VAGUENESS AND FAILING SENTENCES

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1. NEITHER TRUE NOR FALSE. If we consider sentences of the sort used to make assertions, there are different «levels» of requirements that a fully satisfactory sentence must meet. These include (in ascending order) the requirements that the sentence be (a) grammatical (well-formed), (b) significant, (c) true or false, (d) true. A sentence which is successful at a given level is successful at all lower levels, but may not satisfy the higher-level requirements. So a sentence can be grammatical but not significant, or it can be grammatical and significant but neither true nor false. If it is false, it is both grammatical and significant. And, clearly, a true sentence is also grammatical, significant, and true or false.

Some questions have been raised about the distinction between (a) and (b). It is possible to construct an artificial formal language which contains grammatical sentences that are not significant. But it seems somewhat pointless to allow this; one could as easily make the grammatical requirements more stringent, so that grammatical and significant sentences coincide. With respect to natural languages, it is an arbitrary matter whether one allows a difference between sentences which are grammatical and those which are significant. It all depends on what one chooses to call grammatical. The sentence 'Colorless green ideas sleep furiously.' was once a favorite candidate for a grammatical sentence that is not significant. But Chomsky has suggested that an adequate grammar will not leave room for such sentences. (See, for example, p. 75, [1].)

The distinction between (a) and (b) is less important than that between (b) and (c). Frege may have been the first logician to recognize that a significant sentence, of the kind that is true or false (as opposed to questions or commands), might fail to be either. At least, Frege may have been the first logician to do this, who regarded the truth or falsity of a sentence as independent of the time when it is uttered. Russell certainly missed this point, as is evident from his discussion of Frege's views in «On Denoting.» However, I don't agree with Frege's choice of significant sentences that are neither true nor false. In a natural language, the failure of a singular term to refer to a real object doesn't *automatically* lead to truth-value failure. The sentence 'Hitler wasn't much like Hamlet.' is a straightforward and true English sentence-it doesn't require rewriting.

Even though Frege's choice of failing sentences is wrong, the recognition that some significant sentences fail is right. I think the clearest, though not the only, cases of such failure are provided by sentences containing vague predicates. In a natural language, there are criteria associated with a predicate «by the language.» A thing must satisfy these criteria before the predicate can be truly applied to that thing. A vague predicate has associated criteria that give rise to borderline cases — a borderline case for a predicate neither clearly satisfies nor clearly does not satisfy the associated criteria. The word 'tall' as applied to persons is vague. Some people are clearly tall, some people are clearly not tall, and some people are neither tall nor not tall. If Jones is a borderline case for the predicate 'tall,' then the sentence 'Jones is tall.' is significant, but it is not either true or false. (The same holds, of course, for the sentence 'Jones is not tall.')

Natural languages contain significant sentences that fail. So natural languages are at least three valued. If someone wished to take account of different sorts of failure, he might recognize more than three values. For the present, I wish to lump all failing sentences together. I will consider only the values truth, falsity, and failure. Sentences that fail do not very often make trouble for nonphilosophers. There seem to be at least two reasons for this. One is that we can often eliminate failing sentences when it is important to do so. A vague predicate can either be redefined to eliminate (some) borderline cases, or it can be replaced by a different one. (Instead of saying «Jones is tall,» we might say, «Jones' height is five feet, eleven inches.») A second reason is that it frequently does no harm to ignore failing sentences. Many purposes for using and studying language can be successfully achieved if we make the simplifying assumption that significant sentences are either true or false. However, in this paper I will try to get a better understanding of the logical aspects of natural languages by abandoning this assumption.

It is even an oversimplification to regard significant sentences (the ones that belong in the true-or-false ballpark) as capable of a neat division into the true, the false, and the failing. The predicates 'true,' 'false,' and 'failing' are themselves vague and will have borderline cases. (I have discussed this possibility in connection with self-reference in [4].) However, it will prove helpful to make this oversimplification in the present paper. To gain a better understanding of the three-valued character of natural languages, I will employ the logician's strategy. I will develop an artificial formal language that has some resemblance to a natural language, but which is much simpler than a natural language. The artificial language is an instrument we can use to gain a better understanding of natural languages. Artificial languages can also serve other purposes; they might be used as languages in their own right. But it is important not to confuse an artificial formal language with a natural language.

2. THE FORMAL LANGUAGE L. This is a conventional first-order language. Its building blocks are the following:

Punctuation: (,), [,], the comma: ,
Individual variables: x_0, y_0, z_0, x_1, y_1,...
Individual constants: a_0, b_0, c_0, a_1, b_1,...

For $n > 0$, n-adic predicates: F_0^n, G_0^n, H_0^n, F_1^n,...
Connectives: v, ~
Quantifier component: $\forall$

The *well-formed formulas* (*wffs*) are given by:

(1) If $\alpha_1, \ldots, \alpha_n$ are individual variables or constants and φ^n is an n-adic predicate, then $\varphi^n(\alpha_1,\ldots,\alpha_n)$ is a(n) (atomic) wff.

(2) If A, B are wffs, so are $\sim A$ and $[A \vee B]$.

(3) If A is a wff containing free occurrences of individual variable α, then $(\forall \alpha)A$ is a wff.

(4) All wffs are constructed according to (1) - (3).

A *sentence* is a wff without free occurrences of individual variables.

The following define connectives used as abbreviations:

$$[A \,\&\, B] \;=\; (\text{def.})\;\; \sim[\sim A \vee \sim B]$$
$$[A \supset B] \;=\; (\text{def.})\;\; [\sim A \vee B]$$
$$[A \equiv B] \;=\; (\text{def.})\;\; [[A \supset B] \,\&\, [B \supset A]]$$
$$(\exists \alpha)A \;=\; (\text{def.})\;\; \sim(\forall \alpha)\sim A$$

Brackets will abbreviated according to the conventions of [2].

The semantics for L is three valued; I will use 'T,' 'F,' and 'O' for truth, falsity, and failure. Since the connectives of L are familiar ones, it is reasonable to require the semantics to assign the ordinary values when the components are true or false. We need only determine how to evaluate compound sentences with failing components. The most satisfactory three valued semantics seems to be provided by the following:

A	B	~ A	A v B	A & B	A⊃B	A≡B
T	T	F	T	T	T	T
T	F	F	T	F	F	F
T	O	F	T	O	O	O
F	T	T	T	F	T	F
F	F	T	F	F	T	T
F	O	T	O	F	T	O
O	T	O	T	O	T	O
O	F	O	O	F	O	O
O	O	O	O	O	O	O

The semantics for ~, v, & is intially plausible, but it might be felt that a different truth-table should be used for the horseshoe. (Of course, given the definition above, the horseshoe must have the table shown. But it might be thought preferable to take the horseshoe as primitive, and give it a different interpretation.) The only plausible alternative for the horseshoe would change its value in the last row to yield:

A	B	$A \supset B$
O	O	T

I have several reasons for opposing this alternative. In the first place, I think a truth-functional connective that yields a true sentence given (exclusively) failing components is more like a predicate of sentences than it is like a conventional connective. Such connectives are not so useful for studying natural languages. A second reason for avoiding the alternative interpretation is that the redefined horseshoe would have an even poorer match with natural language conditionals than does the horseshoe I have chosen. I regard an inference from $A \supset [A \supset B]$ to $A \supset B$ as entirely reasonable; it is invalid for the proposed alternative interpretation. A third reason concerns the difference in status between failure on the one hand and truth or falsity on the other. The failure of a sentence can in some cases be removed by changes in the language — by, for example, redefining a predicate. It is a suitable regulative principle that such changes have a minimal effect on the (already) true or false sentences. But if $A \supset B$ were taken as true when both A and B fail, a change in the values of A and B could convert the true sentence into a false one.

If we extend the «idea» of the truth-table above to quantifiers, we get:

(1) A sentence $(\forall \alpha)A$ is true iff A is true for every value of α; it is false iff A is false for some value of α; it fails otherwise.

(2) A sentence $(\exists \alpha)A$ is true iff A is true for some value of α; it is false iff A is false for every value of α; it fails otherwise.

A more precise account of the semantics of L is as follows.

Let ϑ be a nonempty domain. Then a *valuation of* L *for* ϑ is a function f such that

(i) f assigns an individual of ϑ to each individual constant of L.

(ii) If φ^n is an n-adic predicate of L, then f assigns an ordered pair $\langle R, S\rangle$ to φ^n, where R and S are disjoint sets of ordered n-tuples of individuals of ϑ.

(iii) If $\varphi^n(\alpha_1,\ldots,\alpha_n)$ is an atomic sentence of L and $f(\varphi^n) = \langle R, S\rangle$, then this sentence has value T for f iff $\langle f(\alpha_1),\ldots,f(\alpha_n)\rangle \in R$. It has value F for f iff $\langle f(\alpha_1),\ldots,f(\alpha_n)\rangle \in S$. It has value O for f otherwise.

(iv) If A is a sentence of L, then $f(\sim A) = \mathrm{T}$ iff $f(A) = \mathrm{F}$. $f(\sim A) = \mathrm{F}$ iff $f(A) = \mathrm{T}$. $f(\sim A) = \mathrm{O}$ iff $f(A) = \mathrm{O}$.

(v) If A, B are sentences of L, then $f(A \vee B) = \mathrm{T}$ iff either $f(A) = \mathrm{T}$ or $f(B) = \mathrm{T}$. $f(A \vee B) = \mathrm{F}$ iff $f(A) = f(B) = \mathrm{F}$. $f(A \vee B) = \mathrm{O}$ otherwise.

(vi) Let $(\forall\alpha)A$ be a sentence of L. Let β be the first individual constant in alphabetic order not occurring in A. Let A' be obtained from A by replacing all free occurrences of α in A by β. Then $f[(\forall\alpha)A] = \mathrm{T}$ iff $f'(A') = \mathrm{T}$ for every valuation f' of L for ϑ that agrees with f on the individual constants and predicates of L with the possible exception of β. (Such an f' is a β-*variant* of f.) $f[(\forall\alpha)A] = \mathrm{F}$ iff there is some β-variant f' of f such that $f'(A') = \mathrm{F}$. $f[(\forall\alpha)A] = \mathrm{O}$ otherwise.

The semantics for L is the same as that for the strong three-valued logic presented by Kleene in [5]. However, Kleene applied this logic to other situations than vagueness, and allowed some sentences with the third value to simultaneously be either true or false. For this reason, he insisted that the third value should not be regarded as on a par with truth and falsity. Stephan Körner has adopted Kleene's logic for dealing with vague predicates — he calls them *inexact* predicates. (See, for example, chapter III of [6].) He was retained Kleene's insistence that failure not be counted as a genuine third value. Körner's reason for this is that a vague predicate is always subject to a redefinition which eliminates borderline cases.

Such redefinition is sometimes required for scientific or legal purposes. Since true and false sentences have permanent values, and failing sentences can become true or false, we should regard failure as not on a par with truth and falsity. (Körner seems to regard a two valued language as an ideal which we attempt to realize.)

Kleene's reasons for regarding the third value as not quite a genuine value do not apply to the case of sentences which fail because of vagueness (or to any instance of a significant sentence that is neither true nor false). Körner seems to me to make too strong a separation between failure and the two classical values. At any given time, a natural language contains (or permits) failing sentences. These sentences are neither true nor false. The three-valued semantics corresponds to a natural language at a given time. The changed situation at a later time can be represented by a different valuation. It is unrealistic to think that all vagueness can be eliminated; to do so would probably make a language unusable for conversational purposes. And vagueness is not the only cause of sentence failure — *some* sentences with empty singular terms do fail. For these other sentences, failure is not removable at a later date. Finally, even true and false sentences are liable to being reevaluated due to changes in the language, though this is much rarer than for failing sentences. Two examples are the following:

(1) Whales are fish.
(2) The Earth is a planet.

So three valued L can be used to represent a natural language at a given time. None of the three values can be eliminated.

3. THE LOGICALLY DISTINGUISHED ITEMS OF L. One of the most striking features of L is that it contains no logically true sentences. Since every atomic sentence is subject to failure, we cannot construct a sentence that is true for all valuations of L. However, the importance of logically true sentences has been overrated in modern logic. These sentences are im-

portant largely because they can be used to present what Church has called the *leading principles* of valid inferences. (See, for example, Exercise 15.9, pp. 104-105, [2].) Since there are no logical truths in L, we simply need another way to focus on valid inferences.

An inference is a human act (or activity). Although sentences can be used to perform an inference, these sentences do not constitute the inference. I will say that a sequence of $n + 1$ sentences ($n \geqq 0$) is an *inference sequence,* and I will write such a sequence in this way: $A_1,\ldots,A_n / B$. The slant line separates the *premisses* of the inference sequence from its *conclusion.* Logic does not study human acts, but the logician is properly concerned with inference sequences. Logic uncovers (discovers) norms that can be used to evaluate (some) human acts.

An inference sequence $A_1,\ldots,A_n / B$ is *valid* iff every valuation which makes all of the premisses true also makes the conclusion true. A set of sentences X *logically implies* a sentence *B*, and *B* is a *logical consequence* of X (in symbols: $X \Vdash B$) iff every valuation which makes all the members of X true also makes *B* true. So $A_1,\ldots,A_n / B$ is valid iff $\{A_1,\ldots,A_n\} \Vdash B$.

Since L contains no logical truths, there are no valid inference sequences with zero premisses. However, the following present the forms of valid inference sequences of L.

(1) $A / A \vee B$
(2) $B / A \vee B$
(3) $A \& B / A$
(4) $A \& B / B$
(5) $A, B / A \& B$
(6) $A / {\sim}{\sim} A$
(7) ${\sim}{\sim} A / A$
(8) $A, {\sim} A / B$
(9) ${\sim} A, A \vee B / B$
(10) ${\sim}[A \& B] / {\sim}A \vee {\sim}B$
(11) ${\sim}[A \vee B] / {\sim}A \& {\sim}B$
(12) $A / B \supset A$
(13) $A, A \supset B / B$
(14) $A \supset B, {\sim}B / {\sim}A$
(15) $A \supset B / {\sim}B \supset {\sim}A$
(16) ${\sim}B \supset {\sim}A / A \supset B$
(17) $A \& B \supset C / A \supset .B \supset C$
(18) $A \supset .B \supset C / A \& B \supset C$
(19) $(\forall \alpha)A / \text{Ş}^{\alpha}_{\beta}A|$ Here β is any individual constant. The notation for substitution is from [2]. It indicates the substitution of β for all free occurrences of α in A.
(20) $\text{Ş}^{\alpha}_{\beta}A| / (\exists \alpha)A$

The differences between L and a two valued language are minimized if we consider only valid inference sequences. To get a clearer picture of the differences, it is helpful to consider more complex inferences. I will consider deductions carried out by means of tree proofs. In such a proof, the sentences at the tops of branches are the *hypotheses*. An *inference figure* is that part of a tree proof constituted by the premisses of a particular inference, the line separating the premisses from the conclusion, and the conclusion. If a kind of inference figure is *elementary*, all the premisses of such a figure must be sentences occurring on the line. In a nonelementary inference figure, some of the premisses may consist of tree (sub)proofs. In nonelementary inference figures, it is common to have some of the hypotheses of the tree-proof premisses *cancelled*, so they no longer count as hypotheses of the main proof. (A more elaborate explanation of such proofs is found in [3].)

An inference figure is valid if it is truth preserving. There is a valid (elementary) inference figure corresponding to every valid inference sequence. In addition, the following are valid inference figures.

v Elimination

$$\frac{A \vee B \quad \overset{[A]}{C} \quad \overset{[B]}{C}}{C}$$

This is a nonelementary inference figure. The bracketed expressions are hypotheses in (two) tree proofs leading to the occurrences of C. Occurrences of the bracketed hypotheses are cancelled by this inference figure.

∀ Introduction

$$\frac{A}{(\forall \alpha) \text{Ș}^{\beta}_{\alpha} A|}$$

Here β is an individual constant that occurs in A but does not occur in a wf part $(\forall \alpha)B$ of A. And β does not occur in any uncancelled hypothesis of a subproof which concludes on the line of this inference figure.

∃ Elimination

$$\frac{(\exists\alpha)A \qquad \begin{matrix}[\text{Ş}^{\alpha}_{\beta}A|]\\ B\end{matrix}}{B}$$

Here β is an individual constant that does not occur in A, B or in any uncancelled hypothesis other than $\text{Ş}^{\alpha}_{\beta}A|$ in the subproof whose conclusion is B.

Examples of inference figures valid for a two-valued language but not for L are the following.

∼ Introduction

$$\frac{\begin{matrix}[A]\\ \sim A\end{matrix}}{\sim A}$$

∼ Elimination

$$\frac{\begin{matrix}[\sim A]\\ A\end{matrix}}{A}$$

⊃ Introduction

$$\frac{\begin{matrix}[A]\\ B\end{matrix}}{A \supset B}$$

In L, being contradictory (leading to a contradiction) is no guarantee of falsity. Similarly, if we can reason from A to B, the sentence $A \supset B$ may still fail. In L, the horseshoe is not a very conditional connective.

4. A FORMAL SYSTEM. The formal system F is a natural deduction system using tree proofs. The rules of inference are the following:

v Introduction

$$\frac{A}{A \vee B} \qquad \frac{B}{A \vee B}$$

v Elimination

$$\frac{A \vee B \qquad \begin{matrix}[A]\\ C\end{matrix} \qquad \begin{matrix}[B]\\ C\end{matrix}}{C}$$

∼∼ Introduction

$$\frac{A}{\sim\sim A}$$

∼∼ Elimination

$$\frac{\sim\sim A}{A}$$

Contradiction Elimination

$$\frac{A \qquad \sim A}{B}$$

$\forall$ Introduction $\dfrac{A}{(\forall\alpha)Ş^{\beta}_{\alpha}A|}$ The restrictions on this rule are in § 3.

$\forall$ Elimination $\dfrac{(\forall\alpha)A}{Ş^{\alpha}_{\beta}A|}$ β is a constant.

$\exists$ Elimination

$$\dfrac{(\exists\alpha)A \qquad \begin{matrix}[Ş^{\alpha}_{\beta}A|]\\ B\end{matrix}}{B}$$

The restrictions on this rule are in § 3.

$\sim$v Eimination $\dfrac{\sim[A \vee B]}{\sim A}$ $\dfrac{\sim[A \vee B]}{\sim B}$

$\sim$v Introduction $\dfrac{\sim A \qquad \sim B}{\sim[A \vee B]}$

$\sim\forall$ Introduction $\dfrac{\sim Ş^{\alpha}_{\beta}A|}{\sim(\forall\alpha)A}$ β is a constant

$\sim\forall$ Elimination $\dfrac{\sim(\forall\alpha)A}{(\exists\alpha)\sim A}$

$\forall$v Interchange $\dfrac{(\forall\alpha)\,[A \vee B]}{A \vee (\forall\alpha)B}$ α does not occur free in A.

If there is a proof of B from uncancelled hypotheses $A_1,\ldots,A_n$, then $A_1,\ldots,A_n\,/\,B$ is a *theorem* of F. I will write either $\vdash A_1,\ldots, A_n\,/\,B$ or $A_1,\ldots,A_n \vdash B$ to indicate that $A_1,\ldots, A_n\,/\,B$ is a theorem of F. The system F is clearly sound with respect to the semantics of L. We can also show that it is complete for the valid inference sequences of L. To do this, we can establish the following results. (Proofs are omitted, because they are straightforward.)

(4.1) If $A_1,\ldots,A_n\,/\,B$ is an inference sequence of L, then $\vdash A_1,\ldots, A_n\,/\,B$ iff $\vdash A_1 \,\&\, \ldots \,\&\, A_n\,/\,B$.

(4.2) If A, B are sentences of L, then $\sim[A \vee B] \vdash \sim A \,\&\, \sim B$, $\sim A \,\&\, \sim B \vdash \sim[A \vee B]$, $A \vee B \vdash \sim[\sim A \,\&\, \sim B]$, and $\sim[\sim A \,\&\, \sim B] \vdash A \vee B$.

(4.3) If A, B are sentences of L, then $\sim[A \,\&\, B] \vdash \sim A \vee \sim B$, $\sim A \vee \sim B \vdash \sim[A \,\&\, B]$, $A \,\&\, B \vdash \sim[\sim A \vee \sim B]$, and $\sim[\sim A \vee \sim B] \vdash A \,\&\, B$.

(4.4) If A, B, C are sentences of L, then $A \vee [B \,\&\, C] \vdash [A \vee B] \,\&\, [A \vee C]$, $[A \vee B] \,\&\, [A \vee C] \vdash A \vee [B \,\&\, C]$, $\sim[A \vee [B \,\&\, C]] \vdash \sim[[A \vee B] \,\&\, [A \vee C]]$, and $\sim[[A \vee B] \,\&\, [A \vee C]] \vdash \sim[A \vee [B \,\&\, C]]$.

(4.5) If A, B, C are sentences of L, then $A \,\&\, [B \vee C] \vdash [A \,\&\, B] \vee [A \,\&\, C]$, $[A \,\&\, B] \vee [A \,\&\, C] \vdash A \,\&\, [B \vee C]$, $\sim[A \,\&\, [B \vee C]] \vdash \sim[[A \,\&\, B] \vee [A \,\&\, C]]$, and $\sim[[A \,\&\, B] \vee [A \,\&\, C]] \vdash \sim[A \,\&\, [B \vee C]]$.

(4.6) Let B be a sentence obtained from a sentence A by replacing zero or more occurrences of a wff M in A by a wff N. Let $\alpha_1,\ldots,\alpha_n$ be the distinct individual variables occurring free in either M or N. Let $\beta_1,\ldots,\beta_n$ be the first n individual constants in alphabetic order not occurring in A, B, M, or N.

Let M', N' be $\text{Ş}^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_n} M|$, $\text{Ş}^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_n} N|$.

Let $M' \vdash N'$, $N' \vdash M'$, $\sim M' \vdash \sim N'$, and $\sim N' \vdash \sim M'$. THEN $A \vdash B$, $B \vdash A$, $\sim A \vdash \sim B$, and $\sim B \vdash \sim A$.

It follows from (4.2) - (4.6) that every quantifier-free sentence A can be put into disjunctive normal form A', and that $A \vdash A'$ and $A' \vdash A$. (For the purpose of obtaining the disjunctive normal form, the connective '&' is regarded as a genuine expression rather than a definitional abbreviation. In disjunctive normal form, each disjunct need not contain every atomic component or its negation.) This, together with (4.1) and (4.7), establishes that F contains all valid quantifier-free inference sequences.

(4.7) Let B be a sentence without quantifiers. Let $A_1,\ldots,A_n$ be distinct atomic sentence, among which are all the atomic components of B. Let $v_1,\ldots,v_n$ be a sequence of the values T, F, O such that B has value T or F for these values of $A_1,\ldots,A_n$. Let $C_1,\ldots,C_n$ be constructed: (a) if v_i = T, then C_i is A_i; (b) if v_i = F, then C_i is $\sim A_i$; (c) if v_i = O, then C_i is the null formula. THEN if B has value T for the values $v_1,\ldots,v_n$ of $A_1,\ldots,A_n$, then $C_1,\ldots,C_n \vdash B$. If B has value F for this assignment, then $C_1,\ldots,C_n \vdash \sim B$.

To show that F is complete with respect to the valid inference sequences of L, we can establish that F is equivalent to system PP of Wang [7]. In Wang's system, a sentence $A \rightarrow B$ is best understood as a one-premiss inference sequence. The following results, together with the preceding ones, are sufficient to establish that F is equivalent to PP, and hence complete.

(4.8) If $(\exists\alpha)A$ is a sentence of L, then $(\exists\alpha)\sim A \vdash \sim(\forall\alpha)A$, $\sim(\forall\alpha)A \vdash (\exists\alpha)\sim A$, $\sim(\exists\alpha)\sim A \vdash (\forall\alpha)A$, and $(\forall\alpha)A \vdash \sim(\exists\alpha)\sim A$.

(4.9) If $(\forall\alpha)A$ is a sentence of L, then $(\forall\alpha)\sim A \vdash \sim(\exists\alpha)A$ and $\sim(\exists\alpha)A \vdash (\forall\alpha)\sim A$.

(4.10) If $(\forall\alpha)\,[A \vee B]$ is a sentence of L in which α does not occur free in A, then $(\forall\alpha)\,[A \vee B] \vdash A \vee (\forall\alpha)B$, $A \vee (\forall\alpha)B \vdash$ $\vdash (\forall\alpha)\,[A \vee B]$, $\sim(\forall\alpha)\,[A \vee B] \vdash \sim[A \vee (\forall\alpha)B]$, and $\sim[A \vee (\forall\alpha)\,B] \vdash \sim(\forall\alpha)\,[A \vee B]$.

(4.11) If $(\exists\alpha)\,[A \vee B]$ is a sentence of L in which α does not occur free in A, then $(\exists\alpha)\,[A \vee B] \vdash A \vee (\exists\alpha)B$, $A \vee (\exists\alpha)B \vdash (\exists\alpha)\,[A \vee B]$, $\sim(\exists\alpha)\,[A \vee B] \vdash \sim[A \vee (\exists\alpha)B]$, and $\sim[A \vee (\exists\alpha)B] \vdash \sim(\exists\alpha)\,[A \vee B]$.

These results show that every sentence A can be put into prenex normal form A', and that $A \vdash A'$ and $A' \vdash A$. So the completeness theorem described by Wang in [7] also applies to F.

5. STRONG COMPLETENESS. The system F and the language L don't lend themselves to a strong completeness theorem like Henkin's. It is possible to obtain such a result if L is enlarged by adding the monadic connective η. This yields the language L^+. The truth-table for η is:

A	ηA
T	F
F	F
O	T

In the language L^+, there are logically true sentences. For example, sentences having these forms are logically true: $A \vee {\sim}A \vee \eta A$, ${\sim}\eta\eta A$.

The system F^+ is obtained from F by adding these rules:

$\sim\eta$ Introduction $\dfrac{A}{}$ $\dfrac{\sim A}{\sim\eta A}$ $\sim\eta\eta$ Introduction $\dfrac{\eta\eta A}{\sim\eta\eta A}$

Ɐɥ~

$\sim\vee\eta$ Introduction $\dfrac{\begin{array}{c}[A]\\ \sim A\end{array}}{\sim A \vee \eta A}$ $\eta\vee\eta$ Introduction $\dfrac{\eta[A \vee B]}{\eta A \vee \eta B}$

$\eta\forall$ Elimination $\dfrac{\eta(\forall\alpha)A}{(\exists\alpha)\eta A}$

I will use the symbol $|\!\vdash_{+}$ to indicate implication in L^+. If X is a set of sentences of L^+ and B is a sentence of L^+, then B is *deducible from* X *by means of* F^+ ($X \vdash_{+} B$) iff there are sentences $A_1,\ldots,A_n$ which are elements of X such that $A_1,\ldots,A_n / B$ is a theorem of F^+. We can adapt the proof of Henkin's completeness theorem for standard systems (as found, say, in [2]) to give the following result about F^+.

(5.1) Let X be a set of sentences of L^+ and let A be a sentence of L^+ such that $X |\!\vdash_{+} A$. Then $X \vdash_{+} A$.

Since L is contained in L^+, we can clearly deduce all the consequences of a set of sentences of L. So L is compact — i.e. if $X \Vdash A$, then there is a finite set $Y \subseteq X$ such that $Y \Vdash A$.

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