THE MAKINSON COMPLETENESS OF TENSE LOGIC

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In this paper we offer an alternative to Cocchiarella's semantic tableaux proof of the completeness of tense logic. Our results, which are adapted from a proof of D. Makinson for modal logic, not only are somewhat stronger than Cocchiarella's but are simpler as well ([1]).

1. Syntax and Semantics

The primitive signs of TL (tense logic) shall be '$\sim$', '$\supset$', '(',')', the tense operators 'F' and 'P', and a denumerable infinity of sentence letters, say 'p', 'q', 'r', 'p'', 'q'', etc. The wffs of TL are the sentence letters plus all formulas (i.e. strings of signs) of the following four sorts:

(i) $\sim$A, where A is a wff,

(ii) (A$\supset$B), where A and B are wffs,

(iii) FA, where A is a wff, and

(iv) PA, where A is a wff.

Henceforth we shall use 'A', 'B', and 'C' to refer exclusively to wffs of TL, 'S' to refer to sets of wffs of TL, 'G' as short for '$\sim$F$\sim$', 'H' as short for '$\sim$P$\sim$', '(A & B)' and '(A$\vee$B)' as short for '$\sim$(A$\supset\sim$B)' and '($\sim$A$\supset$B)', respectively, and we shall omit all sundry parentheses.

Five sublanguages of TL-called TL^1, TL^2, TL^3, TL^4, and TL^5 — shall be characterized below which are equivalent to five of the best known tense calculi ([2]). A1-A3 and B1-B8 reading as follows, the axiom schemata for each TL^i ($1 \leqslant i \leqslant 5$) are given in the accompanying table.

A1. $A \supset (B \supset A)$

A2. $(\sim B \supset \sim A) \supset (A \supset B)$

A3. $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$

B1. $G(A \supset B) \supset (GA \supset GB)$

B2. $PGA \supset A$

B3. GA, where A is an axiom.

B4. MI(A) [3], where A is an axiom and MI(A) is the result of simultaneously replacing each occurrence of 'F' in A by 'P' and each occurrence of 'P' by 'F'.

B5. $GA \supset GGA$

B6. $(FA \,\&\, FB) \supset (F(A \,\&\, B) \vee (F(A \,\&\, FB) \vee F(B \,\&\, FA)))$

B7. $GA \supset FA$

B8. $GGA \supset GA$

TABLE OF AXIOM SCHEMATA [4]

TL^1: A1-A3 and B1-B4

TL^2: A1-A3 and B1-B5

TL^3: A1-A3 and B1-B6

TL^4: A1-A3 and B1-B7

TL^5: A1-A3 and B1-B8

In addition, each TL^i has *modus ponens* as a rule of inference.

Presuming the notions of a proof and a derivation in TL^i to be understood, we shall say that a wff A is *provable* in TL^i — $\vdash_i$ A, for short — if there is a proof of A in TL^i and that a wff A is *derivable* from a set S in TL^i — S $\vdash_i$ A, for short — if there is a derivation of A from S in TL^i. Furthermore, if '$\sim(p \supset p)$' is derivable from S in TL^i, we shall say that S is *syntactically inconsistent* in TL^i and, otherwise, *syntactically consistent* in TL^i.

By a *truth-value assignment* for TL we understand any function from the set of sentence letters of TL to $\{1, 0\}$, where 1 is the truth-value "true" and 0 the truth-value "false". Mimicking Cocchiarella (4), we shall take any pair of the sort $<E,R>$ to be a *history* of TL if E is a family of truth-value assignements for TL whose index set is I and R is a dyadic relation on I. Where a_n is a term of E, we shall say that a_n is a *moment* of the history $<E,R>$. The histories of the sublanguages TL^i are to be distinguished by the properties of R. P1-P4 being as below, the relation R in the histories of TL^1 is unrestricted, R in the histories of TL^2 has P1, R in the histories of TL^3 has P1 and P2, R in the histories of TL^4 has P1, P2, and P3, and R in the histories of TL^5 has all of P1-P4.

P1. $(\forall x)(\forall y)(\forall z)((R(x,y) \& R(y,z)) \supset R(x,z)$

P2. $((\forall x)(\forall y)(\forall z)((R(x,y) \& R(x,z)) \supset ((y = z) \vee (R(y,z) \vee R(z,y))))$ and $(\forall x)(\forall y)(\forall z)((\breve{R}(x,y) \& \breve{R}(x,z)) \supset ((y = z) \vee (\breve{R}(y,z) \vee \breve{R}(z,y))))$ (5)

P3. $(\forall x)(\exists y) R(x,y)$ and $(\forall x)(\exists y) \breve{R}(x,y)$

P4. $(\forall x)(\forall y)(R(x,y) \supset (\exists z)(R(x,z) \& R(z,y)))$

We shall take a wff A to be *true* at a moment a_n of a history $<E, R>$ of TL^i if:

(i) in case A is a sentence letter, $a_n(A) = 1$,

(ii) in case A is a negation $\sim$B, B is not true at a_n,
(iii) in case A is a conditional B$\supset$C, either B is not true at a_n or C is,
(iv) in case A is of the sort FB, B is true at some b_p of $\langle E, R\rangle$ such that R(n,p), and
(v) in case A is of the sort PB, B is true at some b_p of
(v) $\langle E, R\rangle$ such that $\breve{R}$(n,p).

Finally, a wff A shall be said to be *valid* in TL^i if A is true at every moment of every history of TL^i; a set S shall be said to be *semantically consistent* in TL^i if there is a moment of a history of TL^i at which all the members of S are true, otherwise S shall be said to be *semantically inconsistent* in TL^i; and S shall be said to *entail* A in TL^i if S U{ $\sim$A} is semantically inconsistent in TL^i.

2. Soundness and Completeness Theorems

The proof of our (strong) soundness theorem for TL is similar in all respects to the modal case and shall be left to the reader ([6]).

Theorem 1. If $S \vdash_i A$, then S entails A in TL^i.

Turning then to matters of completeness, our first prefatory lemma is the tense logic analogue of "Lindenbaum's Lemma".

Lemma 1. If S is syntactically consistent in TL^i there is a set K such that:

(a) S$\subset$K,
(b) K is syntactically consistent in TL^i,
(c) For any wff A, $\sim$A is a member of K iff A is not a member of K,
(d) For any wffs A and B, A$\supset$B is a member of K iff either A is not a member or B is,
(e) For any wff A not a member of K, K U { $\sim$A} is syntactically inconsistent in TL^i.

Proof: Let S_0 be S and define S_n, for each n from 1 on, as follows: A_n being the alphabetically n-th wff of TL^i, let S_n be S_{n+1} U {A_n} if syntactically consistent in TL^i, and otherwise let S_n be S_{n+1}. Then let K be the union of S_0, S_1, S_2, etc. By arguments now familiar from the literature it is easily verified that (a) — (e).

We shall henceforth refer to sets such as K by the name *Lindenbaum sets* of TL^i and, where S and K are as above, shall call K the *Lindenbaum extension* of S in TL^i.

Where K is a Lindenbaum set of TL^i S_{B_1} is the set consisting of all wffs B such that GB is in K, S_{B_2} is the set consisting of all wffs B such that HB is in K, and FA and PA' are members of K, we shall call the Lindenbaum extension of $S_{B_1} \cup \{A\}$ a *future attendant* of K and the Lindenbaum extension of $S_{B_2} \cup \{A'\}$ a *past attendant* of K. Note that there is a future (past) attendant of K for each wff of the sort FA (PA') in K.

With these definitions in hand, we pass to our next lemma.

Lemma 2. If K, S_{B_1}, S_{B_2}, FA, and PA' are as above, then:

(a) $S_{B_1} \cup \{A\}$ is syntactically consistent in TL^i, and

(b) $S_{B_2} \cup \{A'\}$ is syntactically consistent in TL^i.

Proof: (a) Suppose, for a reductio, that $S_{B_1} \cup \{A\}$ is syntactically *in*consistent in TL^i and let $\{B_1, B_2, \ldots, B_k\}$ be a finite subset of S_{B_i} such that $\{B_1, B_2, \ldots, B_k\} \vdash_i \sim A$. Then $\vdash_i B_1 \supset (B_2 \supset \ldots (B_k \supset \sim A) \ldots)$ by A1-A3, and $\vdash_i GB_1 \supset (GB_2 \supset \ldots (GB_k \supset G\sim A) \ldots)$ by A1-A3 and B1 and B3. Hence, by A1-A3 again, $\{GB_1, GB_2, \ldots, GB_k\} \vdash_i G \sim A$. But, by hypothesis, $\{GB_1, GB_2, \ldots, GB_k\}$ is a subset of K. Hence, $K \vdash_i G\sim A$ (= $\sim$FA). However, by hypothesis, $K \vdash_i FA$ (since FA was assumed a member of K). Therefore, $S_{B_1} \cup \{FA\}$ is syntactically consistent in TL^i (7).

(b) By (a) and B4.

Hence, by our first two lemmata,

Lemma 3. If K is a Lindenbaum set of TL^i, then the future and past attendants of K are Lindenbaum sets of TL^i.

As in the classical case, corresponding to each Lindenbaum set K there is a truth-value assignment which assigns 1 to all the sentence letters in K and 0 to all others. Henceforth we shall call this truth-value assignment the *associated truth-value assignment* to K.

Beginning with a syntactically consistent set S we next construct a family E_L of Lindenbaum sets of TL^i. E_L is to be the least family containing K_n — the Lindenbaum extension of S — all of its future and past attendants, all of their future and past attendants, etc. The index set of E_L shall be I ([8]). On I a dyadic relation R is defined as follows:

For any two terms K'_i and K''_j of E_L, if A is a member of K''_j for every wff of the sort GA in K'_i, then R(i,j).

The pair $<E_L, R>$ shall be called an L-history of TL^i.

It is crucial for our forthcoming completeness theorem that the relation R in the L histories of TL^i, constructed as above, have the properties appropriate to the specific system. This is established by the following lemma.

Lemma 4. (a) In the L-histories of TL^2, R has P1.
(b) In the L-histories of TL^3, R has P1 and P2.
(c) In the L-histories of TL^4, R has P1, P2, and P3.
(d) In the L-histories of TL^5, R has P1-P4.

Proof: The arguments verifying (a) - (d) are similar to the modal case and are by now familiar from the literature. We give the proof that TL^3 histories have P2 as an example.

Since B6 is an axiom of TL^3 it is a member of each term of the L-history $<E_L, R>$ of TL^3. Suppose then that K_n is a term of E_L and that there are two other (not necessarily distinct) terms K'_p and K''_r of E_L such that R(n,p)and R(n,r). Furthermore, let A belong to K'_p and B belong to K''_r. Then if F(A & B) is a member of K_n, p = r; if F(A & FB) is a member of K_n, R(p,r); and if F(FA & B) is a member of K_n, R(r,p). By the same reasoning and axiom B4, if $\check{R}$(n,p) and $\check{R}$(n,r) then ne of p = r, $\check{R}$(p,r), and $\check{R}$(r,p) will hold. Hence R in the L-histories of TL^3 has P2.

A *history* of TL^i can be constructed to parallel the L-history $<E_L, R>$ of TL^i by forming the family E of all the associated truth-value assignments to the terms of E_L and using the same index for the associated truth-value assignment as was used for the term of E_L. Carrying the relation R over, we shall say that $<E, R>$ corresponds to $<E_L, R>$, if it is formulated as above. And this brings us to our crucial lemma.

Lemma 5. Where S is a set of TL^i, if S is syntactically consistent in TL^i, then S is semantically consistent in TL^i.

Proof: If S is syntactically consistent in TL^i, then S extends by Lemma 1 to the Lindenbaum set K. Out of K the L-history $<E_L, R>$ can be constructed as described above, and corresponding to $<E_L, R>$ is a history $<E, R>$ of TL^i. By mathematical induction on the length of a wff A, it is easily established that A is a member of a term K'_p of E_L if and only if A is true at the moment a_p of $<E, R>$ (where a_p is the associated truth-value assignment to K'_p), and hence that K'_p is semantically consistent in TL^i. Thus, since S is a subset of a term K_n of $<E_L, R>$, and all the terms of $<E_L, R>$ are semantically consistent in TL^i, S is semantically consistent in TL^i as well. *Base Case.* Let A be a sentence letter of TL^i. Then by the definition of a_p, A is true at a_p if and only if A is a member of K'_p. *Inductive Case.* Suppose, for every wff A' shorter than A, that A' is true on a_p if and only if A is a member of K'_p. Then, by the standard arguments, if A is either of the sort $\sim B$ or $B \supset C$, A is true at a_p if and only if A is a member of K'_p. Or, suppose A is of the sort FB and is a member of K'_p. Then there is a set K''_r such that B is a member of K''_r and R(p,r). Hence by the hypothesis of the induction, B is true at a'_r and FB is true at a_p, since R(p,r). On the other hand, suppose FB is true at a_p. Then B is true at some a'_r such that R(p,r). Again by the hypothesis of the induction it follows that B is a member of K''_r, and in view of the fact that R(p,r), FB is a member of K'_p Hence, if A is of the sort FB, A is true at a_p if and only if A is a member of K'_p. And the case where A is of the sort PB is similar. Consequently, K'_p is semantically consistent in TL^i as are all the terms of E_L. Hence S is semantically consistent in TL^i (9).

Thus our strong completeness theorem for tense logic is now at hand.

Theorem 2. If S entails A in TL^i, then $S \vdash_i A$.

Proof: Suppose S entails A in TL^i. Then $S \cup \{\sim A\}$ is semantically inconsistent in TL^i. Hence by the contrapositive of Lemma 5, $S \cup \{\sim A\}$ is syntactically inonsistent in TL^i, and, hence $S \vdash_i A$. (10, 11).

NOTES

(1) Cocchiarella's proof is given in full detail for the quantificational case in N. B. Cocchiarella, TENSE LOGIC: A STUDY IN THE TOPOLOGY OF TEMPORAL REFERENCE (Ph. D. Thesis, University of California at Los Angeles, 1965). Also see his abstract "A Completeness Theorem for Tense Logic," in THE JOURNAL OF SYMBOLIC LOGIC, vol. 31 (1966), pp. 689-690. The Makinson proof is found in D. Makinson "On Some Completeness Theorems in Modal Logic," ZEITSCHRIFT FUR MATHEMATISCHE LOGIK UND GRUNDLAGEN DER MATHEMATIK, Band 12 (1966), pp. 379-384.

(2) To be precise, TL^1 is equivalent to K_t, TL^2 to CR, TL^3 to CL, TL^4 to CS, and TL^5 to GH1. For additional information on the origin of these calculi and alternative axiomatizations see M. K. Rennie, "Postulates for Temporal Order," THE MONIST, vol. 53 (July 1969), pp. 457-459 and A. N. Prior, PAST, PRESENT, AND FUTURE (Oxford, 1967), Appendix A.

(3) Schemata B3 and B4 replace the more customary rules of inference RG:

$$\text{If} \vdash A, \text{ then } \vdash GA$$

and RMI:

$$\text{If} \vdash A, \text{ then } \vdash MI(A)$$

Including these principles among the axiom schemata makes for vastly simpler proofs of some of the meta-theory of TL, especially Soundness and the Deduction Theorem (on this, also see footnote 10, below).

(4) Cocchiarella, *op. cit.*

(5) By $\check{R}$, we mean the converse of R.

(6) For the full proof, see Robert P. McArthur, TENSE LOGIC, Chapter 5 (forthcoming).

(7) This proof owes some to Hugues Leblanc.

(8) See McArthur, *loc. cit.*, for the details of this indexing.

(9) Some of the minor details of this proof have been left out for the sake of brevity. For the complete proof see McArthur, *loc. cit.*

(10) A comprehensive account of the move from "$S \cup \{\sim A\}$ is syntactically inconsistent in TL^i" to "$S \vdash_i A$" requires (among other things) the Deduction Theorem, i.e., If $S \cup \{A\} \vdash B$, then $S \vdash A \supset B$. Since the proof of this theorem for TL is similar in all respects to the classical case (in light of axiom schemata B3 and B4 in place of extra rules of inference) it has been omitted.

(11) I am indebted to a reader for LOGIQUE ET ANALYSE for several helpful criticisms and suggestions on the penultimate draft of this paper.