ON THE ADDITION OF CLASSES TO SET THEORIES

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The process of adding classes to a set theory is familiar to most logicians and a deailed account of it can be found in [3]. If T is a set theory, then by the *theory of classes over T* we mean the theory that is obtained from T by making appropriate technical changes to add proper classes and by taking $\varnothing^{\vee}$ (i.e. $\varnothing$ relativised to $\exists X\ Y \in X$) as an axiom whenever $\varnothing$ is an axiom of T. Such theories are usually strengthened in two ways:

(i) by the addition of the impredicative, or the predicative, class existence schema, and
(ii) by replacing the axiom schema of T by axioms, in a natural way.

Details of these procedures can be found in [3].

Using the notation of [3], $T^{\cup}$ ($T^{þ}$) is the theory of classes over the set theory T in which the impredicative (predicative) class existence schema is added and all the axiom schema of T are replaced by axioms. Then two of Schock's results (in [3]) show that $T^{þ}$ is equiconsistent with T and if $ZF \subseteq T$, then $T^{\cup}$ is strictly stronger than T.

$T^{\cup}$ and $T^{þ}$ do not exhaust the ways of getting a theory of classes over a set theory T, and the purpose of this note is to give some examples where the theory of classes over T that contains the impredicative class existence schema and has all but one of T's axiom schemas replaced by axioms (in a natural way, of course) is a conservative extension of T. Theorem 1 gives this result for T = ZF and this shows, perhaps somewhat surprisingly, that the extra strength of ZF^{U} is not only due to the presence of the impredicative class existence schema. Then theorem 2 gives an analogous result for T = ZM, where ZM is obtained by adding the following schema to ZF (see [1]).

(M) If $\varnothing$ represents a normal function, then $\varnothing$ has an inaccessible fixed point.

Theorem 1. The theory of classes over ZF that has the impredicative class existence schema and all the axiom schema of ZF, except replacement, replaced by axioms is a conservative extension of ZF.

Proof. It suffices to show that whenever $\varnothing$ is a sentence in the language of ZF and $\varnothing^{V}$ is provable in the theory described in the theorem, then ZF $\vdash \varnothing$.

We will make use of Ackermann's set theory, A*, and we refer the reader to [2] for details of this theory. Firstly, we note that when the axioms of the theory of classes described above are relativised to $\mathscr{P}(V)$, the collection of all subclasses of V, they become provable in the theory A*. This follows from axiom A2 of A* (see [2]) and theorems 1.5 and 3.1 of [2]. Now, if $\varnothing^{V}$ is provable in the theory of classes defined above and $\varnothing$ is a sentence in the language of ZF, then A* $\vdash \varnothing^{V}$. Theorem 3.1 of [2] then gives ZF $\vdash \varnothing$, as required. □

Theorem 2. The theory of classes over ZM that has the impredicative class existence schema and all the axiom schema of ZM, except (M), replaced by axioms is an conservative extension of ZM.

Proof. The proof is very similar to that given for theorem 1, except that A^*_0 (see [2]) is used instead of A*, and theorem 5.2 of [2] is used instead of theorem 3.1. □

It is straightforward to check that the method of proof used for theorems 1 and 2 also can be used to obtain analogous results for all the set theories ZM that are described in [1].

This suggests the following open problem.
Question 3. Is it possible to prove a result (similar to those given in [3]) which generalises all our examples ?

REFERENCES

[1] A. Levy, Axiom schemata of strong infinity in axiomatic set theory, Pacific Journal of Maths, 10 (1960) 223-238.

[2] W. Reinhardt, Ackermann's set theory equals ZF, Annals of Mathematical Logic, 2 (1970) 189-249.

[3] R. Schock, On standard models of set theory, Logique et Analyse, 63 (1973) 413-433.

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