LOGICAL COMPLETENESS OF DIRECTED RESOLUTION

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1. Introduction

Resolution was introduced originally as a proving theorem technique operating on sets of clauses. Robinson showed that given a formula A of first order logic it is possible to construct a set S of clauses such that A is logically valid if and only if the empty clause can be generated from S using resolution as a derivation rule. Several modifications of the original method were proposed by Robinson himself and others.

We present in this paper an extension of resolution that also operates on set of clauses but not to obtain the empty clause but rather to generate a clause that weakly subsumes a given fixed clause E. The notion of weak subsumption is decidable and will be defined later.

The interest of this technique is that the clause E itself can be used to direct the order in which resolutions are carried out during the computation. More precisely the given clause E allows to define preferences that operate as a filter on the set of resolutions that can be performed at any level of the computation. We call the resulting technique E-directed resolution.

Directed resolution is a generalization of both hyper-resolution and consequence-finding resolution (see references [2] and [4]). This will be discussed in the conclusion of the paper.

2. Directed resolution

In this section we shall describe the directed resolution method. The completeness of the procedure is proved in the next section.

We shall assume familiarity with the notation and results of [1]. The set union of the clauses C and D is denoted (C,D). When this notation is used it is assumed that C and D are disjoint. A clause containing only one literal is called a *unit clause*. Most of the time we identify such clause with its unique literal.

If C is a clause and s is some substitution then Cs is called an *instance* of C. In case s replaces variable by variable in a one-to-one manner we say that Cs is a *variant* of C. It follows in this case that C is also a variant of Cs.

A clause C is a *weak instance* of a clause D in case every literal in C is an instance of some literal in D. If there is some substitution s such that Cs is a weak instance of D we say that *C weakly subsumes D.*

The complement of a literal A is denoted by $\overline{A}$. Two clauses C and D are *potentially complemented* in case there is a substitution s such that Cs and Ds are complemented literals.

A *valuation* V is a function such that for every literal A the value V(A) is a truth value true (T) or false (F), and V(A) is different from $V(\overline{A})$. If C is a clause and V is a valuation the *V-residual* of C is the subclause of C (possibly empty) containing exactly those literals which are given value F by the valuation V.

If S is a set of clauses and A is a literal in some clause of S we say that A is an *S-literal*. A *preference* for a set S of clauses is a valuation V such that whenever B is an instance of an S-literal A then V(B) = V(A).

We are now in position to define directed resolution. Let S be a set of clauses, V a preference for S and E some fixed clause. To apply E-directed resolution with preference V to the clauses C and D we require:

i) There is a variant (C_1,C_2) of C and a variant (D_1,D_2) of D such that C_1 and D_1 are potentially complemented; and let s be a most general substitution such that C_1s and D_1s are complemented literals.
ii) $V(C_1s) = F$ and the V-residual of D_2 weakly subsumes E.

In case these two conditions are satisfied the result of applying E-directed resolution with preference V to the clauses C and D is the clause $(C_2,D_2)s$. (We do not require here that C_2 and D_2 be disjoint).

We note that this rule is a form of ordinary resolution as defined in [1], with some special requirements depending on the preference V and the clause E. In the case the clause E is empty and V is the valuation that gives value T to all unnegated literals we have the rule which is called P_1-resolution in [2].

We intend to apply the rule in the following way. Given a set S of clauses and a clause E we select some preference for S and then we try to generate by E-directed resolution with preference V some clause C that weakly subsumes the clause E. In the next section we discuss the conditions S and E must satisfy in order that such derivation exists for any preference V.

3. Completeness

What we need obviously is some semantic relation between S and E. One possibility is to introduce a definition expressing that E is some kind of logical consequence of S. For the purposes of this paper we prefer rather to extend the definition of satisfiability in [1] as follows.

Let S be a set of clauses and E some clause. Let S_1 be obtained by adding to S all unit clauses obtained by taking the complement of any literal in E. If S_1 is unsatisfiable we shall say that S is *E-unsatisfiable.*

Theorem 1. Let S be a set of clauses and E a clause such that it is possible to obtain by resolution (in the general sense of [1]) a clause C that weakly subsumes E. Then S is E-unsatisfiable.

Informally the argument is the following. Let D be the instance of C which is a weak instance of E. If M is a model for S then it is also a model for D. Any value given to the variable will produce some true literal in D. Such literal is an instance of a literal in E. Hence there is a value for the variables that makes a literal of E true, hence the negation false. It follows that M is not a model for S_1.

To prove the converse of Theorem 1 (but with resolution restricted to directed resolution) we need some results on the cut rule. This rule is of the form

$$\frac{(A,C) \qquad\qquad (\bar{A},D)}{(C,D)}$$

where A is any literal and C, D are arbitrary clauses, not necessarily disjoint.

Lemma. Let S be a set of clauses and D a clause that can be obtained from S by using the cut rule. Let V be some valuation. Then there is a subclause D' of D which can be obtained from S by using the cut rule in such a way that every cut rule is of the form

$$\frac{(A,C_1) \qquad\qquad (\bar{A},C_2)}{(C_1,C_2)}$$

where V(A) = F and the V-residual of C_2 is included in (the V-residual of) D.

The proof is by induction in the number of cut rules used to generate D. In case no cut rule is used (hence D is an element of S) the result is trivial. Assume some cut rules have been used and let the last be of the form

$$\frac{(B,D_1) \qquad\qquad (\bar{B},D_2)}{(D_1,D_2)}$$

with V(B) = F. This means of course that (D_1,D_2) is D.

By the induction hypothesis we may assume there is a derivation of (B,D'_2) from S (here D'_2 is some subclause of D_2) and in which every cut rule is of the form

$$\frac{(A,C_1) \qquad (\overline{A},C_2)}{(C_1,C_2)}$$

where $V(A) = F$ and the V-residual of C_2 is included in (B,D_2), hence in $(D_1,D_2) = D$, since $V(B) = T$.

Also by the induction hypothesis there is a derivation of (B,D_1') from S (here D_1' is some subclause of D_1) and in which every ground resolution is of the form

$$\frac{(A,C_1) \qquad (\overline{A},C_2)}{(C_1,C_2)}$$

where $V(A) = F$ and the V-residual of C_2 is included in (B,D_1). In case C_2 contains B, say $C_2 = (B,C_2')$ we modify the derivation as follows

$$\frac{(A,C_1) \qquad \dfrac{(B,\overline{A},C_2') \qquad (\overline{B},D_2')}{(\overline{A},C_2',D_2')}}{(C_1,C_2',D_2')}$$

This transformation replaces C by (C_2',D_2'). If this is carried out everywhere we end with a derivation of either (D_1'',D_2') or (B,D_1'',D_2') where D_1'' is a subclause of D_1. In the second case we apply the cut rule again with (B,D_2') to obtain (D_1'',D_2'). This completes the proof.

Theorem 2. Let S be E-unsatisfiable and V some preference for S. Then either there is a clause of the form $(A,\overline{A})$ that weakly subsume E, or it possible to generate from S using E-directed resolution with preference V a clause C that weakly subsumes E.

Let S_1 be obtained by adding to S as unit clauses the complement of literals in E. Since S_1 is unsatisfiable we can generate from S_1 the empty clause using ordinary resolution. This means we can generate the empty clause using the cut rule from instances of the clauses in S_1. If in this derivation using the cut rule we avoid all applications of the rule involving the unit clauses which are instances of complements of literals in E we get a new derivation using the cut rule of a clause D which is a weak instance of E. This derivation can be transformed in another satisfying the conditions in the Lemma and the result in a clause D' which is also a weak instance of E. Finally the whole derivation can be lifted to a derivation from S using E-directed resolution with preference V. The result is a clause C such that D' is an instance of C. So C weakly subsumes E.

4. Conclusions

We have already mentioned that P_1-resolution is a special case of directed resolution. So the main result of [2] can be obtained as a consequence of Theorem 2. Given a computation by E-directed resolution with preference V it is possible to consider those units (involving in general many resolutions) which produce clauses C such that the V-residual of C is included in E. Following [2] we may call those units E-directed hyper-resolution with preference V.

The results of [4] are also special case of this paper. For suppose E is a logical consequence of the set S as defined in [4]. Let E_1 be obtained by replacing in E every variable by a new constant. It follows that E is E_1-unsatisfiable. Hence using E_1-directed resolution with any preference V we can obtain a clause C that weakly subsumes E_1. Clearly C subsumes (in the sense of [1]) E. So we get the result of [4] in the stronger form that whenever E is a logical consequence of S then some clause that subsumes E can be obtained from S using E_1-directed resolution with any preference V. Actually we can eliminate the reference to E_1 by modifying the notion of directed resolution and replacing condition ii) by the following:
ii') $V(C_1s) = F$ and the V-residual of D_2 subsumes E.

Finally we remark that the idea of preference in this paper is similar to that of model in [3]. But actually it is more general since we may have preference giving different truth values to literals P(f(x)) and P(gx)).

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