A PARTIALLY TRUTH-FUNCTIONAL MODAL CALCULUS

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In any normal assertoric propositional calculus, it is easy to determine which formulae are valid. Since the operators of such a calculus are truth-functional, the valid formulae will be those which are truth table tautologies. We may also use this characteristic of assertoric calculi to judge various axiomatizations of such calculi. Ideally, the theses of any such axiomatization (i.e., its axioms and theorems) will consist of all and only those formulae which are truth table tautologies. It is generally held that such a procedure will not work when modal operators are introduced. The reason given for this is that modal operators are not truth-functional. A formal definition of validity for a calculus containing modal operators is normally given in terms of a «possible worlds» interpretation. I shall show that at least in one case, the «possible worlds» approach is not necessary, and that validity for a modal propositional calculus can be defined by means of what I call a quasi-truth-table (QTT) decision procedure. I have no particular bone to pick concerning the «possible worlds» approach, but I do think the QTT procedure presented in this paper, along with the associated modal calculus, is highly consistent with our intuitions about modality.

The modal system to be presented in this paper, and the QTT decision procedure, can neither be properly understood without first considering an overlooked truth about the concept of truth-functionality. It is customary to distinguish logical operators as either truth-functional or non-truth-functional. But it would be more precise to distinguish three categories of operators: 1) truth-functional, 2) non-truth-functional, and 3) partially truth-functional. The usual assertoric operators (∨, ⊃, etc.) obviously belong in the first category. Certain epistemic operators belong in the second category. There is, for example, no truth-functional relation between «today is Tuesday» and «Henry believes that today is Tuesday». Modal operators should be classified in the third category. As an example, □ p cannot be true if p is false. So while the relation of □ p to p is not completely truth-functional (since □ p is not a dependent variable of p), the truth value of □ p is not wholly independent of the truth value of p. The most significant thing about this proposed classification of operators is that partially truth-functional operators are amenable to a QTT decision procedure; an important consideration since the QTT method can be used to decide validity as satisfactorily and as easily as normal truth tables. This will become evident when the QTT procedure is presented and explained in section III. Section II is devoted to the presentation of a modal calculus, and in section IV the significance of the system is briefly discussed.

II

The axiomatized modal system presented below is constructed as an addition to an assertoric system. The idea of axiomatizing modal logic by adding to an assertoric system originated with Godel. The assertoric system I have chosen as a base is the system of *Principia Mathematica.* I call the system Nl, and its construction requires, in addition to its assertoric base, one each of the following: a primitive idea, a formation rule to deal with the new primitive, and an axiom. Two definitions are also introduced.

PRIMITIVE IDEA: □.
«□ p» will be read «it is necessary that p».

FORMATION RULE: «□ p» is a wff iff p is either a single propostional variable or the negation or the negation of the negation of a single propositional variable. The significance of this admittedly unusual rule is discussed in section IV.

AXIOM N 1.1: $\Box p \supset p.$

DEFINITION N 1.2: $\Diamond p = df \sim \Box \sim p.$
«$\Diamond$p» will be read «it is possible that p».

The addition of the modal components listed above to the system of *Principia* yields indefinitely many further theor ems, the most noteworthy of which are given below. The demonstrations, although elementary, are included to show how Nl makes use of its assertoric base. Starred numbers refer to the theorems of *Principia*.

THEOREM N 1.21: $\Diamond \sim p \equiv \sim \Box p.$

Demonstration:	1. $p \equiv p.$	*4.2.
	2. $\Diamond p \equiv \Diamond p.$	1, $\Diamond$p/p.
	3. $\Diamond p \equiv \sim \Box \sim p.$	2, N 1.2.
	4. $\Diamond \sim p \equiv \sim \Box \sim \sim p.$	3, $\sim$ p/p.
	5. $\Diamond \sim p \equiv \sim \Box p.$	4, *4.13.

THEOREM N 1.22: $\sim \Diamond \sim p \equiv \Box p.$

Demonstration:	1. $\Diamond \sim p \equiv \sim \Box p.$	N 1.21.
	2. $\sim \Diamond \sim p \equiv \sim \sim \Box p.$	1, *4.11.
	3. $\sim \Diamond \sim p \equiv \Box p.$	2, *4.13.

THEOREM N 1.23: $\sim \Diamond p \equiv \Box \sim p.$

Demonstration:	1. $\sim \Diamond \sim p \equiv \Box p.$	N 1.22.
	2. $\sim \Diamond \sim \sim p \equiv \Box \sim p.$	1, $\sim$ p/p.
	3. $\sim \Diamond p \equiv \Box \sim p.$	2, *4.13.

Theorems N 1.21, N 1.22, and N 1.23, along with definition N 1.2, are the four forms of what is usually called «modal operator exchange».

THEOREM N 1.3: $p \supset \Diamond p$.
This theorem is called *ab esse ad posse.*

Demonstration:		
1.	$\sim \Diamond p \equiv \Box \sim p$.	N 1.23.
2.	$(\sim \Diamond p \supset \Box \sim p) . (\Box \sim p \supset \sim \Diamond p)$.	1, *4.01.
3.	$\sim \Diamond p \supset \Box \sim p$.	2, *3.26.
4.	$\Box p \supset p$.	N 1.1.
5.	$\Box \sim p \supset \sim p$.	4, $\sim p/p$.
6.	$\sim \Diamond p \supset \sim p$.	3,5, *2.06.
7.	$p \supset \Diamond p$.	6, *2.17.

THEOREM N 1.4: $\Box p \supset \Diamond p$.

This theorem may be called *a necesse ad posse.*

Demonstration:		
1.	$\Box p \supset p$.	N 1.1.
2.	$p \supset \Diamond p$.	N 1.3.
3.	$\Box p \supset \Diamond p$.	1,2, *2.06.

THEOREM N 1.5: $\Box p \supset (\sim p \supset p)$.

Demonstration:		
1.	$p \supset (q \supset p)$.	*2.02.
2.	$p \supset (\sim p \supset p)$.	1, $\sim p/q$.
3.	$\Box p \supset p$.	N 1.1.
4.	$\Box p \supset (\sim p \supset p)$.	2,3, *2.06.

THEOREM N 1.51: $\sim \Diamond p \supset (p \supset \sim p)$.

Demonstration:		
1.	$\Box p \supset (\sim p \supset p)$.	N 1.5.
2.	$\Box \sim p \supset (\sim \sim p \supset \sim p)$.	1, $\sim p/p$.
3.	$\Box \sim p \supset (p \supset \sim p)$.	2, *4.13.
4.	$\sim \Diamond p \supset (p \supset \sim p)$.	3, N 1.23.

Theorems N 1.5 and N 1.51 are the modal forms of *conse-*

quentia mirabilis.

THEOREM N 1.6: $(\Box p \cdot (p \supset q)) \supset q.$

This is the modal form of *modus ponens.*

Demonstration:	1.	$p \supset ((p \supset q) \supset q).$	*2.27.
	2.	$\Box p \supset p.$	N 1.1.
	3.	$\Box p \supset ((p \supset q) \supset q).$	1,2, *2.06.
	4.	$(\Box p \cdot (p \supset q)) \supset q.$	3, *3.31.

THEOREM N 1.61: $(\sim \Diamond q \cdot (p \supset q)) \supset \sim p.$

Ths is the modal form of *modus tollens.*

Demonstration:	1.	$\sim q \supset ((p \vee q) \supset p).$	*2.56.
	2.	$\sim q \supset ((\sim p \vee q) \supset \sim p).$	1, $\sim p/p$.
	3.	$\sim q \supset ((p \supset q) \supset \sim p).$	2, *1.01.
	4.	$\Box p \supset p.$	N 1.1.
	5.	$\Box \sim q \supset \sim q.$	4, $\sim q/p$.
	6.	$\sim \Diamond q \supset \sim q.$	5, N 1.23.
	7.	$\sim \Diamond q \supset ((p \supset q) \supset \sim p).$	3,6, *2.06.
	8.	$(\sim \Diamond q \cdot (p \supset q)) \supset \sim p).$	7, *3.31.

It is interesting to note that N 1.61 will not hold if we substitute «$\sim \Diamond p$» for «$\sim p$» as the final consequent. In fact, N1 was designed to prevent such inferences from holding — a point which is discussed in part IV.

DEFINITION N 2.1: $\nabla p =$ df $(\sim \Box p \cdot \Diamond p).$

«∇p» will be read «it is contingent that p».

THEOREM N 2.2: $\nabla p \equiv \nabla \sim p.$

I call N 2.2 «complementarity of contingency». If any proposition is contingent, then its negation is contingent.

Demonstration:
1. $p \equiv p.$ — *4.2.
2. $(\sim \Box p . \Diamond p) \equiv (\sim \Box p . \Diamond p).$ — 1, $\sim \Box p . \Diamond p/p.$
3. $\nabla p \equiv (\sim \Box p . \Diamond p).$ — 2, N 2.1.
4. $\nabla \sim p \equiv (\sim \Box \sim p . \Diamond \sim p).$ — 3, $\sim p/p.$
5. $\nabla \sim p \equiv (\Diamond p . \Diamond \sim p).$ — 4, N 1.2.
6. $\nabla \sim p \equiv (\Diamond p . \sim \Box p).$ — 5, N 1.21.
7. $\nabla \sim p \equiv (\sim \Box p . \Diamond p).$ — 6, *4.3.
8. $\nabla \sim p \equiv \nabla p.$ — 7, N 2.1.
9. $\nabla p \equiv \nabla \sim p.$ — 8, *4.21.

THEOREM N 2.3: $(\Box p \vee \sim \Diamond p) \equiv \sim \nabla p.$

Theorem N 2.3 is called «modal trichotomy». The reason for that label is obvious.

Demonstration:
1. $\Box p \equiv \sim \Diamond \sim p.$ — N 1.22.
2. $\sim \Diamond p \equiv \Box \sim p.$ — N 1.23.
3. $\Box p \vee \sim \Diamond p \equiv \sim \Diamond \sim p \vee \Box \sim p$ — 1,2, *4.39.
4. $\Box p \vee \sim \Diamond p \equiv \sim (\Diamond \sim p . \sim \Box \sim p).$ — 3, *4.53.
5. $\Box p \vee \sim \Diamond p \equiv \sim (\sim \Box p . \sim \Box \sim p).$ — 4, N 1.21.
6. $\Box p \vee \sim \Diamond p \equiv \sim (\sim \Box p . \Diamond p).$ — 5, N 1.2.
7. $(\Box p \vee \sim \Diamond p) \equiv \sim \nabla p.$ — 6, N 2.1.

The theorems demonstrated above are sufficient to show the general character of the system N1.

III

The QTT decision procedure presented in this section was designed to reflect the idea that validity is truth-preserving. In other words, only those expressions which are true under every consistent assignment of truth values to their components will count as valid. In order to do this, it was necessary to construct the QTT procedure in such a way that all normal assertoric tautologies remain so. In addition, all of the theorems of section II are QTT tautologies. In fact, it is easy to see that the property of being a QTT tautology is hereditary with respect to the system N1. So what we really have is a modified truth table which, while preserving the status of ordinary assertoric tautologies, also shows which modal expressions are to be counted as tautologies.

We begin by noting that the heuristic model of modality we have in mind is such that a false proposition cannot be necessary, whereas a true propostion may or may not be necessary. In order to accomodate this fact, the normal truth table column for a single propositional variable is simply doubled:

p
T
F
T
F

□ p, our basic modal unit, can now be assigned the column:

p	□ p
T	F
F	F
T	T
F	F

The rule for generating the column for □ p can be given a strictly mechanical formulation; namely, when p is false, □ p is false; in the first half of the rows where p is true, □ p is false; and in the second half of the rows where p is true, □ p is true.

A glance at the columns for p and □ p shows that the relation between the two is not truth-functional. Where p (the independent variable) is true, □ p (the dependent variable) may be either true or false. What is more important, however, is the fact that this variance is governed by a rule. Furthermore, the rows of the table represent every assignment of values consistent with our heuristic model. For the necessity operator alone we need only one row where p is false, but two such rows have been included to handle the case where $\sim p$ is necessary (i.e., where p is impossible). That case is illustrated in the table below:

p	$\sim p$	$\Box p$	$\Box \sim p$
T	F	F	F
F	T	F	F
T	F	T	F
F	T	F	T

It should be observed that the same rule was used for generating the column for $\Box \sim p$ as was used for $\Box p$. When $\sim p$ is false, $\Box \sim p$ is false. In the first half of the rows where $\sim p$ is true, $\Box \sim p$ is false. In the second half of the rows where $\sim p$ is true, $\Box \sim p$ is true also. The normal truth-functional operators are treated in the usual manner. Thus, the column for $\sim p$ is the opposite of the column for p.

To obtain the column for $\Diamond p$ we simply negate the column for $\Box \sim p$ since $\Diamond p =\mathrm{df} \sim \Box \sim p$. The column for «it is possible that p» is:

$\Box \sim p$	$\sim \Box \sim p$	$\Diamond p$
F	T	T
F	T	T
F	T	T
T	F	F

Using the normal procedures for . and $\vee$, $((\Box \sim p) \,.\, (\Diamond p))$ is a contradiction, and $(\Diamond p \vee \Box \sim p)$ is a tautology. These are, of course, the results we would expect (or at least hope for).

The column for ∇p is also derived from its definition:

$\sim \Box p$	$\Diamond p$	$\sim \Box p \,.\, \Diamond p$	∇p
T	T	T	T
T	T	T	T
F	T	F	F
T	F	F	F

The column for $\nabla \sim p$ comes out to be identical to the column for ∇p. This again is as it should be.

A QTT table for two variables requires sixteen rows. This is necessary so that p can vary independently of the modal configurations of q and *vice versa*. The rules used for generating the columns for the modal configurations of q are the same as those used in the case of p. Thus, $\Box q$ is false where q is false, false in the first half of the rows where q is true, and true in the second half of the rows where q is true. The table below summarizes tables for two variables:

p	$\Box p$	$\sim \Diamond p$	$\Diamond p$	∇p	q	$\Box q$	$\sim \Diamond q$	$\Diamond q$	∇q
T	F	F	T	T	T	F	F	T	T
F	F	F	T	T	T	F	F	T	T
T	T	F	T	F	T	F	F	T	T
F	F	T	F	F	T	F	F	T	T
T	F	F	T	T	T	T	F	T	F
F	F	F	T	T	T	T	F	T	F
T	T	F	T	F	T	T	F	T	F
F	F	T	F	F	T	T	F	T	F
T	F	F	T	T	F	F	F	T	T
F	F	F	T	T	F	F	F	T	T
T	T	F	T	F	F	F	F	T	T
F	F	T	F	F	F	F	F	T	T
T	F	F	T	T	F	F	T	F	F
F	F	F	T	T	F	F	T	F	F
T	T	F	T	F	F	F	T	F	F
F	F	T	F	F	F	F	T	F	F

IV

The motivation behind the construction of the system N1 was a desire to have a modal logic which is interpretable in such a way that the following intuitively plausible conditions are met:

1. All normal assertoric tautologies should be valid.
2. Validity should be truth-preserving.
3. Theophrastus' rule should hold.
4. Trichotomy should hold between necessity, impossibility, and contingency.
5. The negation of any contingent proposition should be itself contingent.
6. The necessity operator should be interpretable in such a way that it is not superfluous.

Let us examine each of these conditions in turn.

The first condition is guaranteed by constructing N1 on the base of a system which is adequate to the demonstration of all assertoric tautologies. The second condition is insured by constructing N1 in such a way that the property of being a QTT tautology is hereditary within the system.

Theophrastus' rule was to the effect that the conclusion of any modal argument cannot have a modality «stronger» than the modality of the «weakest» premiss. Most modal systems do not preserve this rule. In S2, for instance, theorem 18.53 reads:

$$((p \prec q) \cdot \Box p) \prec \Box q.$$

If we break 18.53 into an argument, the premiss which contains q is assertoric. But the conclusion has q as necessary. It is certainly not my intention to argue that S2 is somehow faulty because it allows inferences of this sort. But it is interesting that we can construct a system such as N1 which does preserve Theophrastus' rule. Incidentally, the N1 analogue of 18.53 (which is the same expression with horseshoes substituted for the horns) does not hold; it is not a QTT tautology, and if asserted leads to a contradiction.

Conditions 4 and 5 are shown by the fact that the appropriate theorems are demonstrated (N 2.2 and N 2.3).

The final condition needs some explanation. Every system of logic contains an implicit necessity operator in the sense that every asserted expression is necessary (i.e., follows necessarily from the axioms). The introduction of an explicit necessity operator which does nothing more than reiterate this implicit necessity would seem to be somewhat superfluous. It is intuitively obvious, for instance, that placing a square in front of the expression $(p \vee \sim p)$ hardly adds anything of significance to the expression. By restricting the scope of the modal operators in N1 to propositional variables and their negations, such a situation is avoided. In effect, the necessary propositions of N1 are «atomic» with respect to the system; they are unanalyzed in terms of the structure of the system. We are, of course, free to think of $\Box p$ in terms of the same logical necessity implicit in the system, but we need not do so. Nor is such an interpretation the most obvious one. In fact, two very interesting interpretations which are immediately suggested are that the necessary propositions of N1 are either semantically analytic or synthetically necessary. If we choose either of these interpretations, then the necessity of the necessity operator is not superfluous. I am not, however, prepared to argue for either of these interpretations since I have philosophical reservations about both sorts of propositions.

The question «Which system of modal logic is the correct one ?» seems to be misguided since the different systems are not really in competition. N1 is a system which meets the conditions specified above. But there is no reason to accept those conditions as definitive of a «correct» modal system. N1 is simply one interesting system of many which are possible.

I should like to add one final observation. Since the time of Lewis' work there has been a tendency to identify modal logic and the logic of strict implication. But modal logic need not include any particular sort of implication. N1 is content with material implication. In fact, it must be since to introduce strict implication would necessitate allowing the scope of the necessity operator to extend over compound expressions. It may be true that any system which includes strict implication must be a modal system, but the converse is certainly not true.