ON CREATIVE DEFINITIONS IN THE *PRINCIPIA MATHEMATICA*

A recent paper by E.Z. and E.A. Nemesszeghy (*Mind,* April 1971, pp. 282-3) discusses the definition

*1.01 $p \supset q \quad . = . \quad \sim p \vee q$ Df.

of *Principia Mathematica.* Their paper contains an incorrect citation, a valuable insight, and a logical error. All three points deserve discussion.

1. An Incorrect Citation. The Nemesszeghy's state:

> «The idea that definitions should not strengthen the theory in any significant way finds expression in the following two criteria first formulated by the Polish logician S. Leśniewski:
>
> (1) a defined symbol should be always eliminable
>
> (2) a definition should not permit the proof of previously unprovable relationships among the old symbols.
>
> These two criteria are called by some logicians the criteria for a proper definition.»

Although these two criteria have been credited to deśniewski at least twice before (1) he never formulated these conditions as conditions a definition should satisfy, either in his published work or in his lectures (2). He did discuss — although he never published his views on the matter — definitions that *fail* to satisfy condition (2). These he called *creative definitions,* and he used frequently in his three logical system: Protothetic, Ontology, and Mereology. Thus not only did Leśniewski not formulate condition (2), but also he would not accept it as a condition which all definitions should satisfy.

Leśniewski did make significant contributions to the theory of definitions and I shall review them briefly here (3). He viewed definitions as theses which occur in the object language, and as such they have the same status as axioms and theorems. The most natural way to state definitions, he felt, was as equivalences. Consequently he desired to use equivalence as the primitive term of his Protothetic. This became possible when his student Tarski (4) showed that conjunction could be defined in terms of equivalence and the universal («general» is the word Leśniewski preferred) quantifier. The use of equivalence in stating definitions avoids the use of « = Df.», which Leśniewski felt must be regarded as an additional primitive term (5).

Since definitions are in the object language Leśniewski realized — and this is a valuable contribution — that it is necessary to have rules for introducing definitions. In «Über Definitionen in der sogenannten Theorie der Deduktion» (6) he formulated, in a very precise way, the rules of substitution, detachment and definition for the propositional calculus. Creative definitions were neither defined nor discussed in this paper. However, some of the definitions introducible according to that rule are creative (relative to the particular axiom system chosen) (7). Leśniewski also formulated rules of definition for his systems. This was done in such a way that the condition (1), but not (2), was satisfied and of course so that the addition of definitions to the system preserved consistency (8).

The exact origin and development of conditions (1) and (2) is unknown to me. The following quotations should suffice to show that they are not of recent origin (9).

That definitions are abbreviations is explicit in a remark of Galileo: (10).

> «Note by the way the nature of mathematical definitions which consist merely in the imposition of names, or, if you prefer, abbreviations of speech.»

The non-creativity condition can be read into the writings of Pascal: (11).

> «... if in the demonstration one always substitutes mentally the definitions in place of the things defined, the invincible force of the conclusion will not fail to have its full effect.»

Condition (2) is even more explicit in Mill: (12).

> «... I point out one of the absurd consequences flowing from the supposition that definitions, as such, are premises in any of our reasonings, except such as relate to words only.»

2. *A Valuable Insight.* The main point of the Nemesszeghy's paper is that the definition *1.01 does not satisfy the condition (2) of non-creativity and hence is not a proper definition. They win this point by misconstruing what is said in the *Principia* about definitions. In the second edition we read:

> «It is to be observed that a definition is, strictly speaking, no part of the subject in which it occurs. ... Theoretically, it is unnecessary ever to give a definition ... the definitions are no part of our subject, but are, strictly speaking, mere typographical conveniences. ... theoretically, all definitions are superfulous.» (p. 11).
>
> «The Implication Function is a propositional function with two arguments p and q, and is the proposition that either not-p or q is true, that is, it is the proposition $\sim p \vee q$.» (p. 7).
>
> «Hence in this book p . q is merely a shortened form of symbolism for $\sim(\sim p \vee \sim q)$.» (p. 6).

From these passages it is not surprising that the common belief is that «an expression containing the definiendum must be understood as if it had the definiens in place of the definiendum — not only semantically, i.e., in regard to meaning, but also syntactically.» [13]. If we accept this view then the axioms [14] of *P.M.* are not

(1) $(p \vee p) \supset p$
(2) $q \supset (p \vee q)$
(3) $(p \vee q) \supset (q \vee p)$
(4) $(q \supset r) \supset [(p \vee q) \supset (p \vee r)]$

but rather

(5) $\sim(p \vee p) \vee p$
(6) $\sim q \vee (p \vee q)$

(7) $\sim(p \vee q) \vee (q \vee p)$
(8) $\sim(\sim q \vee r) \vee [\sim(p \vee q) \vee (p \vee r)]$.

Also the rule of procedure, besides substitution, is not
R1 $\alpha,\ \alpha \supset \beta \vdash \beta$

but rather

R2 $\alpha,\ \sim\alpha \vee \beta \vdash \beta$.

It is well known that from (5) — (8) we can derive all tautologies involving $\sim$ and $\vee$ by using the rules of substitution and detachment (R2). Thus if we accept this view of definitions neither the definition *1.01 nor any other is creative with respect to this axiomatization.

The Nemesszeghy's, however, have chosen to reject this view of definitions and to regard the inscriptions (1) — (4) *as* the axioms. What then are we to make of the definition *1.01 ? It can no longer be regarded as a pure abbreviation. Rather, I feel, the definition should be considered as a *rule* which permits the interchange of (a substitution instance of) the definiens with (the same substitution instance of) the definiendum. What the authors show is that this (rule) definition is needed in the proof of $\sim p \vee p$ from (1) — (4) and is thus a creative definition. Let us review their proof.

Consider the matrices: ([15])

$\vee$	0	1	$\sim$
*0	0	0	1
1	0	1	1

$\supset$	0	1
0	0	1
1	0	0

Since $\vee$ and $\sim$ are primitive it seems strange that we give matrices for $\vee$, $\sim$ and $\supset$. But this is necessary since our proof calls for showing that the rule permitting interchange of $p \supset q$ and $\sim p \vee q$ is not satisfied. This cannot be done without a matrix for $\supset$. That the (rule) definition *1.01 fails is seen since $1 \supset 1 = 0$, whereas $\sim 1 \vee 1 = 1$, i.e., $p \supset q$ and $\sim p \vee q$ sometimes take different values. It is easy to check that these matrices verify (1) — (4) and the rule R2 (and also R1), and reject $\sim p \vee p$ (for $p = 1$). Thus if $\sim p \vee p$ can be proved at all (and of course it can) then the definition must be used in the proof. Hence if we accept axioms (1) — (4) and use the rules of substitution and R2 (or even R1 and R2) then the (rule) definition *1.01 is creative.

This important property of (rule) definitions has been pointed out before. H. Rasiowa ([16]) has shown that $p \supset p$, and hence $\sim p \vee p$, can be derived from the axioms of Götlind:

$$p \supset (p \vee q)$$
$$(p \vee p) \supset p$$
$$(p \supset r) \supset [(q \vee p) \supset (r \vee q)].$$

Her proof makes essential use of the definition *1.01 as was pointed out by Rescher ([17]) in a review of a related paper.

3. *A Logical Error*. The Nemesszeghy's state that in order for *1.01 to be a proper definition, i.e., to satisfy conditions (1) and (2), that «the axioms should be written differently, for example without «$\vee$» like this»

(9) $(\sim p \supset p) \supset p$

(10) $q \supset (\sim p \supset q)$

(11) $(\sim p \supset q) \supset (\sim q \supset p)$

(12) $(q \supset r) \supset [(\sim p \supset q) \supset (\sim p \supset r)]$

This is stated without proof; incorrectly as the following matrices show:

$\vee$	0	1	2	$\sim$
*0	0	1	0	1
1	0	1	1	0
2	0	1	1	0

$\supset$	0	1	2
0	0	1	1
1	0	0	0
2	0	0	2

These matrices verify (9) — (12) and R1 (also R2), but reject the definition (for $p = q = 2$) as well as

(13) $\sim(\sim\sim p \vee p) \vee p$
(14) $\sim q \vee (\sim\sim p \vee q)$
(15) $\sim(\sim\sim p \vee q) \vee (\sim\sim q \vee p)$
(16) $\sim(\sim q \vee r) \vee [\sim(\sim\sim p \vee q) \vee (\sim\sim p \vee r)]$

which are the translations of (9) — (12) (18). Since these formulas are easily provable using *1.01 we see that the definition is creative for these formulas with respect to the axiom system (9) — (12).

The axiom system (9) — (12) suffers from an even more serious flaw; for even if we allow this creative use of the definition, there are tautologies which are not provable as is shown by the following matrices: (19)

$\vee$	0	0'	1	2	$\sim$
*0	0	0	0	0	1
*0'	0	0	0	2	1
1	0	0	1	1	0
2	x	x	x	x	0'

$\supset$	0	0'	1	2
0	0	0	1	1
0'	0	0	1	1
1	0	0	0	0
2	0	0	0	2

Where 0 and 0' are the designated values and the x's are arbitrary values. These matrices verify (9) — (12) and the definition (and hence the translations (13) — (16)), as well as R1 and R2. However $\sim p \vee p$ (and $p \supset p$) is rejected for $p = 2$ and is thus not provable, even with the definition. They have fallen into the trap that Hiż warns about (20).

4. *A Suggestion,* Łukasiewicz required, and his suggestion was adopted by Leśniewski, (21) that the axioms of a system be stated using only the primitive terms. If this were done in the axiom systems considered here then the definition *1.01 would not play a creative role.

FOOTNOTES

(1) P. Suppes, *Introduction to Logic,* Van Nostrand, 1957, p. 153. G. Kelley, *General Topology,* Van Nostrand, 1955, p. 251.

(2) I would like to thank Professor Bolesław Sobociński for verifying all of the comments I have made about Leśniewski and in particular for recollecting Leśniewski's views on definitions.

(3) For more details see C. Lejewski, On Implicational Definitions, *Studia Logica,* vol. VIII (1958), pp. 189-208. Also I am preparing a paper debating the advantages and disadvantages of the different styles of definition.

(4) A Tarski, O wyrazie pierwotnym logistyki (On the primitive term of Logistic), *Przegląd Filozoficzny,* vol. 26 (1923). English translation in *Logic, Semantics, Metamathematics,* Oxford 1956, pp. 1-23.

(5) Leśniewski, Grundzüge eines neuen Systems der Grundlagen der Mathematik, *Fundamenta Mathematicae,* vol. 14 (1929), pp. 1-18, cf. p. 11.

(6) *Comptes Rendus des séances de la Société des Sciences et des Lettres de Varsovie,* Classe III, vol. 24 (1931), pp. 289-309. English translation in *Polish Logic* (Oxford 1967, edited by S. McCall) pp. 170-187.

(7) See my paper on "Creative Definitions in Propositional Calculi", *The Notre Dame Journal of Formal Logic,* vol. XVI (1975), pp. 273-294.

(8) For a statement of the rule of procedure for Protothetic (including definitions) see Leśniewski 1929 op. cit. and my Axiomatic Inscriptional Syntax, Part II: The Syntax of Protothetic, *The Notre Dame Journal of Formal Logic,* vol. XIV (1973), pp. 1-52. For the rule of procedure for Ontology see Leśniewski, Über die Grundlagen der Ontologie, *Comptes Rendus ... Varsovie,* vol. 23 (1930) pp. 111-132. On the provability of condition (1) in Leśniewski's systems see J. Słupecki's papers: St. Leśniewski's Protothetics, *Studia Logica,* vol. 1 (1953), pp. 44-111 especially p. 61 ff. and S. Leśniewski's calculus of names, *Studia Logica* vol. 3 (1955), pp. 7-70, especially p. 94. Examples of creative definitions in Leśniewski's systems are also pointed out in these papers.

(9) I would like to thank Thomas W. Scharle for these references.

(10) *Discorsi e dimostrazioni matematiche intorno a due nuove scienze attenti alla mercanica e ai movimenti Locali* (1638), as quoted by D.T. Struik in *A Source Book in Mathematics, 1200-1800,* Cambridge, Mass. 1969, p. 203.

(11) «... si dans la demonstration on substitue toujours mentalement les definitions à la place des definis, la force invincible des consequences ne peut manquer d'avoir tout son effet.» See Pascal's essay «De l'Art de persuader,» *Œuvres de Blaise Pascal,* edited by L. Brunschvicg, P. Boutroux, and F. Gazier, Paris 1914, vol. 9, p. 278.

(12) *A System of Logic,* Book I, Chapter VIII.

(13) A. Church, *Encyclopedia Britanica,* 1967 edition, vol. 7, pp. 173-4.

(14) The superfluous axiom has been omitted.

(15) The Nemesszeghy's matrices are three valued. The use of the more

usual ones makes the proof more perspicuous and does, I feel, full justice to their point.

(16) Sur un certain système d'axioms du calcul des Propositions, *Norsk Matematisk Tidsskrift*, vol. 31 (1949) pp. 1-3.

(17) *Journal of Symbolic Logic*, vol. 17, pp. 66-67. Rescher neglects to give a matrix for negation, but most any one will do. It is perhaps worth pointing out that if the Götlind axioms are written in terms of $\vee$, $\sim$ and rule R2 is used that the proof of Rasiowa can be rewritten so that the definition is not used at all.

(18) (13) - (16) take the undesignated value 1 whenever the last variable in the formula takes the value 1. Similarily $\alpha \vee p$ is always rejected. Also note that (13) - (16) are not the same as (5) - (8).

(19) The construction of these matrices may be of some interest. Knowing that the three valued matrices stated above (Rescher's $\supset$, my $\sim$) verify (9) - (12) and reject $p \supset p$, I tried to trace back thru the definition to get a matrix for $\vee$ which verifies the definition. This fails since

$$0 = 1 \supset 2 =_{Df.} \sim 1 \vee 2 = 0 \vee 2$$
$$2 = 2 \supset 2 =_{Df.} \sim 2 \vee 2 = 0 \vee 2.$$

Thus I «split» the designated value 0 into 0 and 0' and put $0 = 0 \vee 2$ and $2 = 0' \vee 2$. Then one can trace back back thru the definition to get the four valued matrix for $\vee$. Observe that the two sets of matrices for $\sim$ and $\vee$ are isomorphic to the usual two valued matrices (i.e., they verify and reject the same formulas and rules).

(20) A Warning About Translating Axioms, *The American Mathematical Monthly*, vol. 65 (1958), pp. 613-4.

(21) B. Sobociński, On Well Constructed Axiom Systems, *Polskiego Towarzystwa Naukowego na Obczyznie, Rocznik VI (Yearbook VI of the Polish Society of Arts and Sciences Abroad)*, London, 1955-56 pp. 54-65.

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