(X) - G'(X)) > \aleph_0$$

so there are $\mathcal{A}_0$, $\mathcal{A}_1 \in G(X) - G'(X)$ such that $\mathcal{A}_0 \not\equiv \mathcal{A}_1$.

Now $\mathcal{A}_0$, $\mathcal{A}_1$ are both g-generic so that $\mathcal{A}_0 \not\Rightarrow (\exists_1)\ \mathcal{A}_1$ and hence there is some $\exists_1$-sentence σ_0 such that $\mathcal{A}_0 \models \sigma_0$, $\mathcal{A}_1 \models \neg \sigma_0$.

But then (since $\mathcal{A}_1$ is e.c.) there is some $\exists_1$-sentence σ_1 such that $\mathcal{A}_1 \models \sigma_1$ and $T \vdash \sigma_1 \rightarrow \neg \sigma_0$. This gives us (ii) so it remains to show (i).

Let $i = 0,1$. Notice that $\mathcal{A}_i \in G(X \cup \{\sigma_i\})$.
If $j(G(X \cup \{\sigma_i\})) \leqslant \aleph_0$ then $\sigma_i \in \Sigma$ and so $\mathcal{A}_i \in G'(X)$. By choice this is not so, and hence we get (i).

Let Ψ be the complete binary tree i.e. Ψ is the set of finite sequences of 0 and 1 (including the empty sequence) ordered by extension.

10.5. *LEMMA. Let* T *be a theory such that* $j(G_T) > \aleph_0$. *Then* *there is a system* $\mathbf{X} = \{X_\nu : \nu \in \Psi\}$ *of sets of sentences with the following properties.*

(i) *For each node* ν *of* Ψ, X_ν *is a finite set of* $\exists_1$*-sentences such that* $j(G(X_\nu)) > \aleph_0$.

(ii) *For each two nodes* ν, ν' *of* Ψ, *if* $\nu \leqslant \nu'$ *then* $X_\nu \subseteq X_{\nu'}$.

(iii) *For each two nodes* ν, ν' *of* Ψ, *if* $\nu | \nu'$ *then* $T^g \cup X_\nu \cup X_{\nu'}$ *is inconsistent.*

PROOF. We first put $X_\varnothing = \varnothing$ and then construct the rest of the system $\mathbf{X}$ by induction up the tree using 10.4.

We can now complete the proof of 10.3(g).

Consider the system $\mathbf{X}$ given by 10.5. For each branch β of Ψ let $X_\beta = \bigcup\{X_\nu : \nu \in \beta\}$. By 10.5 (i, ii) X_β is a set of $\exists_1$-sentences consistent with T^g, so there is some $\mathcal{B}_\beta \in Sb(T)$ such that $\mathcal{B}_\beta \models X_\beta$. But then there is some $\mathcal{A}_\beta \in G_T$ such that $\mathcal{B}_\beta \subseteq \mathcal{A}_\beta$ and so (since X_β is a set of $\exists_1$-sentences) $\mathcal{A}_\beta \models X_\beta$. Finally 10.5 (iii) shows that for each two branches β, β' if $\beta \neq \beta'$ then $\mathcal{A}_\beta \not\equiv \mathcal{A}_{\beta'}$, and hence $j(G_T) = 2^{\aleph_0}$

This completes the proof of 10.3(g). We now look at the modifications required to prove 10.3(e.f).

First consider 10.3(f). Here we may carry out the above proof up to the existence of the model $\mathcal{B}_\beta$ of X_β. But F_T is not cofinal in $Sb(T)$ so there may be no $\mathcal{A}_\beta \in F_T$ such that $\mathcal{A}_\beta \models X_\beta$. Next consider 10.3(e). Here we do not have, for $\mathcal{A}_0, \mathcal{A}_1 \in E_T$,

$$\mathcal{A}_0, \Rrightarrow(\exists_1)\ \mathcal{A}_1 \Rightarrow \mathcal{A}_0 \equiv \mathcal{A}_1$$

so part of the proof of 10.4 breaks down. In fact we can construct a corresponding system **X** but we can not assume that its members are sets of $\exists_1$-sentences. Thus again we have a a problem to obtain a suitable model of X_β.

In both cases the system must be constructed with a little more care so that the following holds.

(?) For each branch β of Ψ there is some

$$\mathcal{A} \in E_T \text{ (or } F_T\text{) such that } \mathcal{A} \models X_\beta$$

To do this we use the fact that E_T (F_T) is exactly the class of models of T^e (T^f) which omit a certain countable set of types. We then interweave an omitting types construction into the construction of **X** to ensure that each of the relevant types can be omitted in a model of each $T^e \cup X_\beta$ $(T^f \cup X_\beta)$.

The details of this construction are similar to those of the proof of 10.2 given in [11].

The results of this and the previous section tell the whole story concerning the relative values of $j(E_T)$, $j(F_T)$, $j(G_T)$. Modulo these results anything which can happen does happen. Various examples to show this can be found in [4].

§ 11. A topological proof

In this section we derive 10.2 and 10.3(f) from a well known result of descriptive topology. This method can be modified to give a proof of 10.3(e) also.

We are concerned with four properties of a topological space S.

(S_1). S *is second countable.*
(S_2). S *is haudsdorff.*

(S_3). S *is compact.*
(S_4). S *has a basis of clopen sets.*

Any space satisfying ($S_{1,2,3}$) is induced by a complete metric. so we have the following result.

11.1 *THEOREM. Let* S *be a space satisfying* ($S_{1,2,3}$) *and let* Φ *be a* G_δ *subset of* S *such that either*

(i) $|\Phi| > \aleph_0$

or (ii) Φ *has no isolated points.*

Then Φ *includes a perfect set, in particular* $|\Phi| = 2^{\aleph_0}$.

The space satisfying ($S_{2,3,4}$) are, of course, exactly the dual space of boolean algebras. The particular space we use is constructed as just such a dual space.

From now on let T be a fixed theory. Let A be the sentence algebra of T^f (i.e. the boolean algebra of all sentences modulo T^f) and let S be the dual space of A. Thus S satisfies ($S_{2,3,4}$), and since the undelying language is countable S also satisfies (S_1). We call S the f-space of T.

The points of S are essentially the complete extensions P of T^f, in particular each $\mathcal{A} \in F_T$ gives us a point $P = Th(\mathcal{A})$ of S. We call such points the f-points of S (not every point of S is an f-point) and let Φ be the set of f-points. Notice that two members of F_T are elementarily equivalent if and only if they give the same f-point ,so that $j(F_T) = |\Phi|$. Thus to use 11.1 we must show that Φ is a G_δ subset of S.

For each formula φ let $\Omega(\varphi)$ be the type

$\{\varphi\} \cup \{\neg\vartheta : \vartheta$ is an $\exists_1$-formula, consistent with T, such that $fv(\vartheta) \subseteq fv(\varphi)$ and $T^f \vdash \vartheta \rightarrow \varphi\}$.

These types enable us to characterize the f-points of S.

11.2. *THEOREM. Let* T *be a theory and* S *its f-space. A point* P *of* S *is an f-point if and only if for each formula* φ *the type* $\Omega(\varphi)$ *is non-principal over* P.

PROOF. Suppose first that P is an f-point of S, so that there is some $\mathcal{A} \in F_T$ with $P = Th(\mathcal{A})$. Suppose also that φ is a formula such that the type $\Omega(\varphi)$ is principal over P i.e. there is some formula ψ, consistent with P, such that $fv(\psi) \subseteq fv(\Omega(\varphi)) = fv(\varphi)$ and

$$P \models \psi \rightarrow \bigwedge \Omega(\varphi).$$

Since ψ is consistent with P there is some point **a** of $\mathcal{A}$ such that $\mathcal{A} \models \psi(\mathbf{a})$. Thus (since $P \vdash \psi \rightarrow \varphi$) $\mathcal{A} \models \varphi(\mathbf{a})$ and so (since $\mathcal{A} \in F_T$) there is some formula ϑ with $\neg \vartheta \in \Omega(\varphi)$ and $\mathcal{A} \models \vartheta(\mathbf{a})$. But $\neg \vartheta \in \Omega(\varphi)$ gives $P \vdash \psi \rightarrow \neg \vartheta$ so that $A \models \neg \vartheta(\mathbf{a})$, which is a contradiction.

Conversely suppose that for each formula φ the type $\Omega(\varphi)$ is non-principal over the point P of S. Thus, by the classical omitting types theorem, there is some model $\mathcal{A}$ of P which omits each $\Omega(\varphi)$. We easily verify that $\mathcal{A}$ is a completing model of T^f i.e. $\mathcal{A} \in F_T$, so that $P = Th(\mathcal{A})$ is an f-point.

For each sentence σ we let

$$\beta(\sigma) = \{P : P \text{ is a point of S such that } P \vdash \sigma\}$$

so that $\beta(\sigma)$ is a typical member of the canonical basis of S.

11.3. *COROLLARY. Let* T *be a theory and* S *its f-space. Then the set* Φ *of f-points is a* G_δ *subset of* S.

PROOF. For each two formulas $\psi(\mathbf{v})$, $\varphi(\mathbf{v})$ (so that $fv(\psi) \subseteq fv(\varphi)$) let $U_{\psi,\varphi}$ be the union of $\beta(\neg(\exists \mathbf{v})\,\psi(\mathbf{v}))$ and

$$\bigcup \{\beta((\exists \mathbf{v})\,[\psi(\mathbf{v}) \wedge \neg\, \omega(\mathbf{v})]) : \omega \in \Omega(\varphi)\}$$

so that $U_{\psi,\varphi}$ is an open set of S. The theorem 11.2 shows that

$$\Phi = \bigcap \{U_{\psi,\varphi} : \psi,\varphi \text{ are formulas such that } fv(\psi) \subseteq fv(\varphi)\}$$

which gives the required result.

We now see that 10.3(f) immediately follows from 11.1(i). To obtain 10.2 from 11.1(ii) we prove the following.

11.4. *THEOREM.* *Let* T *be a theory and let* σ *be any* $\exists_1$*-sentence, consistent with* T, *such that* $T + \sigma$ *has JEP. Then* $T^f + \sigma$ *is an isolated point of the set* Φ *of f-points of* T.

PROOF. By 9.3(f) we have $T^f + \sigma = (T + \sigma)^f$ so that (since $T + \sigma$ has JEP) $T^f + \sigma$ is complete and hence is a point of the f-space S of T. But there is some $\mathcal{A} \in F_T$ *such that* $\mathcal{A} \models \sigma$ and hence $T^f + \sigma \subseteq Th(\mathcal{A})$. The completeness of $T^f + \sigma = Th(\mathcal{A})$, so that $T^f + \sigma \in \Phi$.

Now $\beta(\sigma)$ is an open set of S and, for each point P of S,

$$\begin{aligned} P \in \Phi \cap \beta(\sigma) &\Rightarrow T^f \subseteq P \text{ and } \sigma \in P \\ &\Rightarrow T^f + \sigma \subseteq P \\ &\Rightarrow P = T^f + \sigma \end{aligned}$$

so that $\Phi \cap \beta(\sigma) = \{T^f + \sigma\}$, which gives the required result.

The method of this section can be modified to obtain 10.3(e) To prove 10.3(e) we look at the space S of complete extensions of T^e i.e. the dual space of the sentence algebra of T^e. Certain of the points of S (the e-points) correspond to the $\equiv$-blocks of E_T. There is a characterization of these e-points corresponding to 11.2 (we simply use the types $\Omega(\varphi)$ for $\forall_1$-formulas φ only), and so (corresponding to 11.3) the set of e-points form a G_δ subset of S. Thus 10.3(e) follows from 11.1(i). There appears to be no reasonable way of stating the corresponding result which follows from 11.2 (ii).

CHAPTER D

The space controlling generic structures

§ 12. The dual space of a distributive lattice

In section 11 we used spaces constructed as the dual space of boolean algebras. In section 14 we will use a space constructed as a dual space of a distributive lattice. This section contains the required lattice theoretic results.

The duality theory for boolean algebras was developed by M.H. Stone and then extended (by him) to distributive lattices. Unfortunately he made the wrong extension. In this section we give an account of the right extension, which is due to H.A. Priestley. The proofs and more details of these results can be found in [8], [9].

Throughout we are concerned with distributive lattices which have a top 1 and a bottom 0 such that $0 \neq 1$. For simplicity we refer to such structures as d-lattices. The smallest d-lattice is, of course, the two element lattices **2**.

The dual space of a d-lattice A can be built with the points as either the prime ideals of A or the prime filters of A. There are advantages and disadvantages to both approaches, however I prefer a third approach.

12.1. *DEFINITION.* Let A be a d-lattice. A *character* of A is a morphism

$$A \xrightarrow{\quad p \quad} \mathbf{2}$$

We let βA be the set of characters of A.

These three concepts (prime ideal, prime filter, character) are, of course, equivalent. Thus, for a character p of the d-lattice A, the corresponding prime ideal is $p^{-1}[0]$ and the corresponding prime filter is $p^{-1}[1]$.

Every non-trivial result of boolean duality follows from the boolean prime ideal theorem, so we must expect to use some generalization of this theorem here. The following is the appropriate generalization.

12.2 *THEOREM. Let* A *be a d-lattice,* I *an ideal of* A, *and* F *a filter of* A *such that* $I \cap F = \emptyset$. *Then there is some character* p *of* A *such that* $p[I] = \{0\}$, $p[F] = \{1\}$.

We now turn βA into a space in such a way that A is isomorphic to a subalgebra of the algebra of clopen subsets of

βA. We do this in the obvious way. For each $x \in A$ we put

$$\beta(x) = \{p \in \beta A : p(x) = 1\}$$

and topologize βA by taking

$$\{\beta(x) : x \in A\} \cup \{\beta A - \beta(x) : x \in A\}$$

as a subbase. We call this space the dual space of A.

Notice that if A is a boolean algebra then the above defined dual space is the usual dual space.

12.3. *THEOREM. Let* A *be a d-lattice. The dual space* βA *of* A *is a boolean space. If* A *is countable the* βA *is second countable.*

Since the dual βA of A is boolean it must be the dual space of the boolean algebra $Co(\beta A)$ of clopen subsets of βA. Notice that, for $x \in A$, $\beta(x) \in Co(\beta A)$. Also we easily check that, for $x,y \in A$,

$$\begin{aligned} \beta(x \wedge y) &= \beta(x) \cap \beta(y) \\ \beta(x \vee y) &= \beta(x) \cup \beta(y) \end{aligned}$$

and that β is $1 - 1$. Thus we have an embedding

$$A \xrightarrow{\quad\beta\quad} Co(\beta A).$$

To determine the range of this embedding we use the natural ordering of βA, i.e. for each $p, q \in \beta A$ we let

$$p \leqslant q \qquad \text{mean} \qquad (\forall x \in A)\ [q(x) \leqslant p(x)].$$

The simplicity of the following result should be compared with Stone's original result.

12.4. *THEOREM. Let* A *be a d-lattice. Then* β *is an embedding of* A *into the algebra* $Co(\beta A)$ *of clopen subsets of the dual* *space* βA *of* A. *The range of* β *is the set of clopen initial sections of* βA. *In fact* $Co(\beta A)$ *is the boolean closure of* A.

We now come to some lattice theoretic results which, although they have independent interest, are motivated by the application to the study of generic structures.

12.5. *DEFINITION.* Let A be a d-lattice. Then μA is the set of maximal elements of βA.

Notice that the elements of μA correspond to the maximal prime ideals of A or the minimal prime filters of A.

We easily check that the ordering of βA is inductive so we have the following lemma.

12.6. *LEMMA.* *For each* $p \in \beta A$ *there is some* $q \in \mu A$ *such that* $p \leqslant q$.

To study μA it is useful to introduce another idea. For each $a \in A$ let

$$a^+ = \{x \in A : a \vee x = 1\}.$$

We easily check that a^+ is a filter of A, and that a^+ is principal if and only if a has a dual pseudo complement in A. Notice also that for $p \in \beta A$,

$$p(a) = 0 \Rightarrow p[a^+] = \{1\}.$$

These filters a^+ enable us to characterize μA inside βA, but first we require a weakened version of 12.2.

12.7. *LEMMA.* *Let* J *be an ideal and* a *an element of the d-lattice* A *such that* $J \cap a^+ = \emptyset$. *Then there is some* $q \in \mu A$ *with* $q[I] = \{0\}$ *and* $q(a) = 0$.

PROOF. Consider the subset oφ A

$$I = \{x \vee y : x \in J,\ y \leqslant a\}.$$

We easily check that I is an ideal of A and that $J \cup \{a\} \subseteq I$.

Now $1 \notin I$, for otherwise there is some $x \in J$ such that $x \vee a = 1$ i.e. $x \in J \cup a^+$.
Thus $I \cap \{1\} = \emptyset$ so that 12.2 gives us some $p \in \beta A$ with $p[I] = \{0\}$. But then 12.6 gives us some $q \in \mu A$ such that $p \leqslant q$ so that (by the definition of the ordering of βA) $q[I] = \{0\}$. Hence we have the required result.

We can now caracterize μA inside βA.

12.8. *THEOREM. Let* A *be a d-lattice. For each* $p \in \beta A$ *the following are equivalent.*

(i) $p \in \mu A$.
(ii) *For each* $a \in A$, *if* $p[a^+] = \{1\}$ *then* $p(a) = 0$.

PROOF. (i) $\Rightarrow$ (ii) Suppose $p \in \mu A$ and $a \in A$ is such that $p[a^+] = \{1\}$. Let $J = p^{-1}[0]$ so that J is an ideal with $J \cap a^+ = \emptyset$. By 12.7 there is some $q \in \mu A$ such that $q[J] = \{0\}$ and $q(a) = 0$. The choice of J gives $p \leqslant q$ so that (since $p \in \mu A$) $p = q$, which gives the required result.

(ii) $\Rightarrow$ (i). Suppose p satisfies (ii) and consider any $q \in \beta A$ such that $p \leqslant q$. Then, for each $a \in A$, we have

$$\begin{aligned} q(a) = 0 \quad &\Rightarrow q[a^+] = \{1\} && \text{by above remark} \\ &\Rightarrow p[a^+] = \{1\} && \text{since } p \leqslant q \\ &\Rightarrow p(a) = 0 && \text{by (ii)} \end{aligned}$$

so that $q \leqslant p$, which gives $p = q$, as required.

12.9. *COROLLARY. Let* A *be a countable d-lattice. Then* μA *is a* G_δ *subset of* βA.

PROOF. For each $a \in A$ let

$$U_a = \{\beta A - \beta(a)\} \cup \{\beta A - \beta(x) : x \in a^+\}$$

so that U_a is an open set of βA. From 12.8 we have

$$\mu A = \cap \{U_a : a \in A\}$$

which gives the required result.

The particular d-lattices we consider in the following sections will, of course, be countable so 12.9 will be applicable. Certain of these lattices A will also satisfy the following property.

(12.10). For each $a \in A$, if $a \neq 0$ then there is some $p \in \mu A$ with $p(a) = 1$.

This enables us to apply the following theorem.

12.11. *THEOREM. Let* A *be a d-lattice which has* (12.10). *Then* μA *is dense in* βA.

PROOF. It is sufficient to show that each non-empty member of the canonical subbase of βA meets μA. So we consider two cases.

Suppose first that $a \in A$ is such that $\beta(a) \neq \emptyset$ i.e. $a \neq 0$. Then (12.10) gives us some $p \in \mu A \cap \beta(a)$.

Secondly suppose that $a \in A$ is such that $\beta(a) \neq \beta A$ i.e. $a \neq 1$ so there is some $p \in \beta A$ with $p(a) = 0$. But then 12.6 gives us some $q \in \mu A$ such that $p \leqslant q$, in particular $q(a) = 0$. Thus $q \in \mu A \cap (\beta A - \beta(a))$, as required.

We now wish to look at a particular factor lattice of A. In order to describe the congruence used we require a couple of results.

12.12. *LEMMA. Let* a *be an element of the d-lattice* A. *For each* $x \in A$ *the following are equivalent.*

(i) $x \in a^+$.
(ii) *For each* $p \in \mu A$, *if* $p(x) = 0$ *then* $p(a) = 1$.

PROOF. (i) $\Rightarrow$ (ii). Suppose $x \in a^+$ and consider any $p \in \mu A$. Then $p(x) \vee p(a) = p(x \vee a) = p(1) = 1$ so that if $p(x) = 0$ then $p(a) = 1$.

(ii) $\Rightarrow$ (i). Suppose (i) does not hold, so that

$$(x] \cap a^+ = \emptyset$$

where $(x]$ is the principal ideal of A generated by x. Then

12.7 gives us some $p \in \mu A$ such that $p(x) = 0$, $p(a) = 0$, which contradicts (ii).

12.13. *THEOREM. Let* a, b *be elements of the d-lattice* A. *The following are equivalent.*

(i) $a^+ \subseteq b^+$
(ii) *For each* $p \in \mu A$, $p(a) \leqslant p(b)$.

PROOF. (i) $\Rightarrow$ (ii). Suppose (i) holds and consider any $p \in \mu A$. If $p(b) = 0$ then $p[b^+] = \{1\}$ so (i) gives $p[a^+] = \{1\}$. But then 12.8 gives $p(a) = 0$, and so $p(a) \leqslant p(b)$, as required.

(ii) $\Rightarrow$ (i). Suppose (ii) holds and consider any $x \in a^+$. We use 12.12 to show that $x \in b^+$ (and hence $a^+ \subseteq b^+$).

Consider any $p \in \mu A$ such that $p(x) = 0$. Then 12.12 ((i) $\Rightarrow$ (ii)) (applied to a) gives $p(a) = 1$, so that (ii) gives $p(b) = 1$. But then 12.12 ((ii) $\Rightarrow$ (i)) (applied to b) gives $x \in b^+$, as required.

The factor lattice we construct will have property (12.10), in fact it will have a stronger property, namely the following. (12.14). For each two elements a, b of A, $a \leqslant b$ if and only if $a^+ \subseteq b^+$.

12.15. *LEMMA. Each d-lattice with* (12.14) *also has* (12.10).

PROOF. Suppose A is a d-lattice with (12.14) and consider any $a \in A$, $a \neq 0$. Since $a \nleqslant 0$, (12.14) gives $a^+ \nsubseteq 0^+$ so that 12.13 gives us some $p \in \mu A$ with $p(a) \nleqslant p(0) = 0$, i.e. $p(a) = 1$, as required.

Now let $\sim$ be the relation on A defined by

$$a \sim b \Leftrightarrow (\forall p \in \mu A)\ [p(a) = p(b)].$$

We easily check that $\sim$ is a congruence on A; 12.13 shows that $a \sim b$ holds exactly when $a^+ = b^+$.

12.16 *DEFINITION.* For each d-lattice A we let A° be the factor lattice $A/\sim$ of A, and let

$$A \xrightarrow{\quad f \quad} A^{o}$$

be the canonically associated epimorphism.

To show that A^o has (12.14) we need three easy results.

12.17. *LEMMA. Let* a, b, c *be elements of the d-lattice* A. *The following hold.*

(i) $a \leqslant b \Rightarrow a^+ \subseteq b^+$.
(ii) $c \sim 1 \Rightarrow c = 1$.
(iii) $(a \wedge b)^+ = a^+ \cap b^+$.

PROOF. (i). Suppose $a \leqslant b$ and $x \in a^+$, so that $a \vee b = b$ and $x \vee a = 1$. Then

$$x \vee b = x \vee a \vee b = 1 \vee b = 1$$

so that $x \in b^+$, as required.

(ii). Suppose $c \neq 1$ so that (by 12.2 and 12.6) there is some $p \in \mu A$ such that $p(c) = 0 \neq p(1)$, which gives $c \sim 1$, as required.

(iii). For each $x \in A$ we have

$$\begin{aligned} x \in (a \wedge b)^+ &\Leftrightarrow (a \wedge b) \vee x = 1 \\ &\Leftrightarrow (a \vee x) \wedge (b \vee x) = 1 \\ &\Leftrightarrow (a \vee x = b \vee x = 1 \\ &\Leftrightarrow x \in a^+ \cap b^+ \end{aligned}$$

as required.

12.18. *THEOREM. For each d-lattice* A *the factor lattice* A^o *has* (12.14) *and hence* (12.10).

PROOF. By 12.17(i) it is sufficient to show that, for each $a, b \in A$,

$$f(a)^+ \subseteq f(b)^+ \Rightarrow f(a) \leqslant f(b).$$

Let $a, b \in A$ be such that $f(a)^+ \subseteq f(b)^+$. Consider any $x \in a^+$, so that (since

$$f(x) \vee f(a) = f(x \vee a) = f(1) = 1)$$

$f(x) \in f(a)^+ \subseteq f(b)^+$, and hence $f(x \vee b) = 1$. But now 12.17 (ii) gives $x \vee b = 1$, i.e. $x \in b^+$. This shows that $a^+ \subseteq b^+$.

Now 12.17 (iii) gives $(a \wedge b)^+ = a^+ \cap b^+ = a^+$ so that $a \wedge b \sim a^+$ i.e. $f(a \wedge b) = f(a)$. Hence

$$f(a) \wedge f(b) = f(a \wedge b) = f(a)$$

which gives $f(a) \leqslant f(b)$, as required.

The dual spaces βA, βA° are, in general, different. Nevertheless the morphism f induces, in a natural way, a map

$$\beta A^\circ \xrightarrow{\quad\varphi\quad} \beta A$$

which gives us a comparison between βA° and βA.
The following theorem can now be proved.

12.19. *THEOREM. The above map* φ *is a* $1 — 1$, *continuous, increasing map, and sets up a* $1 — 1$ *correspondence between* μA° *and* μA. *In particular* μA *and* μA° *are homeomorphic.*

§ 13. The components of a theory

In the next section we will set up (for each theory T) a certain space which controls the $\equiv$ - blocks of G_T and F_T. To do this we will use the lattice theoretic results of the last section and some model theoretic results. This section, which is based on [3], contains these required model theoretic results.

13.1. *DEFINITION.* Let T be a theory. A *component* of T is a theory P such that

(i) P is $\forall_1$-axiomatizable,
(ii) $T \cap \forall_1 \subseteq P$,

(iii) P has JEP,
and is minimal with respect to these properties.

For each theory T we write $T_\forall$ for the theory axiomatized by $T \cap \forall_1$.

The following theorem gives the fundamental result concerning components.

13.2. *THEOREM. Let* T *be a theory. For each theory* P *the following are equivalent.*

(i) *There is some* $\mathcal{A} \in E_T$ *such that* $P = \mathrm{Th}(\mathcal{A})_\forall$

(ii) P *is a component of* T.

(iii) *There is some* $\mathcal{A} \in G_T$ *such that* $P = \mathrm{Th}(\mathcal{A})_\forall$

PROOF. (i) $\Rightarrow$ (ii). Suppose $P = \mathrm{Th}(\mathcal{A})_\forall$ where $\mathcal{A} \in E_T$. Clearly P satisfies 13.1 (i, ii, iii) so it is sufficient to show the minimality of P.
Let S be any theory satisfying the properties of 13.1 (i, ii, iii) such that $S \subseteq P$, and let $\mathcal{B}$ be any model of S. Since $\mathcal{A} \models S$, S has JEP, and $\mathcal{A} \in E_T$ we have $\mathcal{A} \Rrightarrow(\forall_1)\, \mathcal{B}$. Thus (using the $\forall_1$-axiomatizability of P and S) we have $\mathcal{B} \models P$ and hence $S = P$, as required.

(ii) $\Rightarrow$ (iii). Suppose that P is a component of T and consider any $\mathcal{B} \in E_P$. Since P is $\forall_1$-axiomatizable we have $P \subseteq \mathrm{Th}(\mathcal{B})$ so, since P has JEP, $P = \mathrm{Th}(\mathcal{B})_\forall$.

Now $T \cap \forall_1 \subseteq P$ so there is some $\mathcal{A} \in G_T$ with $\mathcal{B} \subseteq \mathcal{A}$. This gives $T \cap \forall_1 \subseteq \mathrm{Th}(\mathcal{A})_\forall \subseteq \mathrm{Th}(\mathcal{B})_\forall = P$ hence (since $\mathrm{Th}(\mathcal{A})_\forall$ is a component of T) we get $P = \mathrm{Th}(\mathcal{A})_\forall$, as required.

(iii) $\Rightarrow$ (i) is trivial.

The next theorem should be compared with 9.3. It is proved in more or less the same way.

13.3. *THEOREM. Let* T *be a theory and let* P *be a component of* T. *Then the following hold.*

(e) $P^e = T^e + P, \quad E_P = E_T \cap Md(P).$
(f) $P^f = T^f + P, \quad F_P = F_T \cap Md(P).$
(g) $P^g = T^g + P, \quad G_P = G_T \cap Md(P).$

Suppose $\mathcal{A}, \mathcal{B} \in E_T$. Clearly $\mathcal{A}, \mathcal{B}$ give rise to the same component of T if and only if $\mathcal{A} \equiv_1 \mathcal{B}$. In particular if $\mathcal{A}, \mathcal{B}$ are both f-generic or both g-generic for T then they give the same component if and only if $\mathcal{A} \equiv \mathcal{B}$. In other words there is a $1 - 1$ correspondence between the components of T and the $\equiv$ - blocks of G_T, and some components of T correspond to the $\equiv$ - blocks of F_T. (This gives us another proof of $j(F_T) \leqslant j(G_T)$.)

In the next section we construct a space μT out of components of T. This space will be, in fact ,the space μA of some d-lattice A associated with T. The corresponding lattice A^o (and the congruence $\sim$ which gives us A^o from A) will be located using the following lemma.

13.4. *LEMMA. Let* T *be a theory and* α, β *two* $\forall_1$*-sentences. The following are equivalent.*

(i) $T^o \vdash \alpha \leftrightarrow \beta$
(ii) *For each* $\mathcal{A} \in E_T$, $\mathcal{A} \models \alpha$ *if and only if* $\mathcal{A} \models \beta$.

PROOF. Simply remember that T^o is axiomatized by $T^e \cap \forall_2$.

§ 14. The component space of a theory

Throughout this section T is a fixed theory and A is the lattice of $\forall_1$-sentences modulo T. Thus A is a d-lattice and so all the machinery of section can be used in connection with T. The next theorem shows the connection between these lattice theoretic methods and the components of T.

14.1. *THEOREM. There is a 1 — 1 correspondence between the theories* P *satisfying* 13.1 (i, ii, iii) *and the points of* βA. *Under this correspondence the components of* T *are paired with the points of* μA.

PROOF. The theories satisfying 13.1 (i, ii) are essentially the (proper) filters of A so, using 2.18, the theories satisfying 13.1 (i, ii, iii) are essentially the prime filters of A and these are essentially the elements of βA.

The ordering imposed on βA is essentially the ordering of ideals by inclusion, or the ordering of filter by anti-inclusion. Thus the components of T (= minimal prime filters of A) correspond to the maximal elements of βT i.e. the elements of μT.

Let μT be the space μA (i.e. the space βA relativized to μA). Thus, by 12.3 and 12.9, μT if a G_δ subset of the second countable boolean space βA. Also, using 13.2, the elements of μT are essentially the $\equiv$ - blocks of G_T and some of these points correspond to the $\equiv$ - blocks of F_T. These f-points form a large part of μT.

14.2. *THEOREM. The set of f-points of* μT *form a dense subset of* μT.

PROOF. Let us identify the points of μT with the components of T.

After a few moments thought we see that typical members of the cannonical subbase of μT are

$$X_\alpha = \{P \in \mu T : \alpha \in P\} \qquad Y_\alpha = \{P \in \mu T : \neg\alpha \in P\}$$

where α is an $\forall_1$-sentence. For each $\exists_1$-sentence σ let

$$U_\sigma = \{P \in \mu T : \sigma \in P\}$$

so that $U_\sigma = Y_{\neg\sigma}$ i.e. U_σ is an open subset of μT. Also, remembering 13.2, we soon see that

$$X_\alpha = \cup \{U_\sigma : \sigma \text{ is an } \exists_1\text{-sentence such that } T \vdash \sigma \rightarrow \alpha\}$$

so that $\{U_\sigma : \sigma$ an $\exists_1$-sentence$\}$ is a subbase (in fact a basis) of μT.

This shows that it is sufficient to show that each non-empty U_σ contains some f-point. But this is trivial since U_σ is non-empty if and only if σ is consistent with T, and each $\exists_1$-sentence consistent with T holds in some member of F_T.

As defined μT appears to depend on T, whereas it should depend only on the cotheoretic class of T. We show that, in fact, this is so. We do this by showing that the factor lattice A° of A is the lattice of $\forall_1$-sentences modulo T°, and then applying 12.19

14.3. *THEOREM. Let* A *be the* $\forall_1$*-lattice of the theory* T. *Then the* $\forall_1$*-lattice of* T° *is the factor lattice* A°.

PROOF. Let B be the $\forall_1$-lattice of T°. There is an obvious epimorphism

$$A \xrightarrow{\quad g \quad} B$$

(simply send the T-equivalence class of each sentence onto the T°-equivalence class of that sentence). To show that $B = A^\circ$ it is sufficient to show that g has the right congruence kernel. But this follows immediately from 13.4, which gives the required result.

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