THE CONSISTENCY OF THE AXIOM OF CONSTRUCTIBILITY IN ZF WITH NON-PREDICATIVE ULTIMATE CLASSES

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ZF, ZF^p, and ZF^u are the set theories of Zermelo, Fraenkel, and Skolem, of von Neumann, Bernays, and Gödel, and of von Neumann and Morse respectively. For axiomatizations and model-theoretic terminology, the reader is referred to [3]. The metalanguage is assumed to be a definitional extension of ZF without the axiom of choice. In the metalanguage, $x \cup y$ and $x \cap y$ are the union and intersection of x and y respectively while x^+, $x\cup$, and $x\mathscr{P}$ are $x \cup \{x\}$, $\{a\colon a \in b \text{ for some } b \in x\}$, and $\{a\colon a \subseteq x\}$ respectively. L is the level of construction operation of [3] while V is von Neumann's level operation: $V_n = \{V_m\mathscr{P}\colon m \in n\} \cup$ for ordinals n. $\lesssim$ and $<$ are the relations of smaller or equal power and of smaller power respectively and $x^{\#}$ is $\{n\colon n \text{ is an ordinal and } n \lesssim x\}$. y card just when $y = \{n\colon n \text{ is an ordinal and } n < x\}$ for some x and y con just when $x \in L_n{}^+$ for some ordinal n. y inac just when y is an ordinal such that the set ω of finite ordinals $\in y$, $x^{\#} < y$ if $x \in y$, and $x\cup \in y$ if $x \subseteq y$ and $x < y$. If in addition $x \in y$ implies $x\mathscr{P} < y$, then y sinac. If a dot is placed above one of these defined expressions, the expression denotes the corresponding notion among the constructible sets. For example, $x \dot{\lesssim} y$ just when x con, y con, and there exists a one-to-one function f from x into y such that f con, and $x^{\dot{\#}} = \{n\colon n \text{ is an ordinal and } n \dot{\lesssim} x\}$ for x such that x con (since every ordinal is constructible). C is the formula of ZF^p and ZF^u asserting that every class is constructible: for any set x and class y, there is an ordinal n such that $x \cap y \in L_n{}^+$. Similarly, c is the assertion that, for any set x, there is an ordinal n such that $x \in L_n{}^+$, AC is the assertion that every class is well-ordered, ac is the assertion that every set is well-ordered, gch is the generalized continuum hypothesis among sets, and in is the assertion that there is a set n such that n inac.

In [1] and [2], Gödel showed that, if ZF and ZF^p are consistent, then so are $ZF+c$ and ZF^p+C respectively. Since $ZF+c$ implies gch and ac while ZF^p+C implies gch and AC, the consistency of $ZF+gch+ac$ and of $(ZF+gch)^p+AC$ follows from the consistency of ZF and ZF^p respectively. But now consider ZF^u. Here, the schema of comprehension over sets holds not just for formulas with only bound set variables, but for arbitrary formulas; that is, non-predicative ultimate classes exist. Intuitively, it seems that C should fail for such classes. Consequently, the consistency of gch and AC with ZF^u seems to be questionable. It is here shown that these doubts can be dispelled in the sense of

Theorem 1. If there is an n such that n inac, then $ZF^u + C$ has a standard complete model.

Corollary 1. If there is an n such that n inac, then $(ZF+gch)^u + AC$ is consistent.

Corollary 2. If there is an n such that n inac, then ZF^u+C does not imply ***in***.

Theorem 2. If there is an n such that n in$\dot{a}$c, then ZF^u+C has a denumerable standard complete model $M = \langle U, \in_U \rangle$ such that, if $M' = \langle U', \in_{U'} \rangle$ is a standard complete model of ZF, $m = \{k\colon k \in U'$ and k is an ordinal$\}$, and $\omega^{\dot{\#}}{}'' \dot{<} U'$, then $U \subseteq U'$.

In other words, the assumption that there is a weakly inaccessible ordinal implies that any standard complete model of Zf whose ordinals set is constructively more powerful than the smallest constructively uncountable cardinal harbors a certain model of ZF^u+C The stronger condition $\omega^{\#} < m$ could also be used here [1].

Lemma 1. If x inac, then x inac.

Assume the antecedent. Clearly, $\omega \in x$ and $w \in x$ implies $w^{\dot{\#}} \dot{<} x$ since every ordinal is constructible and $x \dot{\lesssim} w^{\dot{\#}} \subset w^{\dot{\#}} < x$ is impossible. Also, x is an ordinal and x card since $w \in x$ implies $w < w^{\#} < x$. Hence, if $w \subseteq x$ and $w \dot{<} x$, there is an ordinal y such that $w \dot{\lesssim} v \dot{<} x$ since w is well-ordered by $\in$. But $v \in x$ since v and x are ordinals. Hence, $w < x$ since x card and $w \cup \in x$ by assumption.

That is, any inaccessible ordinal is an inaccessible ordinal among the constructible sets. As Gödel observed in note 10 on p. 69 or [2], this is why $(\mathrm{ZF}+in)^{\mathrm{p}} + C$ is consistent if $(\mathrm{ZF}+in)^{\mathrm{p}}$ is.

Lemma 2. If x con for every x, then the axiom of choice and the generalized continuum hypothesis hold and so x sinac just when x inac.

This lemma is well known.

Lemma 3. If n sinac and V_n is well-ordered, then $V_m < n$ for $m \in n$.

The proof is by transfinite induction. If $m \in n$ and m is empty, there is no problem. Assume that $m = k^{+} \in n$ and $V_k < n$. Since $V_k \subset V_n$, there is an ordinal c such that $V_k \lesssim c \in n$ and so $V_m = V_k \mathscr{P} \lesssim c\mathscr{P} < n$ since n sinac. On the other hand, if m is a limit ordinal and $V_k < n$ for $k \in m$, then $x = \{V_k^{\#} : k \in m\} \subseteq n$ since n inac and $x \lesssim m \in n$ since m is well-ordered. But $V_m \lesssim x \cup$ $V_k \subseteq V_n$, there is an ordinal c such that $V_k \lesssim c \in n$ and so $V_m = x \cup \in n$ by n inac.

Lemma 4. If n sinac and V_n is well-ordered, then $M = \langle V_n{}^+, \in_{V_n{}^+} \rangle$ is a strongly standard complete model of ZF^u.

Assume the antecedent and that a is an axiom of ZF^u. Clearly, M is strongly standard complete. If a is an instance of the shemas of comprehension or foundation or a is the axiom of extensionality, $\models_N a$ since the universe of M is a set $V_m{}^+$ for ordinal n. $\omega \in V_\omega{}^+ \subseteq V_n$ and, if $m \in n$ and $x \in V_m{}^+$; $x \cup \in V_m{}^+ \subseteq V_n$ and $x\mathscr{P} \in V_m{}^{+\,+}$ V_n since an infinite cardinal is a limit ordinal. Hence, $\models_M a$ if a is one of the axioms of infinity, unions, or power sets. Assume finally that $m \in n$, $x \subseteq V_m$, f is a function from V_n into V_n, y is the range of f restricted to x, r is the function which assigns to any $w \in y$ the least ordinal $j \in n$ such that $w \in V_j{}^+$, and z is the range of r. Since V_n is well-ordered, so are x and y. Hence, $z \precsim y \precsim x \subseteq V_m$. But $V_m < n$ by Lemma 3 since $m \in n$ and so $z \subseteq n$ and $z < n$. Since n inac, it follows that $i = z \cup \in n$. Consequenetly, $y \in V_n$ since $y \subseteq V_i{}^+$ and n is a limit ordinal. That is, $\models_M a$ if a is an instance of the schema of replacement and the lemma is proved.

The main ideas of the proofs of Lemma 3 and Lemma 4 are due to Zermelo. They were clarified by Sheperdson in [4] to give the same results for ZF^p.

Lemma 5. If n card, $\omega \subseteq n$, x con, and $x \subseteq L_n$, then $x \in L_n\#$.

This is the main lemma of Gödel's proof in [1] that ZF + c implies gch. The following lemmas seem to be new.

Lemma 6. If x con for every x and n inac, then $V_n = L_n$.

Assume the antecedent. By transfinite induction up to n, it is easy to show that $L_m \subseteq V_n$ for $m \in n^+$. The proof that $V_m \subseteq L_n$ for $m \in n^+$ is also by transfinite induction. Assume that $m \in n^+$. If m is empty or a limit ordinal, then there is no problem. The only remaining possibility is that $m = k^+ \in n$ since n is a limit ordinal. Assume that $x \subseteq V_k \subseteq L_n$. Also, let r be the function which assigns to to every $w \in x$ the least ordinal $j \in n$ such that $w \in L_j{}^+$ and let y be the range of r. Since V_n is well-ordered and n sinac by Lemma 2, $y \precsim x \precsim V_k < n$ by Lemma 3 and $y \subseteq n$.

Hence, $i = y \cup \in n$ since n inac. But $x \subseteq L_i^+ \subseteq L_i\#$ and so $x \in L_i\#\# \subseteq L_n$ since either $i^{\#} \in \omega$ or $\omega \subseteq i^{\#}$, the consequent of Lemma 5 holds, and $i^{\#\#} \in n$ since n inac. Consequently, $V_k^+ \subseteq L_n$.

Lemma 7. If x con for every x and n inac, then $M = \langle V_n, \in v_n^+ \rangle$ is a stroongly standard complete model of $ZF^u + C$.

Assume the antecedent. By Lemma 2 and Lemma 4, M is a strongly standard complete model of ZF^u. Also, if $x \in V_n$, there is an $m \in n$ such that $x \subseteq V_m$. Hence if $y \subseteq V_n$, $x \cap y \subseteq x$ and so $x \cap y \in V_m^+ \subseteq V_k$ since n is a limit ordinal. But $V_n = L_n$ by Lemma 6 and so $x \cap y \in L_k^+$ for some k such that $k^+ \in n$. That is, $\models_M C$.

Now assume the antecedent of Theorem 1. Since all the axioms of ZF hold among the constructible sets, so do Lemma 2 through Lemma 7. In addition, the axiom of constructibility for sets holds and there is an inaccessible ordinal by Lemma 1. Finally, the constructible axioms of $ZF^u + C$ are the axioms of $ZF^u + C$, the notions of being a standard complete model and of satisfaction are absolute, and, given a function f from the set of variables into a set u such that u con and formula F, there is a function g from the set of variables into u such that g con and the restriction of g to the set of variables free in F is the same as that of f. Consequently, a constructible set M is a standard complete model of $ZF^u + C$ among the constructible sets just when M is a standard complete model of $ZF^u + C$. By Lemma 7 among the constructible sets, the consequent of Theorem 1 follows. Corollaries 1 and 2 follow immediately by Lemma 2 within ZF^u and Gödel's theorem on the unprovability of consistency.

Lemma 8. If x con for every x and n inac, then there is a standard complete model $M = <U, \in_U>$ of ZF^u + C such that $\omega \in U < \omega$ and $U \subseteq L_\omega \# \#$.

$\sim$

Assume the antecedent. By Lemma 7 and Lemma 2 together with the proof of Theorem 1 of [3], $ZF^u + C$ has a denumerable standard complete model $M = <U, \in_U>$. Let $m = \{k$: k is an ordinal and $k \in U \cup \}$. Since $\models_M C$ and constructibility is absolute, $U = L_m \cup P$ for some $P \subseteq L_m \mathscr{P}$ disjoint from L_m. Clearly, $\omega \in m \subseteq U \underset{\sim}{\leq} \omega$ and $L_m \subseteq L_\omega \#$. Consequently, $U \subseteq L_m \mathscr{P} \subseteq$ $L_\omega \# \mathscr{P} \subseteq L_\omega \# \#$ by Lemma 5.

Since $L_\omega \# \# \subseteq U'$ if $M' = <U', \in_{U'}>$ is a standard complete model of ZF such that $\omega \# < m$ where $m = \{k$: $k \in U'$ and k is an ordinal$\}$, Theorem 2 follows from Lemma 8 among the constructible sets together with Lemma 1.

REFERENCES

GÖDEL, K.

[1] Consistency proof for the generalized continuum hypothesis, *Proc. Nat. Acad. Sci. U.S.A.* 25, 1939.

[2] *The Consistency of the Continuum Hypothesis*, Princeton, 1940.

SCHOCK, R.

[3] On standard models of set theories, *Logique et Analyse*, 63, 1973. (2).

SHEPERDSON, J.

[4] Inner models for set theory II, *J. Symb. Logic 17*, 1952.

(1) Some time after this study had been submitted to the present journal, consistency results for other versions M of ZF^u with AC have been published elsewhere. Of these, the only non-abstract seems to be R. Chuaqui's «Forcing for the impredicative theory of classes» (*J. Symb Logic* 37, 1972). Here, Chuaqui even asserts without proof that the assumption of the existence of an inaccessible cardinal implies the consistency of the axiom of constructibility with his M. In a footnote, he also mentions that L. Tharp has shown the consistency and independence of gch for M in a thesis of 1965 at M.I.T.

(2) In this paper, the clause «besides the axiom of extensionslity» should be put after «sentence» on line 8 from the botton of p. 415. Also, on line 5 from the botton, put «$\wedge$» for «$\times$».