NO RATIONAL SENTENTIAL LOGIC HAS A FINITE CHARACTERISTIC MATRIX

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Gödel showed (in [2]) that the intuitionist sentential logic has no finite characteristic matrix, i.e. that the logic is not a finitely many valued logic; and Dugundji (in [3]) adapted the matrix argument Gödel used to Lewis modal systems. Since then others have shown, in a likewise rather system-specific way, that other sentential systems such as relevant logics E and R lack a finite characteristic matrix, i.e. a matrix with a finite number of values which verifies all theorems of the given system and falsifies all non-theorems. Meyer's study (in [1]) of metacompleteness enables however a general proof that a large and diverse class of sentential logics lack a finite characteristic matrix.

A sentential logic L is *rational* only if it has a separable sublogic L', with at least connectives $\rightarrow$ and $\vee$, which is M-rational (rational in a modification of the sense of Meyer [1]). A sublogic L' of L is *seperable* iff, for every wff A of L', A is a theorem of L iff A is a theorem of L'. Meyer-rationality is thus far characterised by an extensive listing of optional postulates for rational logics. For present purposes it is enough to offer a necessary condition for Meyer-rationality. Let L be any sentential logic whose improper symbols consist (apart parentheses) of some or all of the following connectives and constants : $\rightarrow$, $\vee$, &, $\neg$, $\Box$, $\Diamond$, t, F.

The *canonical metavaluation* V on L is defined recursively on wff of L as follows :

(i) V(p) = F, for each sentential parameter **P** ;

(ii) V(t) = T ;

(iii) V(F) = F ;

(iv) V(A & B) = T iff V(A) = T = V(B) ;

(v) $V(A \vee B) = T$ iff $V(A) = T$ or $V(B) = T$;

(vi) $V(A \rightarrow B) = T$ iff $A \rightarrow B$ is a theorem of L;

(vii) $V(\neg A) = T$ iff $\neg A$ is a theorem of L;

(viii) $V(\Box A) = T$ iff $\Box A$ is a theorem of L;

(ix) $V(\Diamond A) = T$ iff $\Diamond A$ is a theorem of L.

The connective $\neg$ represents refutability or intuitionistic negation, $\Box$ necessity and $\Diamond$ possibility, and constants F and t can be thought of as the conjunctions of all wff and of all theorems respectively. A *metavaluation* V of L is a function V from wff of L to $\Pi = \{T, F\}$ satisfying those of clauses (ii) — (vii) that apply. A wff A of L is *true on* a metavaluation V iff $V(A) = T$. Logic L is *coherent* iff each theorem of A is true on all metavaluations of L, and is *metacomplete* iff exactly the wff true on the canonical metavaluation of L are theorems. Meyer's main result in [1] for sentential logic L is that *if L is rational then L is metacomplete and accordingly coherent.* It will almost suffice here to take metacompleteness as, what is it, a necessary condition for Meyer-rationality — but not quite. Logic L is *M-rational* only if L is metacomplete, identity (i.e. $A \rightarrow A$) is an axiom scheme of L, and modus ponens (i.e. where A and $A \rightarrow B$ are theorems of L so is B) and uniform substitution are admissible rules of L. (Note that in [1] Meyer requires modus ponens of rational logics and presupposes uniform substitution, but treats identity as an optional extra.)

Theorem. No rational sentential logic L has a finite characteristic matrix.

Proof. Let L be a rational logic. Then L has a separable sublogic L' which is M-rational. Thus L' is metacomplete, and therefore demonstrably prime, i.e.

(1) $A \vee B$ is a theorem of L' iff one of A and B is. Now, to vary Gödel's original argument, consider the wff of L':

$(p_1 \rightarrow p_2) \vee \ldots \vee (p_1 \rightarrow p_n) \vee (p_2 \rightarrow p_3) \vee \ldots \vee \ldots \vee (P_2 \rightarrow p_n) \vee \ldots \vee (P_{n-1} \rightarrow p_n)$ (Vn)

where association is to the left.

(2) (Vn) is a theorem of L' iff $p_i = p_j$ for some i and j, with $n \geqslant 1$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$.
Suppose firstly $p_h = p_k$ for appropriate h and k; then $p_h \rightarrow p_k$ is a theorem of L'. Hence by iterated application of (1), (Vn) is a theorem of L'. In order to respect association to the left there are two steps involved. Firstly (1) is applied to disjoin all disjuncts in (Vn) to the left of $p_h \rightarrow p_k$ at once; and then (1) is repeatedly applied to disjoin one at a time disjuncts to the right of $p_h \rightarrow p_k$ in (Vn). The full proof that (Vn) is a theorem is thus by induction on the length of (Vn) to the right of $p_h \rightarrow p_k$. For the converse we use the fact that

(3) where $h \neq k$, $p_h \rightarrow p_k$ is not a theorem of L'.
Suppose on the contrary $p_h \rightarrow p_k$ is a theorem of L'. Then by uniform substitution, $(A \rightarrow A) \rightarrow p_k$ is a theorem of L', whence p_k is a theorem of L'. But by the metacompleteness of L' and (i), no sentential parameter is a theorem of L'. Thus (3) is established. Next suppose (Vn) is a theorem of L'. Then by repeated application of (1) at least one of the disjuncts $(p_1 \rightarrow p_2), \ldots, (p_{n-1} \rightarrow p_n)$ is a theorem of L' (the full argument is again by induction). But this contradicts (3) given that all the parameters of (Vn) are distinct. Thus (2) is established.

(4) L' has no finite characteristic matrix.
Suppose that **M** is any finite matrix, with j values say. Consider (Vn) for any $n \geqslant j$, say $n = j + 1$. Then any assignment of values from **M** to (Vn) assigns the same value to at least two parameters in (Vn). Hence **M** validates (Vn), since by (2) (Vn) with two parameters identified is a theorem of L'. But also by (2), (Vn) is not a theorem of L' since no variables need coincide. Hence **M** is not characteristic for L'.

(5) L has no finite characteristic matrix.
Suppose it has, and that **M** is such a matrix; that is, wff B is a theorem of L iff **M** validates B. But then **M** is characteristic for L', contradicting (4). For where A is a wff of L', A is a theorem of L' iff A is a theorem of L, i.e. iff **M** validates A.

It follows that a great many sentential logics have no finite characteristic matrices; for example not only intuitionist, minimal and absolute logics, many Lewy systems, and a range of modal logics, but also all the relevant logics of [4] and their negation extensions, and all the relevant implicational systems of [5] — indeed (as Meyer [1] will show) any logic which contains as a separable sublogic a logic axiomatised using the postulates t (where applicable) and $A \rightarrow A$, the rules of modus ponens and, where, applicable adjunction and disjunctive addition (i.e. if A or B is a theorem so is $A \vee B$), and any selection of (at least) the axiom and rule schemata listed below :

$A \& B \rightarrow A,\ A \& B \rightarrow B,\ (A \rightarrow B) \& (A \rightarrow C) \rightarrow . A \rightarrow (B \& C)$
$A \rightarrow . A \vee B,\ B \rightarrow . A \vee B,\ (A \rightarrow C) \& (B \rightarrow C) \rightarrow . (A \vee B) \rightarrow C,$
$A \& (B \vee C) \rightarrow . (A \& B) \vee (A \& C),\ A \rightarrow B \rightarrow . B \rightarrow C \rightarrow . A \rightarrow C,$
$B \rightarrow C \rightarrow . A \rightarrow B \rightarrow . A \rightarrow C,\ A \rightarrow (A \rightarrow . B) \rightarrow . A \rightarrow B,$
$((A \rightarrow A) \rightarrow B) \rightarrow B,\ A \rightarrow . (A \rightarrow B) \rightarrow B,\ t \rightarrow . A \rightarrow A,$
$A \rightarrow . t \rightarrow A,\ (t \rightarrow A) \rightarrow A,\ A \rightarrow (B \rightarrow C) \rightarrow . A \rightarrow B \rightarrow . A \rightarrow C,$
$A \rightarrow B \rightarrow . C \rightarrow D \rightarrow . A \rightarrow B,\ A \rightarrow B \rightarrow . C \rightarrow . A \rightarrow B,$
$(A \rightarrow B \rightarrow . A \rightarrow . B \rightarrow C) \rightarrow . A \rightarrow B \rightarrow . A \rightarrow C,$
$(A \rightarrow (B \rightarrow C) \rightarrow (A \rightarrow B)) \rightarrow . A \rightarrow (B \rightarrow C) \rightarrow . A \rightarrow C,$
$(A \rightarrow B \rightarrow . B \rightarrow C) \rightarrow . A \rightarrow B \rightarrow . A \rightarrow C,$
$(B \rightarrow C \rightarrow . A \rightarrow B) \rightarrow B \rightarrow C \rightarrow A \rightarrow C,$
$(A \rightarrow C \rightarrow . A \rightarrow B) \rightarrow . A \rightarrow C \rightarrow . A \rightarrow B \& C$
$(C \rightarrow B \rightarrow . A \rightarrow B) \rightarrow . C \rightarrow B \rightarrow . (C \vee A) \rightarrow B$
$A \rightarrow B \rightarrow . A \rightarrow C \rightarrow . A \rightarrow (B \& C)$
$A \rightarrow C \rightarrow . B \rightarrow C \rightarrow . A \vee B \rightarrow C,\ F \rightarrow A,$
$A \& (A \rightarrow B) \rightarrow B,\ (A \rightarrow B) \& (B \rightarrow C) \rightarrow . A \rightarrow C,$
$A \rightarrow F \rightarrow . \neg A,\ \neg A \rightarrow . A \rightarrow F;\ A \rightarrow \Diamond A,\ \Diamond\Diamond A,$ etc. ;
$A \& B \rightarrow A, B;\ A \rightarrow B,\ B \rightarrow C \rightarrow . A \rightarrow C,$
$A \rightarrow B \rightarrow B \rightarrow C \rightarrow . A \rightarrow C;\ B \rightarrow C \rightarrow A \rightarrow B \rightarrow . A \rightarrow C;$
$C \rightarrow D \rightarrow A \rightarrow B \rightarrow . C \rightarrow D;\ A \rightarrow B \rightarrow A;$
$A \rightarrow t \rightarrow A;\ A \rightarrow B,\ B \rightarrow A,\ C \rightarrow D,\ D \rightarrow C \rightarrow A \rightarrow C \rightarrow . B \rightarrow D,$

where $A_1, A_2, \ldots, A_n \rightarrow B_1\ B_2$ here abbreviates :
if A_1 and A_2 and ... A_n are theorems so are B_1 and B_2, etc.,

It is sometimes irksome however to have to cope with logics which include a normal or classical negation by first determining an appropriate separable sublogic. The troublesome step can very often be avoided by adding a negation clause

(x) V(~A) =T iff V(A) = F

to the definition of a metavaluation, and working with coherence rather than metacompleteness as a necessary condition of rationality. A logic L is *N-rational* (negation rational) only if it is coherent, and identity, modus ponens and uniform substitution are admissible.

Theorem. No N-rational sentential logic has a finite characteristic matrix.

Proof : varies the proof of the first theorem simply in replacing L' by L and (1) by

(1') (A → B) ∨ (C → D) is a theorem of L iff one of A → B and C → D is.

That (1') holds is an immediate corollary of coherence and disjunctive addition.

As to how coherence of logics which contain connective ~ may be established see Meyer [6] : but thus far it is by no means so easily established as is metacompleteness and coherence of positive-style rational logics, both of which tend to be very straightforward matters.

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