TARSKI-VAUGHT AND LÖWENHEIM-SKOLEM NUMBERS

George Weaver,
Department of philosophy
Bryn Mawr College, Bryn Mawr
Penneylania 19010

1.0 This paper investigates a problem arising from the work of Veblen and Huntington, among others, in the light of the more recent model theoretic discoveries of Löwenheim, Skolem, Tarski and Vaught In the early 1900's considerable interest was shown in characterizing certain mathematical structures uniquely up to isomorphism (e.g. [7] [8] [9] [10] [11] and [25]). From a modern point of view this work essentially involves a member i of a «similarity class» of structures [5] and a class of languages each of which is interpretable in this similarity class; the problem is finding a set of sentences in one of these languages all of whose models are isomorphic to i. The problem here is finding conditions on those languages which are sufficient to guarantee that i is *not* characterizable up to isomorphism.

The results of Tarski-Vaught, Löwenheim and Skolem for first order languages suggest one possible approach — to associate with each language a least cardinal β such that no interpretation of cardinality β or greater can be characterized up to isomorphism.

In the following pages the details of this approach are worked out: (a) several candidates for such a number are defined (by generalizing the relevant features of the Tarski-Vaught Löwenheim-Skolem and upward Löwenheim-Skolem theorems, respectively); (b) conditions necessary and sufficient for the existence of such numbers are discussed; and (c) their relative sizes are investigated.

2.0 In this section a class, $\mathscr{L}$, of denotational semantics is introduced and the Tarski-Vaught, Löwenheim-Skolem, Upward Löwenheim-Skolem, Löwenheim, and Hanf numbers for the members of this class are defined. For each of the numbers conditions which are necessary and sufficient for a sementics to have that number are given and upperbounds are placed on their size. Finally, it is shown that while all semantics in $\mathscr{L}$ have both Hanf and Löwenheim numbers some members have neither Tarski-Vaught, Löwenheim-Skolem nor Upward Löwenheim-Skolem numbers.

The Gödel-Bernays set theory [3] with the axioms of choice and substitution is assumed throughout: in particular, essential use is made of the distinction between sets and proper classes (c.f. [3] p. 3).

By a *language* we mean any set L; the members of L are called *sentences.* A semantics for L is a four-tuple S(L) = $\langle$I, V, T, D$\rangle$; where I is a non-empty class called *interpretations for L;* V: L $\times$ I $\rightarrow$ T, called the *valuation of S(L)*; T is a non-empty set called the *truth values of S(L)*; and D is a non-empty proper subset of T called the *designated values of S(L).* For each interpretation i, *the valuation for i (in S(L))* is the restriction of the valuation of S(L) to L $\times$ $\{$i$\}$. Interpretations i, j are *equivalent in S(L)* provided their valuations, considered as mappings from L to T, are identical.

S(L) is called a *denotational semantics* provided every interpretation i is associated with a unique set (possibly empty) called the *domain of i.* For each interpretation i in a denotational semantics the *cardinality of i* is the cardinality of the domain of i; $|$i$|$ indicates the cardinality of i. $\mathscr{L}$ denotes the class of *denotational semantics.*

$\mathscr{L}$ includes, among other things, the standard semantics for first order languages with and without equality; the semantics for the «non-elementary» languages given by Mostowski [20] (pp. 132-3); the many valued semantics of Chang and Keisler [12]; the «standard» semantics for many-sorted logics (e.g. [13]); free logics (e.g. [14], [17], [6] and [16]); and those semantics which admit empty domains (e.g. [21] and [4]). Explicitly excluded from $\mathscr{L}$ are the truth-valued semantics for various languages (e.g. [15], [18] and [19]); and those semantics which admit of truth-value gaps (e.g. [24]).

Let L be a first order language with equality over the countable set K of non-logical constants. The interpretations for L are pairs $i = \langle u, f \rangle$ when u is a non-empty set and f is a function defined on K taking values in the usual way. The Löwenheim-Skolem theorem establishes that every interpretation of cardinality greater than $\aleph_0$ is equivalent to an interpretation of cardinality $\aleph_0$. Thus, no interpretation greater than $\aleph_0$ can be characterized uniquely up to isomorphism. Tarski and Vaught [23] extended this theorem in two directions by showing that every interpretation of cardinality greater than $\aleph_0$ has equivalent *substructures* (1) in every smaller cardinality not less than $\aleph_0$. The upward Löwenheim-Skolem theorem established that any structure of cardinality $\aleph_0$ or greater has equivalent *extensions* (1) in every greater cardinality. Thus, only interpretations of cardinality less than $\aleph_0$ can be characterized uniquely up to isomorphism.

Obviously, the Tarski-Vaught and upward Löwenheim-Skolem theorems have a «structural» as well as a cardinality aspect. That is, they guarantee that each infinite interpretation, i, is equivalent to interpretations in every other infinite cardinality and, in addition, that among these equivalent interpretations are substructures in each infinite cardinality less than $|i|$, and extensions in every cardinality greater than $|i|$. In generalizing these results, structural aspects are omitted.

To generalize the above first order theorems to all members of $\mathscr{L}$, we introduce the following: for all β a cardinal number.

(1) β is the *Löwenheim-Skolem number of S(L)* provided β is the least cardinal such that every interpretation of cardinality greater than β has equivalent interpretations of cardinality β. (We indicate this by $\beta = LS(S(L))$;

(2) β is the Tarski-Vaught number of S(L) provided β is the least cardinal such that for all cardinals $\beta' > \beta$ every interpretation of cardinality β' has equivalent interpretations in every smaller cardinality not less than β. $(TV(S(L)) = \beta)$; and

(3) β is the *Upward Löwenheim-Skolem number of S(L)* provided β is the least cardinal such that every interpretation of cardinality at least β has equivalent interpretations in every greater cardinality. $(U(S(L)) = \beta)$.

Thus, for S(L) the standard semantics for a countable first order language with equality, TV(S(L)) = U(S(L)) = LS(S(L)) = $\aleph_0$ and for S(L'), the standard semantics for a countable for order language without equality TV(S(L')) = LS(S(L')) = $\aleph_0$, U(S(L')) = 1.

Let C(S(L)) denote the class of $|i|$ for i an interpretation for L. S(L) is *stable* provided there is some cardinal s.t. all cardinals greater than that cardinal are in C(S(L)): *bounded* provided C(S(L)) is bounded from above; and *periodic* provided for every cardinal in C(S(L)) there exists strictly larger cardinals β, β' such that $\beta' < \beta$, $\beta \in$ C(S(L)) and $\beta' \notin$ C(S(L)). Obviously, every semantics falls into one of these categories. Further, a simple cardinality argument shows that C(S(L)) is a proper class if S(L) is stable.

In the following pages, attention is restricted to stable systems unless stated otherwise. There are several reasons for this restriction: (i) all of the semantics mentioned above are stable; (2) all bounded semantics have Tarski-Vaught, Löwenheim-Skolem and Upward Löwenheim-Skolem numbers; (3) no periodic semantics has Tarski-Vaught, or Upward Löwenheim-Skolem numbers; and (4) while bound semantics have appeared in the literature (e.g. Tarski's semantics for first order languages [23] and the weak Löwenheim-Skolem semantics for second order logic [27]), the author knows of no periodic semantics in the literature.

Obviously, no semantics could have any of these numbers should it turn out that all equivalent interpretations were isomorphic. The following remarks imply, for each stable semantics, the existence of equivalent, non-isomorphic interpretations.

Let F be the class of all valuations for members of I. F is itself a set. Let R $\subseteq$ F x C(S(L)) such that R(f, β) if f is a valuation for an interpretation of cardinality β. Note that for each β in C(S(L)) there exists an f such that R(f, β); but, by the axiom of substitution, R is not a function, since otherwise C(S(L)) would be a set. Hence, there exists i, j such that i is equivalent to j, but $|i| \neq |j|$.

We now turn to the problem of the existence of these numbers and some questions concerning their size.

Let S(L) be a semantics, i be an interpretation in S(L). The *spectrum* ([2]) *of i in S(L)* is the class of cardinals β such that i is equivalent to some interpretation of cardinality β. $\hat{\imath}$ denotes the spectrum of i (in S(L)). Note that if i is equivalent to j, then $\hat{\imath} = \hat{\jmath}$.

The spectrum of the interpretations of S(L) plays an important part in the further developments. Let i be an interpretation in S(L), *i is bounded* provided $\hat{\imath}$ is bounded from above; *i is stable* provided there is some cardinal β such that every cardinal greater than or equal to β belongs to $\hat{\imath}$. (σ_i or σ_i (S(L)) denotes the least such cardinal); *i is periodic* provided i is neither bounded nor stable; and *i is minimal* provided $\hat{\imath} = \{|i|\}$. B(S(L)), M(S(L)), S(S(L)) P(S(L)) denote, respectively, the classes of interpretations bounded, minimal, stable and periodic in S(L). B(S(L)) is a proper subclass of I. Further, we can easily verify that $M(S(L)) \subseteq B(S(L))$: that each of the above classes is closed by equivalence in S(L): and, finally, that B(S(L)), P(S(L)) and S(S(L)) are pairwise disjoint and exhaust I.

For each i in B(S(L)), δ_i (S(L)) denotes the least upper bound of $\hat{\imath}$; $\delta(S(L)) = \text{l.u.b.}\ \{\delta_i(S(L)) : i \in B(S(L))\}$ and $\sigma(S(L)) = \text{l.u.b.}\ \{\sigma_i(S(L)) : i \in S(S(L))\}$ (When no ambiguity results we let $\delta = \delta(S(L))$ and $\sigma(S(L)) = \sigma$.) A simple cardinality argument guarantees that both σ and δ exist. Note that for all i and every cardinal β, if $i \in S(S(L))$ and $\beta \geqslant \sigma_i\ \beta \in \hat{\imath}$.

For S(L) the standard semantics for a countable first order language with equality, $\delta = \sigma = \aleph_0$, $B(S(L)) = M(S(L)) = \{i : |i| < \aleph_0\}$, P(S(L)) is empty and hence $S(S(L)) = \{i : |i| \geqq \aleph_0\}$. For S(L) the standard semantics for a countable first order language without equality, $o = \delta$, $\aleph_0 = \sigma$, $S(S(L)) = I$ and P(S(L)), B(S(L)), and M(S(L)) are empty.

The following gives conditions necessary and sufficient for a semantics to have a Tarski-Vaught number and a bound on its size.

Theorem 1: For all S(L), (a) S(L) has a Tarski-Vaught number iff no interpretation is periodic and (b) if S(L) has a Tarski-Vaught number, then TV (S(L)) $\leqq$ max (δ, σ).

Proof: Suppose S(L) has a Tarski-Vaught number but some interpretation i is periodic. Therefore, i is neither bounded nor stable. Since $i \notin B(S(L))$ there exists β, β', β'' such that $TV(S(L)) < \beta < \beta' < \beta'' \in \hat{\imath}$ where hoth $\beta, \beta'' \in \hat{\imath}$ but $\beta' \notin \hat{\imath}$. Therefore, there exists j of cardinality β'' where $|j| \geqslant TV(S(L))$ and j is not equivalent to an interpretation of cardinality β'.

Suppose no interpretation is periodic. Then $TV(S(L)) \leqslant \max(\delta, \sigma)$. Let $i \in I$ be such that $|i| > \max(\delta, \sigma)$. Hence $|i| > \delta$ and i is stable. Further $\delta_i < |i|$. Let β be any cardinal such that $\max(\delta, \sigma) \leqslant \beta \leqslant |i|$. If $\beta \geqslant \max(\delta, \sigma), \beta \geqslant \sigma_i$ and $\beta \in \hat{\imath}$. Q.E.D.

There are semantics which do not have Tarski-Vaught numbers. For example, no second order language with its standard semantics (and hence no standard semantics for languages in the hierarchy of the simple theory of types) has a Tarski-Vaught number. This result (announced in [26]) depends on showing that certain «periodic» features of the cardinal numbers (e.g. «being a limit cardinal») are definable by second order sentences; i.e. there is a sentence A true on all and only interpretations i, provided $|i|$ is a limit cardinal.

The standard semantics for countable first order languages with or without equality have Tarski-Vaught numbers where $TV(S(L)) = \max(\delta, \sigma)$. However, we can easily find semantics having Tarski-Vaught numbers where $TV(S(L)) < \max(\delta, \sigma)$. Let L_1 be that first order language having the binary relational constant R as its only non-logical constants. Let $S(L_1)$ indicate the standard semantics for L_1. Let P be a unary predicate constant, and let A be a sentence involving P but not R. Let $L* = L_1 \cup \{A\}$ and I' be all pairs $i' = \langle u, f' \rangle$ where $f'(P) \subseteq u$ and $f'(R) \subseteq u \times u$. V' is defined as usual for all $B \in L_1$, and $|f'(P)| = \tau_0$ or $|f'(P)| = \varkappa_1$. Let S(L*) denote the resulting se-
$|f'(P)| = \chi_0$ or $|f'(P)| = \chi_1$. Let S(L*) denote the resulting semantics. Obviously, $\delta = \varkappa_1$, and $\sigma = \varkappa_0$ but it can easily be shown that $TV(S(L*)) = \varkappa_0$. Note, however, that if $M(S(L)) = B(S(L))$ or $\delta \leqslant \sigma$ and S(L) has a Tarski-Vaught number, then $TV(S(L)) = \max(\delta, \sigma)$.

Theorem 2: For all S(L), (a) S(L) has an Upward Löwenheim-Skolem number provided no interpretation is periodic; and (b) if S(L) has an Upward Löwenheim-Skolem number, then U(S(L) $\leqslant$ max (δ^+, σ), where δ^+ is the successor of δ.
Proof: Suppose S(L) has an Upward Löwenheim-Skolem number, but some interpretation is periodic. Let i be such an interpretation. Since i is periodic, there exist β, β', such that $U(S(L)) < \beta < \beta'$ where $\beta \in \hat{\imath}$ and $\beta' \notin \hat{\imath}$: contradiction.

Suppose no interpretation is periodic. Let i be an interpretation such that $|i| \geqslant$ max (δ^+, σ). Then $|i| > \delta$ and i is stable. Let β be any cardinal $\geqslant |i|$; since $|i| \geqslant \delta_i$, $\beta \in \hat{\imath}$ and i is equivalent to some structure of cardinality β. Hence, S(L) has an Upward Löwenheim-Skolem number and this number is $\leqslant$ max (δ^+, σ). Q.E.D.

Theorems 1 and 2 yield that a semantics has an Uuward Löwenheim-Skolem number iff it has a Tarski-Vaught number. Hence, from the above remarks it follows that some semantics have no Upward Löwenheim-Skolem numbers.

Note that for the standard semantics for a first order language with or without equality, $U(S(L)) <$ max (δ^+, σ). However, for the semantics S(L*) discussed above we can verify that $U(S(L^*)) =$ max $(\delta^+, \sigma) =$ max $(\aleph_1, \aleph_0)$.

For the standard semantics for a countable first order language with equality we note that $TV(S(L)) = U(S(L))$; and for the standard semantics for a countable first order language without equality we note that $U(S(L)) < TV(S(L))$. Moreover, for the semantics S(L*) discussed above, it can be verified that $TV(S(L^*)) < U(S(L^*))$.

Theorem 3: For all S(L), (a) S(L) has a Löwenheim-Skolem number provided either (i) no interpretation for L is periodic: or (ii) there is a cardinal β satisfying the following: (1) $\beta \in \bigcap \{\hat{\imath}: i \in P(S(L))\}$; (2) $\beta > \sigma_i$; $i \in S(S(L))$: and (3). if $i \in B(S(L))$, $\beta < S_i$, then $\beta \in \hat{\imath}$; and (b) if S(L) has a Löwenheim-Skolem number, then (i), if $P(S(L)) = \wedge$, $LS(S(L)) \leqslant TV(S(L))$; and (ii) if $P(S(L)) \neq \wedge$, then LS(S(L)) is the least cardinal satisying conditions (1), (2) and (3) above.

Interestingly, there are semantics (e.g. the standard semantics for second order languages) which have been shown [27] to have no Löwenheim-Skolem numbers. Further, we can construct semantics which have Löwenheim-Skolem numbers but no Tarski-Vaught numbers. Let $S_1(L^*)$ consist of the language and interpretations of $S(L^*)$. The valuation V_1 is defined as usual for pairs $\langle B, i\rangle$ where $B \in L_1$ and $V_1(A, i) = t$, if $|i| > \aleph_0$ and $|i|$ is a limit cardinal; or if $|i| = \aleph_0$ and $f(P) = \wedge$; and $V_1(A, i) = f$, if $|i| < \aleph_0$; or if $|i| > \aleph_0$ and $|i|$ is not a limit cardinal; or if $|i| = \aleph_0$ and $f(P) \neq \wedge$. We can show that $S_1(L^*)$ contains periodic interpretations and hence has no Tarski-Vaught number. However, it can be verified that $LS(S_1(L^*)) = \aleph_0$.

Obviously, for the standard semantics for a countable first order language with equality, $U(S(L)) = TV(S(L)) = LS(S(L))$, and for the semantics for a countable first order language without equality, $U(S(L)) < LS(S(L))$. However, we can find semantics having Tarski-Vaught numbers where $LS(S(L)) < U(S(L))$. Let $S_2(L^*)$ be composed of the language and interpretations of $S(L^*)$. The valuation V_2 is defined as usual for all pairs $\langle B, i\rangle$ where $B \in L_1$ and $V_2(A, i) = t$ if $|i| = \aleph_1$ or $|i| = \aleph_0$ and $f(P) = \wedge$; and $V_2(A, i) = f$ if $|i| > \aleph_1$ or $|i| < \aleph_0$ or $|i| = \aleph_0$ and $f(P) = \wedge$. We an verify that $LS(S_2(L^*)) = \aleph_1 < TV(S_2(L^*)) = \aleph_2 = U(S_2(L^*))$.

Thus, not every semantics has a Löwenheim-Skolem number. However, a simple cardinality argument (c.f. [1] p. 85) establishes that every system has the following: for all β, *β is the Löwenheim number of S(L)* provided β is the least cardinal such that every interpretation is equivalent to an interpretation of cardinality β or less. ($L(S(L))$ indicates the Löwenheim number of $S(L)$).

Theorem 4: For all $S(L)$, (a) $S(L)$ has a Löwenheim number; and (b) $L(S(L)) = \text{l.u.b.}\{\min \hat{\imath}: i \in I\}$.

We can verify that if $S(L)$ has a Tarski-Vaught number then $L(S(L)) \leqslant LS(S(L)) \leqslant TV(S(L))$. Note that for the semantics for a countable first order language with or without equality, $L(S(L)) = LS(S(L)) = TV(S(L))$, but there are semantics having Tarski-Vaught numbers (e.g. $S_2(L^*)$ above) such that $L(S(L)) <$

TV(S(L)). Further, we can construct a semantics which has a Löwenheim-Skolem number where $L(S(L)) < LS(S(L))$. Let $S_3(L^*)$ contain the language and interpretations of $S(L^*)$. The valuation V_3 is defined in the usual way for all pairs $\langle B, i \rangle$, $B \neq A$. In addition we have the following: $V_3(A, i) = t$ if $|i| = \aleph_1$ and $V_3(A\ i) = f$ otherwise. We can show that $L(S_3(L^*)) = \aleph_2$ but $LS(S_3(L^*)) = \aleph_2$. Note further that $L(S_3(L^*)) < U(S_3(L^*))$ and for the standard semantics for a countable first order language without equality, $U(S(L)) < L(S(L))$, while for the standard semantics for a countable first order language with equality, $U(S(L)) = L(S(L))$.

Earlier we noticed that some semantics have no Upward Löwenheim-Skolem numbers; but a simple cardinality argument (see [1] p. 85) verifies that every system has the «weaker» Han f number.

For all β, β *is the Hanf number of S(L)* provided β is the least cardinal such that every interpretation of cardinality at least β is equivalent to some interpretation of greater cardinality. ($\beta = H(S(L))$).

Theorem 5: For all S(L), (a) S(L) has a Hanf number; and (b) $H(S(L)) = \min. \{\beta : \text{for all } i \in B(S(L), \beta > \delta_i\}$.

Notice that if S(L) has an Upward Löwenheim-Skolem number, then $H(S(L)) \leqslant U(S(L))$. Further, for the semantics for a countable first order language with or without equality $H(S(L)) = U(S(L))$. However, semantics can be constructed which have Upward Löwenheim-Skolem numbers strictly greater than their Hanf numbers. Let $S_4(L^*)$ be composed of the language and interpretations of $S(L^*)$ where $V_4(B, i)$ for $B \neq A$ is defined as usual and $V_4(A, i) = f$ if $|i| < \aleph_1$ or $|i| \geqslant \aleph_2$ and $f(P) \neq \wedge$, and $V_4(A, i) = t$ if $|i| = \aleph_1$ or $|i| \geqslant \aleph_2$ and $f(P) = \wedge$. It can be verified that $H(S_4(L^*)) = \aleph_0 < U(S_4(L^*)) = \aleph_1$.

3.0 In the last setion five candidates for a measure of the ability of a language to distinguish interpretations up to isomorphism were introduced. In this section we briefly discuss which of these candidates would make the best measure. For convenience we group these candidates into the *upward meas-*

ures, i.e. Hanf and Upward Löwenheim-Skolem numbers, and the *downward measures*, i.e. the Tarski-Vaught, Löwenheim-Skolem and Löwenheim numbers. Of the upward measures, the Hanf numbers seem the best candidate, since (i) every semantics has a Hanf number while some semantics have no Upward Löwenheim-Skolem numbers; and (ii) even for those systems having Upward Löwenheim-Skolem numbers $H(S(L)) \leqslant U(S(L))$. Similar remarks apply for the Löwenheim numbers when compared to the other downward measures.

Presumably, the choice between $H(S(L))$ and $L(S(L))$ is a function of their relative sizes, the principle being always to choose the smaller of the two. Notice, however, that their relative sizes are not known (in general) beyond computing both of them; i.e. for the standard semantics for a countable first order language with equality $H(S(L)) = L(S(L))$; however, for the standard semantics for a countable first order language without equality $H(S(L)) < L(S(L))$ and $H(S_2(L^*)) > L(S_2(L^*))$. Interestingly, under the assumption that every bounded interpretation in $S(L)$ is minimal, it can easily be shown that $H(S(L)) \leqslant L(S(L))$.

4.0 Leaving aside this problem, one might wonder whether any of these numebrs could, in general, serve as a necessary and sufficient condition for an interpretation to be determined uniquely up to isomorphism. Tarski ([22] p. 712) has noted that a structure can be characterized uniquely up to isomorphism in the standard semantics for a countable first order language with equality just in case it is finite, i.e., less than the Hanf number of that semantics. However, we can show that there are interpretations of cardinalities less than the Hanf and Löwenheim numbers of the standard semantics for countable second order language which cannot be characterized uniquely up to isomorphism.

Let $S(L)$ be a standard second order semantics whose language contains the binary relational constant R. Let β be any cardinal. β *is describable in S(L)* provided there is some sentence in L true on all and only interpretations of cardinality β. It can easily be shown that $\aleph_0$ is describable (c.f. [26]). It follows from a result of Zykov [28] that $2^{\aleph_0}$, $2^{2^{\aleph_0}}$ and $2^{2^{2^{\aleph_0}}}$ are describable; and hence that both the Hanf and Löwenheim numbers of S(L) are larger than $2^{2^{\aleph_0}}$. Further, we can verify that for every cardinal β which is describable in S(L), the interpretation $i = \langle u, f \rangle$ where $|i| = \beta$ and $f(R) = \wedge$ can be characterized uniquely up to isomorphism in S(L). It follows from a result of Tarski ([22] p. 713) that there are exactly $2^{\aleph_0}$ many non-equivalent interpretations in each infinite cardinality. But it can easily be shown that there are $2^{2^{\aleph_0}}$ non-isomorphic interpretations of cardinality $2^{2^{\aleph_0}}$.

Let u be any set of cardinality $2^{2^{\aleph_0}}$ and let $<$ be some well-ordering on u. For each $a \in u$, let u(a) denote the weak initial segment of $<$ determined by a; and let $i(a) = \langle u, f(a) \rangle$ where f(a) (R) is the restriction of $<$ to u(a). We can easily show that for all a, $b \in u$, if $a \neq b$ then i(a) is not isomorphic to i(b) and hence that there are $2^{2^{\aleph_0}}$ non-isomorphic interpretations of cardinality $2^{2^{\aleph_0}}$. (These arguments can be extended to the entire hierarchy of the elementary theory of types.)

George Weaver
Bryn Mawr College

REFERENCES

1. BELL, J. L. and A. B. SLOMSON: *Models and Ultraproducts*, North Holland Publishing Company (Amsterdam), 1969.
2. COCCHIARELLA, N. A.: «A logic of possible and actual objects», *J. S. L.*, vol. 31, (1966), pp. 688-689.
3. GÖDEL, Kurt: *The consistency of the Continuum Hypothesis: Annals of Mathematics Studies* # 3. Princeton University Press (Princeton), 1940.

Princeton University Press (Princeton), 1940.

4. HAILPERIN, T.: «Quantification theory and empty individual domains», *J. S. L.*, vol. 18, (1953) pp. 197-200.
5. HENKIN, Leon: «Mathematical Foundations for Mathematics», *American Mathematical Monthly*, vol. 18 (1971), pp. 463-87.
6. HINTIKKA, K. J. J.: «Existential presuppositions and existential commitments», *Journal of Phil.*, vol. 56 (1959), pp. 125-137.
7. HUNTINGTON, E. V.: «A Complete Set of Postulates for the Theory of Absolute Continuous Magnitude». *Trans. of the American Mathematical Soc.*, vol. 3 (1902) pp. 264-279.
8. HUNTINGTON, E. V.: «Complete Sets of Postulates for the Theories of Positivie Integral and Positive Rational Numbers», *Trans. of the American Mathematical Soc.*, vol. 3 (1907), pp. 280-284.
9. HUNTINGTON, E. V.: «Complete Sets of Postulates for the Theory of Real Quantities», *Trans. of the American Mathematical Soc.*, vol. 4, (1903) pp. 358-370.
10. HUNTINGTON, E. V.: «A Set of Postulates for Real Algebra», *Trans. of the American Mathematical Soc.*, vol. 6 (1905), pp. 17-41.
11. HUNTINGTON, E. V.: «The Fundamental Propositions of Algebra»: *Monographs on Topics of Modern Mathematics*, editor: J. W. A. Young, Longmans, Green and Company (New York), 1911.
12. KEISLER, H. J. and C. C. CHANG: *Continuous Model Theory: Annals of Mathematics Studies* # 58, Princeton University Press (Princeton), 1966.
13. KREISEL, G. and J. L. KRIVINE: *Elements of Mathematical Logic (Model Theory)*, North Holland Publishing Company (Amsterdam), 1967.
14. LAMBERT, K.: «Free Logic and the Concept of Existence», *Notre Dame Journal of Formal Logic*, vol. 8 (1967), pp. 133-144.
15. LEBLANC, H.: «A Simplified Account of Validity and Implication for Quantificational Logic», *J. S. L.*, vol. 33 (1968), pp. 231-235.
16. LEBLANC, H. and T. HAILPERIN: «Nondesignating Singular Terms», *Phil. Review*, vol. 68 (1959), pp. 129-136.
17. LEBLANC, H. and R. H. THOMASON: «Completeness theorems for Some Presupposition-free Logics», *Fund. Math.*, vol. 62 (1968), pp. 125-164.
18. LEBLANC, H. and R. K. MEYER: «Truth-value semantics for the theory of types»: *Philosophical Problems in Logic: Some Recent Developments:* ed. K. Lambert, D. Reidel Publishing Company (Dorbrecht-Holland), 1970.
19. LEBLANC, H. and G. WEAVER: «Truth-functionality and the ramified theory of types»: *Truth, Syntax and Modality: Proceedings of the 1970 Temple University Conference on Formal Semantics.* North Holland Publishing Company (Amsterdam) 1973, pp. 148-168.
20. MOSTOWSKI, A.: *Thirty Years of Foundational Studies*, Barnes and Noble (New York), 1966.
21. QUINE, W. V.: «Quantification and the empty domain», *J. S. L.*, vol. 19 (1954), pp. 172-177.

22. TARSKI, Alfred: «Some Notions and Methods on the Borderline of Algebra and Metamathematics»: *Proceedings of the International Congress of Mathematicians* 1950, vol. 1 (1951), pp. 202-220.
23. TARSKI, A. and R. VAUGHT: «Arithmatical extensions of relational systems», *Compositio Mathematica,* vol. 13 (1956) pp. 81-102.
24. VON FRASSEN, B. C.: «Singular terms, truth-value gaps and free logic», *Journal of Phil.* vol. 63 (1966), pp. 481-495.
25. VEBLEN, O.: «A system of axioms for geometry», *Trans. of the American Mathematical Soc.,* vol. 5 (1904), pp. 343-384.
26. WEAVER, G.: «Tarski-Vaught and Löwenheim-Skolem like results in higher order logics», (abstract) *J. S. L.* Vol. 36 (1971) pp. 108-9.
27. WEAVER, G.: «A note on compactness and decidability», *Logique et Analyse.* Vol. 14 (1973) pp. 315-316.
28. ZYKOV, A. A.: «The Spectrum problem in the extended predicate calculus», *American Mathematical Society Translations,* vol. 2 (1956), pp. 1-14.

NOTES

(1) $i = \langle u, f \rangle$ is a substructure of $j = \langle v, g \rangle$ provided $u \subseteq v$, f and g agree on all individual constants in K and f on any other member of K is the restriction of g on that member to u. Under the same conditions j *is an extension of i.*

(2) This notion of the spectrum of an interpratation is a generalization of the notion of the spectrum of a set of sentences (c.f. [28], p. 2).

(3) In [26] such numbers are called weak-Löwenheim-Skolem numbers.