A NEW FORMULATION OF PREDICATIVE SECOND ORDER LOGIC (1)

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In what follows, a predicative second order logic is formulated and shown to be complete with respect to the proposed model theoretic semantics. The logic differs in certain fundamental ways from the system formulated by Church in [1], § 58. The more important differences are noted and discussed throughout the present paper. A more specialized motivation for the new formulation is outlined in § 2.

In regard to the motivation for Church's formulation, this will be found in its natural extension to ramified type theory (without the axiom of reducibility). Within this larger framework, the theory of predication represented by such a formulation can be seen to be constructive: higher order entities are constructible from entities of lower order, with real, non-constructed individuals as the entities of lowest order. Set theory, to whatever extent it is representable in the framework, appears in the ramified hierarchy only after propositional functions are allowed to be arguments of third and higher order predicates. To introduce sets as real, non-constructed individuals of lowest order would be antithetical to the framework's constructive theory of predication and in violation of its philosophical motivation.

Ramified type theory, the natural framework for Church's formulation of predicative second order logic, was Russell's alternative to Frege's *Begriffsschrift,* which by Russell's *paradox of membership* was shown to be inconsistent. Russell's *paradox of predication* (2), however, is the form of the paradox to which Russell himself responded in his formulation of ramified type theory. This was because Russell, unlike Frege, came to construe all sentences about sets (classes) and their properties as reducible to (higher order) sentences about propositional functions and their functions, where propositional functions are always indicated by predicates (or open wffs) and never by expressions for individuals.

Frege, however, in radical contradistinction to any such hierarchy, held to a realist theory of predication according to which the entities indicated by predicates (or open wffs) could never themselves be arguments of predicates. (3) Such entities, also called properties or concepts, were said to be "unsaturated." Sets, or the extensions of properties, however, were saturated objects and could themselves be arguments of predicates. In other words, sets in this framework are real, nonconstructed individuals.

Frege's theory (or a variant thereof) finds its natural representation in an applied form of standard second order logic with set-abstraction (for the representation of extensions of properties) as an individual term making operation. No extension to third and higher order logic is permitted, being in strict violation of the framework's theory of predication. Sets are indicated here by (complex) individual expressions and are therefore arguments of predicates. Properties and relations, however, though quantified over, are indicated by predicate expressions (or open wffs) which are not allowed to occupy the nominal or argument positions of predicates ,or in general the (nominal) argument positions of (open) wffs occupied by individual variables. Russell's *paradox of membership,* but not his paradox of predication, is formulable in this framework. However, the derivation of a contradiction requires, as shown in § 2 below, quantification over impredicatively specified properties and relations, which of course is allowed in standard second order logic. On the other hand, with quantifiers ranging over only predicatively specifiable properties and relations as in the predicative second-order logic formulated here, Russell's argument fails to generate a contradiction and merely shows instead that membership is not a predicative relation. This naturally suggests a reformulation of Frege's theory of predication to include a predicative second order logic such as the system developed here with quantifiers distinguished to range over only predicative properties and relations and to restate the included set theory appropriately. (This need not necessarily exclude standard quantifiers ranging over all properties and relations, whether predicatively specified or not.)

Essential to this proposal, however, is a view of the predicative/impredicative distinction radically different from that found in ramified type theory — and hence in Church's formulation of predicative second order logic. The latter framework (barring the axiom of reducibility) represents a constructive theory of predication that rules out all manner of categorial content (indicated by *bound* predicate variables) or logistic efficacy for impredicative contexts. In the proposed, modified Fregean theory, however, impredicative contexts (wffs) are allowed to have logistic efficacy — and perhaps even categorial content if standard quantifiers ranging over all properties and relations are retained as well.

If, on the one hand, only quantifiers for predicatively specifiable properties and relations are allowed, then in this new formulation of predicative second order logic impredicative contexts — which in general will contain free (schematic) predicate variables or certain predicate constants — will be syncategorematic expressions, since they will not then be permissible substituends of generalized predicate variables. This does not mean that they must then be accorded null content. They may instead represent logical or formal content variant to what Frege calls second and third level "concepts". ([4]) This logical content would in effect be the basis of their logistic efficacy. ([5])

If, on the other hand, these impredicative contexts are to be given categorial content by retaining standard quantifiers, then care must be made to distinguish these quantifiers from those ranging over only predicatively specifiable properties and relations. Both kinds of quantifiers will bind the same variables, but impredicative wffs will be permissible substituends only of variables bound by the one quantifier. ([6]) They remain impermissible substituends of variables bound by the quantifiers for predicative properties and relations.

In the system to be formulated here we are concerned only with the first of the above alternatives, although once formulated it is easily extended to the richer framework. (7)

§ 1 *Terminology:*

For our new formulation of predicative second order logic we take as *primitive logical constants* the (material) conditional sign, $\rightarrow$; the negation sign, $\sim$; the universal quantifier for quantification over individuals, $\wedge$; and $\wedge!$,the universal quantifier for quantification over predicative properties and relations. Since indiscernibility with respect to predicative properties will not suffice for full substitutivity, we shall also take $\equiv$ as a primitive identity sign. Other logical constants are understood to be defined (in the syntactical metalanguage) in the usual manner.

We assume a denumerable list of pairwise distinct *individual variables* and, for each positive integer n, a similar list of n-*place predicate variables*. We shall use 'α', 'β', 'γ' (in the syntactical metalanguage) to refer to individual variables, 'π', 'ϱ', 'σ' to refer to predicate variables, and 'μ', 'ν' (occasionally) to refer to both individual and predicate variables.

The only non-logical or descriptive constants we shall concern ourselves with here are *individual* and n-*place predicate constants*, for arbitrary positive integers n. By a *language* we understand a set of such descriptive constants.

Where L is a language, the *terms* of L are the individual variables and the individual constants in L. The atomic wffs of L are either identity formulas of the form $\zeta \equiv \eta$, where ζ, η are terms of L, or simple predications of the form $\pi(\zeta_1,\ldots,\zeta_n)$, where π is either an n-place predicate variable or an n-place predicate constant in L, and $\zeta_1,\ldots,\zeta_n$ are terms of L. The wffs of L constitute the smallest set containing the atomic wffs of L and closed under the formation of conditionals, negations and (universal) generalizations of wffs by either $\wedge$ affixed to an individual variable or $\wedge$! affixed to a predicate variable. We shall use 'ζ', 'η', 'ξ' (in the syntactical metalanguage) to refer to terms and 'φ', 'ψ', 'χ' to refer to wffs. Bondage and freedom of (occurrences of) variables in wffs is understood in the usual way as well as proper substitution of a term for an individual variable in a wff. If ζ can be properly substituted for α, then $\varphi[{}^{\alpha}_{\zeta}]$ is the wff resulting from this substitution; but if ζ cannot be properly substituted for α in φ, then $\varphi[{}^{\alpha}_{\zeta}]$ is just φ itself. In regard to the proper substitution of wffs for predicate variables we adopt the definition and notation given in Church, *op. cit.*, pp. 192f., except for abbreviating

$$\check{S}^{\sigma(\alpha_1,\ldots,\alpha_n)}_{\pi(\alpha_1,\ldots,\alpha_n)}\varphi]$$

by $\varphi[{}^{\sigma}_{\pi}]$, which is used only when n-place predicate variables or constants are substituted for n-place predicate variables.

§ 2 *Motivation:*

Before proceeding to the new formulation of predicative second order logic which is intended to reflect the suggested modified Fregean theory of predication, let us briefly consider the original framework of (pure) standard second order logic with set-abstraction as an individual term making operation. Ignore for the moment $\bigwedge!$, the quantifier for predicative properties and relations, and instead affix $\bigwedge$ to predicate as well as individual variables, understanding it then to range over all properties and relations, whether predicatively specified or not. Impredicative properties and relations are acknowledged in this logic via *the comprehension principle:*

$$\text{(CP)} \quad \bigvee\pi\bigwedge\alpha_1\ldots\bigwedge\alpha_n[\pi(\alpha_1,\ldots,\alpha_n) \leftrightarrow \varphi]$$

where φ is any wff of (pure) standard second order logic in which the n-place (n a positive integer) predicate variable π has no (free) occurrences and $\alpha_1,\ldots,\alpha_n$ are among the distinct individual variables occurring free in φ. If φ contains essential occurrences of bound predicate variables [8], φ is understood to represent an impredicative n-ary relation (or property if n = 'i'); and (CP) posits the (categorial) existence of this n-ary impredicative relation. [9]

The sense in which the (categorially) posited n-ary relation represented by φ, with respect to the variables $\alpha_1,\ldots,\alpha_n$ (as argument indicators), is impredicative (relative to the system in question) can perhaps be best grasped when φ contains essential occurrences of bound n-place predicate variables; for then, on the assumption that these occurrences really are essential, that relation can be comprehended in the system only by a wff which must contain bound occurrences of a variable of which that relation is itself a value, or, equivalently, of which that wff is itself a substituend (with respect to the individual variables in question). [10] If the essential occurrences of bound predicate variables in φ are of a degree k other than n, then the posited n-ary relation's impredicativity can be grasped derivatively through considering φ relative now to the arguments $\alpha_1,\ldots,\alpha_k$ if $k<n$, or if $n<k$ through considering $(\varphi \wedge \alpha_{n+1} \equiv \alpha_{n+1} \wedge \ldots \wedge \alpha_k \equiv \alpha_k)$, where $\alpha_{n+1},\ldots,\alpha_k$ are new to φ. The k-ary relation now posited by (CP) is seen to be impredicative in the preceding direct sense since it is represented by a wff containing an essential occurrence of a variable of which it is a value, and its systematic or provable connection with the n-ary relation in question as the latter's contraction or expansion indicates the sense in which this latter relation must therefore also be impredicative.

This sense of impredicativity, it should be noted, is *immanent* to the logistic system in question, standard second order logic in the present case, since it depends on what is provable in the pure form of that system. [11] Immanence, however, does not signify nullity of content. The point rather is that such content as impredicativity does contain is not independent of the logical structure which is imputed to properties and relations by the logistic system in question.

Accordingly, the restriction to the pure for mof the system is relevant here, since otherwise a predicate constant can always be introduced to (contingently) represent the relation in question (in some particular model), in which case the relation can trivially be represented by a wff containing no bound predicate variables. But whether a property or relation is represented as predicative or impredicative in a logistic system is not a contingent issue involving an "external" relation between the system and that property or relation, the way, for example, whether a property is possessed or not might be. Rather, because of the immanency of impredicativity, the relationship involved is an "internal" one concerning the logical structure which is imputed to the property or relation by the system.

Let us now consider the role of set-abstraction in the present framework. Applied to an (open) wff and a variable (as argument indicator), set-abstraction is intended to result in the extension of the property represented by that wff with respect to that variable. Existentially positing a membership relation satisfying extensionality and the unrestricted form of the conversion principle:

$$\text{(Conv)} \quad \bigwedge\pi\bigwedge\alpha[\alpha\in\{\beta:\pi(\beta)\}\leftrightarrow\pi(\alpha)]$$

results in a system variant to Frege's ([12]), and its inconsistency is the result of Russell's paradox of membership. Note however that an impredicative instance of (CP) is required for the derivation of this paradox. The instance in question posits that non-self-membership is a property ([13]):

$$\bigvee\pi\bigwedge\alpha[\pi(\alpha)\leftrightarrow\alpha\notin\alpha]$$

For it is only this posit which justifies instantiating (Conv) to $\alpha\notin\alpha$ as a substituend of the generalized 1-place predicate variable and thereby obtain Russell's contradiction (via a further instantiation of the individual variable to the term $\{\alpha:\alpha\notin\alpha\}$. ([14]) Observe, however, that the property of non-self-membership is imputed some complexity by the system and that that complexity is at least two-fold: it is the *complement* of the property of self-membership which itself is already imputed complexity in being a (reflexive) *relational property* based upon the impredicatively specified membership relation. That membership is impredicative is obvious since its characterization or imputed structure in the present system involves a wff, (Conv), with an essential occurrence of a bound predicate variable. ([15]) Furthermore, it is a presumed principle that a reflexive relational property is impredicative if the relation it is based upon is impredicative. Accordingly, both self-membership and its complement, non-self-membership, are impredicative properties. Thus Russell's argument, as claimed above, requires an impredicative instance of (CP) for its validation. ([16])

In contrast to the validation procedures of standard second order logic with its commitment to categorial impredicativity, let us consider now the variant of predicative second order logic to be developed here, retaining for the present set-abstraction as an individual term making operation. We return to affixing $\bigwedge!$ rather than $\bigwedge$ to predicate variables and understand variables bound thereby to range over only predicative properties and relations. Free predicate variables, however, and predicate constants as well in applied forms of the system, may refer to either impredicative or predicative properties and relations as their values. Impredicativity is in this way accorded only syncategorial existence.

This semantic distinction between bound predicate variables on the one hand and, on the other, free predicate variables and constants is analogous to certain formulations of first order modal logic such as that of Kripke [4]. In the latter, bound occurrences of individual variables refer only to objects existing in the world in question while free occurrences of individual variables, and individual constants as well, refer to possible individuals that need not exist in that world. In both sorts of context it is false to claim, as for example Kripke has claimed (*op. cit.*, p. 89), that assertion of wffs containing free variables is at best a convenience which can always be replaced by assertion of the universal closure of these wffs.

In addition, because in the present system a wff containing no bound predicate variables may contain free predicate variables or constants which may in a given model refer to impredicative properties or relations, we must not confuse or identify the semantical or ontological predicative/impredicative distinction with the purely syntactical distinction between wffs containing and wffs not containing essential occurrences of bound predicate variables. In Church's formulation, no parallel distinction is to be drawn: free as well as bound predicate variables and predicate constants, one and all, refer only to predicative properties and relations.

In the present formulation, the comprehension principle for predicative properties and relations takes the form:

$$\text{(CP!)} \quad \bigwedge !\sigma_1 \ldots \bigwedge !\sigma_k \bigvee !\pi \bigwedge \alpha_1 \ldots \bigwedge \alpha_n [\pi(\alpha_1,\ldots,\alpha_n) \leftrightarrow \varphi]$$

where φ is a wff in which no predicate constants or the identity sign occur and in which no predicate variable has a bound occurrence, $\sigma_1,\ldots,\sigma_k$ are all the predicate variables occurring (free) in φ, $\alpha_1,\ldots,\alpha_n$ are among the distinct individual variables occurring free in φ, and π is an n-place predicate variable which does not occur in φ. Other than replacing (CP) by (CP!) and using $\bigwedge!$ in place of $\bigwedge$ when affixed to predicate variables, the remaining axioms of second oredr logic are retained in this system. Identity, however, must here be represented by a primitive logical constant (or 2-place predicate constant representing a syncategorial impredicative relation) satisfying reflexivity and full substitutivity. Indiscernibility with respect to predicate properties is of course a necessary but not also a sufficient condition for identity; and, accordingly, indiscernibility cannot suffice here as a definition for identity.

Unlike Church's formulation, in the present system the principle of universal instantiation of wffs for generalized predicate variables has a qualified form even when the substituend contains no bound predicate variables:

$$\text{(UI!)} \quad \bigvee !\sigma \bigwedge \alpha_1 \ldots \bigwedge \alpha_n [\sigma(\alpha_1,\ldots,\alpha_n) \leftrightarrow \varphi] \rightarrow [\bigwedge !\pi\psi \rightarrow$$

$$!\bigwedge_1 \ldots \bigwedge_n [(_1,\ldots,_n)\ ] \rightarrow [\bigwedge ! \rightarrow \check{S}\varphi^{\pi(\alpha_1,\ldots,\alpha_n)}\psi]$$

(where σ does not occur free in φ). The reason for this qualification is that the wff φ may contain free occurrences of predicate variables, or a predicate constant, which refer to impredicative properties or relations. Only by stipulating that φ, relative to the variables $\alpha_1,\ldots,\alpha_n$, (as argument indicators) represents an n-ary predicative relation are we justified in taking φ to be a permissible substituend of a generalized n-place predicate variable — for as such a substituend φ, relative to the variables $\alpha_1,\ldots,\alpha_n$, then represents a value of that variable. This explains, furthermore, the restriction in (CP!) that φ contain no predicate constants (or the identity sign). Were such an instance of (CP!) desired because the embedded constant represented a predicative property or relation after all, then it can be derived from (UI!), on the assumption that the qualifying antecedent of (UI!) is satisfied for that constant. Such an assumption would normally be stipulated in the form of a meaning postulate.

Another variant of the instantiation law for predicate variables is the following consequence of (UI!) and (CP!):

$$(\text{UI!}_2) \quad \bigwedge !\sigma_1 \ldots \bigwedge !\sigma_k[\bigwedge !\pi\psi \rightarrow \check{S}\varphi^{\pi(\alpha_1,\ldots,\alpha_n)}\psi]$$

where φ, π, $\sigma_1,\ldots,\sigma_k$, $\alpha_1,\ldots,\alpha_n$ are as described in (CP!) above. In Church's formulation this last principle, (UI!$_2$), *without the quantifier prefix* on the predicate variables occurring (free) in φ is an axiom schema, and (CP!), without the same initial quantikfier prefix, is a theorem schema of that system. In the present system, however, the quantifier prefix cannot be dropped without assuming

$$\bigvee !\varrho_j \bigwedge \beta_1 \ldots \bigwedge \beta_i[\varrho_j(\beta_1,\ldots,\beta_i) \leftrightarrow \sigma_j(\beta_1,\ldots,\beta_i)]$$

for each $j \leq k$, where ϱ_j and σ_j are distinct i-place predicate variables, for some positive integer i. The point of the distinction once again is that in Church's system all predicate variables, bound or free, refer only to predicative properties and relations.

Let us now return to Russell's argument and reconsider membership as one of the values of the 2-place predicate variables satisfying, besides extensionality, the conversion principle for sets that are the extensions of predicative properties:

$$\text{(Conv!)} \quad \bigwedge !\pi \bigwedge \alpha[\alpha \in \{\beta : \pi(\beta)\} \leftrightarrow \pi(\alpha)]$$

Because of the obvious impredicative character of any relation satisfying (Conv!), we take '$\in$' in this context to be a free 2-place predicate variable not existentially posited to represent a predicative relation. Indeed, instead of inconsistency, all that Russell's argument shows in the present context is that any relation satisfying (Conv!) must for that reason be impredicative. For by (UI!) and (Conv!) it follows that non-self-membership is not a predicative property:

$$\sim \bigvee !\pi \bigwedge \alpha[\pi(\alpha) \leftrightarrow \alpha \notin \alpha]$$

And therefore, by (UI!) and (CP!) membership is not a predicative relation:

$$\sim \bigvee !\varrho \bigwedge \alpha \bigwedge \beta[\varrho(\alpha,\beta) \leftrightarrow \alpha \in \beta]$$

Here we have a striking contrast of the syncategorematic roll of impredicativity in the present system (as represented by means of free predicate variables and constants) with the categorial role of impredicativity in standard second order logic. Of course, were we to actually construct a set theory in the present framework, e.g., by introducing a 2-place predicate constant for membership, we should have to extend (Conv!) so as to allow conversion with respect to at least certain wffs containing '$\in$' and therefore representing impredicative conditions. Such an extension would not be the basis of an argument against the predicative logic formulated here but would rather illustrate one of the special senses in which impredicativity might be said to enter at the very foundations of mathematics. ([17])

It is noteworthy that predicate constants not representing predicative properties or relations in such applications of the logic as envisaged above need not be viewed as descriptive or categorial signs. Since they are not substituends of any bound variable of the logic such constants do not represent a value of any of the bound variables and are in this special sense syncategorematic signs akin to logical constants. Free predicate variables of course can also always be viewed as schema letters and theorems in which they occur as theorem schemata, though, as noted earlier, by no means should the content of the latter be seen to be representable in the object language by their universal closures.

In addition, in such a set theory as envisaged above, where n-tuples of individuals are themselves individuals, we might even exclude bound n-place predicate variables for $n>1$, since predicative n-ary relations can then be construed as predicative properties of n-tuples of individuals. Only predicative or first-order properties or concepts (*Begriffe*) need then remain as the "unsaturated" entities of the original Fregean framework. But to banish these too in the end would be to depart completely from a Fregean theory of predication.

§ 3 *Semantics:*

Where L is a language, i.e., a set of individual and predicate constants, we understand $\mathfrak{A} = \langle A, R, \langle X_n \rangle_{n\in\omega-\{0\}} \rangle$ to be a *model* suited to L if (1) A is a non-empty set (called the *universe* of $\mathfrak{A}$), (2) R is a function with L as domain and such that $R(\zeta) \in A$ whenever ζ is an individual constant in L and, for each positive integer n, $R(\pi) \subseteq A^n$ whenever π is an n-place predicate constant in L, and (3) $\langle X_n \rangle_{n\in\omega-\{0\}}$ is a family of sets indexed by $\omega-\{0\}$, the set of positive integers (natural numbers other than 0) and for each positive integer n, X_n is a set of subsets of A^n. The set X_n represents the predicative n-ary relations of the model. Observe that an n-place predicate constant π need not represent a predicative relation of the model, i.e., $R(\pi)$ need not be a member of X_n. For convenience, we set $U_{\mathfrak{A}} = A$, i.e. $U_{\mathfrak{A}}$ is the universe of the model $\mathfrak{A}$, and whenever δ is an individual or predicate constant in L, we set $\delta_{\mathfrak{A}} = R(\delta)$.

By an *assignement* in $\mathfrak{A}$ of values (drawn from $\mathfrak{A}$) to variables we understand a function $\mathfrak{a}$ with the set of individual and predicate variables as its domain and such that (1) $\mathfrak{a}(\alpha) \in U_{\mathfrak{A}}$ whenever α is an individual variable, and (2) $\mathfrak{a}(\pi) \subseteq U_{\mathfrak{A}}{}^{n}$ whenever π is an n-place predicate variable for some positive integer n. Note that the value of an n-place predicate variable need not be a predicative relation of the model. By $\mathfrak{a}\binom{\mu}{z}$ we understand the assignment which is exactly like $\mathfrak{a}$ except in its assigning z to the variable μ.

Where L is a language, $\mathfrak{A} = \langle A, R, \langle X_n \rangle_{n \in \omega - \{0\}} \rangle$ is a model suited to L, and $\mathfrak{a}$ is an assignment in $\mathfrak{A}$, *the extension* in $\mathfrak{A}$ with respect to $\mathfrak{a}$ of a variable or constant δ in L, in symbols ext $(\delta, \mathfrak{A}, \mathfrak{a})$, is $\mathfrak{a}(\delta)$ if δ is a variable and $\delta_{\mathfrak{A}}$ if δ is a constant in L. *Satisfaction* in $\mathfrak{A}$ by $\mathfrak{a}$ of a wff φ of L is recursively defined in the usual way: (1) for terms ξ, η of L, $\mathfrak{a}$ satisfios $\zeta \equiv \eta$ in $\mathfrak{A}$ iff ext $(\zeta, \mathfrak{A}, \mathfrak{a})$ = ext $(\eta, \mathfrak{A}, \mathfrak{a})$; (2) where $n \in \omega - \{0\}$, π is an n-place predicate variable or constant in L and $\zeta_1, \ldots, \zeta_n$ are terms of L, $\mathfrak{a}$ satisfies $\pi(\zeta_1, \ldots, \zeta_n)$ in $\mathfrak{A}$ iff $\langle$ext $(\zeta_1, \mathfrak{A}, \mathfrak{a}), \ldots,$ ext $(\zeta_n, \mathfrak{A}, \mathfrak{a})\rangle \in$ ext $(\pi, \mathfrak{A}, \mathfrak{a})$; (3) for a wff φ of L, $\mathfrak{a}$ satisfies $\sim\varphi$ in $\mathfrak{A}$ iff $\mathfrak{a}$ does not satisfy φ in $\mathfrak{A}$; (4) for wffs φ, ψ of L, $\mathfrak{a}$ satisfies $(\varphi \rightarrow \psi)$ in $\mathfrak{A}$ iff either $\mathfrak{a}$ does not satisfy φ in $\mathfrak{A}$ or $\mathfrak{a}$ satisfies ψ in $\mathfrak{A}$; (5) for φ a wff of L and α an individual variable, $\mathfrak{a}$ satisfies $\bigwedge\alpha\varphi$ in $\mathfrak{A}$ iff for each $x \in U_{\mathfrak{A}}$, $\mathfrak{a}\binom{\alpha}{x}$ satisfies φ in $\mathfrak{A}$; and for φ a wff and π an n-place predicate variable, n a positive integer, $\mathfrak{a}$ satisfies $\bigwedge!\pi\varphi$ in $\mathfrak{A}$ iff for all $F \in X_n$, $\mathfrak{a}\binom{\pi}{F}$ satisfies φ in $\mathfrak{A}$. Finally, *truth* in $\mathfrak{A}$ as a semantic property of wffs of L is defined as satisfaction in $\mathfrak{A}$ by every assignment in $\mathfrak{A}$. Note that the truth (in $\mathfrak{A}$) of a wff with free predicate variables is not equivalent to the truth (in $\mathfrak{A}$) of its universal closure, since predicate variables may have properties and relations as values that are not predicative (in $\mathfrak{A}$).

Because predicativity is immanent to the logistic system to be formulated, the model theory presented here is not intended to specify the exact meaning of that concept. At best, as with model-theoretic semantics in general, it purports to constitute a set-theoretical representation of the system's logical structure by means of certain invariant semantical features. ([18]) But to do so, a further condition must be imposed: the predicative properties and relations of a model must be closed under all first-order (predicative) logical operations. Or, equivalently, every instance of (CP!) must be true in the model. Those models for which this further condition holds will be said to be *normal*. ([19]) Validity, or the relevant invariant semantic feature, is then identified with truth in every normal model, i.e., a wff of a language L is *valid* if it is true in every normal model suited to L.

The following semantic lemma will be useful in proving completeness. Its proof, which we omit here, is by a simple inductive argument on the structure of φ.

Semantic Lemma: If $\mathfrak{a}$ is an assignment in $\mathfrak{A}$ and $\mathfrak{A}$ is a model suited to a language of which $\varphi[{}^{\alpha}_{\zeta}]$ and $\varphi[{}^{\sigma}_{\pi}]$ are wffs, where the substitutions are proper, then:

1) $\mathfrak{a}({}^{\alpha}_{ext(\zeta,\, \mathfrak{A},\, \mathfrak{a})})$ satisfies φ in $\mathfrak{A}$ iff $\mathfrak{a}$ satisfies $\varphi[{}^{\alpha}_{\zeta}]$ in $\mathfrak{A}$; and

2) $\mathfrak{a}({}^{\sigma}_{ext(\pi,\, \mathfrak{A},\, \mathfrak{a})})$ satisfies φ in $\mathfrak{A}$ iff $\mathfrak{a}$ satisfies $\varphi[{}^{\sigma}_{\pi}]$ in $\mathfrak{A}$.

§ 4. *Axioms and Theorems:*

As the axioms of our formulation we take (CP!) and universal generalization of all wffs of any of the following forms:

(A1) $\varphi \rightarrow (\psi \rightarrow \varphi)$

(A2) $(\varphi \rightarrow [\psi \rightarrow \chi]) \rightarrow ([\varphi \rightarrow \psi] \rightarrow [\varphi \rightarrow \chi])$

(A3) $(\sim\psi \rightarrow \sim\varphi) \rightarrow (\varphi \rightarrow \psi)$

(A4) $\bigwedge\alpha(\varphi \rightarrow \psi) \rightarrow (\bigwedge\alpha\, \varphi \rightarrow \bigwedge\alpha\, \psi)$

(A5) $\bigwedge !\pi(\varphi \rightarrow \psi) \rightarrow (\bigwedge !\pi\varphi \rightarrow \bigwedge !\pi\psi)$

(A6) $\varphi \rightarrow \bigwedge\alpha\varphi$, where α is not free in φ,

(A7) $\varphi \rightarrow \bigwedge !\pi\varphi$, where π is not free in φ,

(A8) $\bigvee\alpha\zeta \equiv \alpha$, where α does not occur in ζ,

(A9) $\zeta \equiv \eta \rightarrow (\varphi \rightarrow \psi)$, where φ, ψ are atomic wffs and ψ is obtained from φ by replacing an occurrence of η in φ by ζ.

Theorems are generated in the usual manner as terminal wffs of proof sequences the constituents of which are either axioms or obtained from preceding wffs of the sequence by *modus ponens*, the only inference rule of the system. A set Γ of wffs is said to *yield* a wff φ, in symbols $\Gamma \vdash \varphi$, if there are a natural number n and n wffs $\psi_0, \ldots, \psi_{n-1}$ in Γ such that $(\psi_0 \rightarrow (\psi_1 \rightarrow \ldots \rightarrow (\psi_{n-1} \rightarrow \varphi)\ldots))$ is a theorem. (When $n = 0$, this last wff is to be φ itself.)

Because of (A1)-(A3), every tautologous wff is a theorem. Moreover, because every (universal) generalization of an axiom is an axiom and, by (A4)-(A5), generalization is preserved by modus ponens, the rule of generalization (for predicate and individual variables) is derivable. Then, utilizing generalization and (A4)-(A7) in a simple inductive argument on the structure of wffs, the interchange law for provably equivalent wffs is derivable. Leibniz' law,

(LL) $\vdash \zeta \equiv \eta \rightarrow (\varphi \leftrightarrow \psi)$

where ψ comes from φ by replacing one or more free occurrences of η by a free occurrence of ζ, is now provable by an induction on the subwffs of φ. In the atomic case, (LL) is obtained from (A9) and sentential logic; otherwise, (LL) is obtained by an inductive argument utilizing sentential logic, (A4)-(A7) and generalization.

Given (LL), the principle of universal instantiation of a term for a generalized individual variable, (UI - 1) $\vdash \bigwedge\alpha\varphi \rightarrow \varphi[{}^{\alpha}_{\zeta}]$

is derivable. For by repeated application of (LL), $[\zeta \equiv \alpha \rightarrow (\varphi \leftrightarrow \varphi[{}^{\alpha}_{\zeta}])]$ is provable, and therefore by generalization on α, sentential logic, axioms (A4), (A6), and (A8), (UI-1) follows. And with (UI-1), generalization, (A4), (A6), sentential logic and the interchange rule for provably equivalent wffs, the rule for rewriting bound individual variables follows.

By a similar argument, the second order analogues of universal instantiation, (UI!), (UI!$_2$), and the rewrite rule for bound predicate variables are also provable. The analogue of (LL) required is:

(LL$_2$) $\vdash \bigwedge\alpha_1 \ldots \bigwedge\alpha_n[\pi(\alpha_1,\ldots,\alpha_n) \leftrightarrow \varphi] \rightarrow [\psi \leftrightarrow \check{S}^{\pi(\alpha_1,\ldots,\alpha_n)}_{\varphi}\psi]$.

which is proved by an inductive argument on the subwffs of ψ. [20] If ψ is atomic, (LL$_2$) follows by n applications of (UI-1); otherwise, (LL$_2$) follows by the inductive hypothesis, sentential logic, generalization, (A4)-((A7) and the fact that the substitution is voided (in which case (LL$_2$) is tautologous) if ψ begins with a universally quantified variable (other than $\alpha_1,\ldots,\alpha_n$) which occurs free in φ. If π does not occur free in φ, then by generalization on (LL$_2$) sentential logic, (A5) and (A7), the following qualified form of (UI!) is obtained:

$$\vdash \bigvee!\pi\bigwedge\alpha_1\ldots\bigwedge\alpha_n[\pi(\alpha_1,\ldots,\alpha_n) \leftrightarrow \varphi] \rightarrow [\bigwedge!\pi\psi \rightarrow \check{S}^{\pi(\alpha_1,\ldots,\alpha_n)}_{\varphi}\psi]$$

This form suffices to establish the rewrite rule for bound predicate variables from which the unqualified form of (UI!) follows by a rewrite of the bound occurrence of π in the antecedent of the qualified form to some n-place predicate variable σ new to φ. To prove the rewrite rule, let φ in the above form be $\sigma(\alpha_1,\ldots,\alpha_n)$, where σ is an n-place predicate variable not occurring in $\bigwedge !\pi\psi$. Then by generalization, (A5) and (A7) applied to this instance of the qualified form,

$$\vdash \bigwedge !\sigma \bigvee !\pi \bigwedge \alpha_1 \ldots \bigwedge \alpha_n[\pi(\alpha_1,\ldots,\alpha_n) \leftrightarrow \sigma(\alpha_1,\ldots,\alpha_n)] \rightarrow (\bigwedge !\pi\psi \rightarrow \bigwedge !\sigma\psi[{}^{\pi}_{\sigma}])$$

and therefore, by (CP!),

$$\vdash \bigwedge !\pi\psi \rightarrow \bigwedge !\sigma\psi[{}^{\pi}_{\sigma}]$$

from which, together with the interchange rule for provably equivalent wffs, the rewrite rule follows, and, accordingly, (UI!) is also thereby established. To obtain (UI!$_2$) where φ, the substituend, contains no predicate constants or the identity sign and no bound predicate variables, we need only apply to (UI!) the rule of generalization (for each predicate variable free in φ) and then by (A5), (CP!) and modus ponens, (UI!$_2$) follows.

Neither (UI!) nor (UI!$_2$) justifies the substitution rule:

(S) if $\vdash \psi$, then $\vdash \check{S}^{\pi(\alpha_1,\ldots\alpha_n,)}\psi$

even when φ contains no bound predicate variables. Yet, whether φ contains bound predicate variables or not, this rule is derivable (whether π is a predicate variable or a predicate constant). For suppose $\chi_1,\ldots\chi_k$ is a proof of ψ. Rewrite each χ_i in this proof to a wff χ'_i which contains no bound variable in common with φ. By selecting enough new variables not occurring in either φ or any χ_i, this rewriting can be done uniformly throughout the entire proof sequence so that bound occurrences of the same variable are rewritten throughout to bound occurrences of the same new variable. The resulting sequence $\chi'_0,\ldots,\chi'_k$ is now a proof of ψ', since where χ_i is an axiom, χ'_i is also an axiom and where χ_i is obtained by modus ponens from preceding wffs of the original sequence χ'_i is obtained by modus ponens from the rewrites of the same wffs. Now replace each wff χ'_i of this new proof sequence by $S^{\pi(\alpha_1,\dots\alpha_n,)}\chi'$ a substitution which is proper in each case since there can be no clash of bound variables between φ and χ'_i. The new sequence is now a proof of $S^{\pi(\alpha_1,\dots,\alpha_n)}_{\varphi}\psi'$, since the substitution of φ (with respect to the variables $\alpha_1,\dots,\alpha_n$) for free occurrences of π in an axiom results is an axiom, and modus ponens, because of the rewriting, preserves the substitution throughout. Finally, by the rewrite rule, $S^{\pi(\alpha_1,\dots,\alpha_n)}_{\varphi}\psi'$ can be transformed to $S^{\pi(\alpha_1,\dots,\alpha_n)}_{\varphi}\psi$, unless there is a clash of variables, in which case the original substitution was void and (S) is a trivial redundancy.

It is noteworthy that adding (S) to Church's formulation of predicative second order logic results in a system equivalent to standard second order logic ([21]), and thus (S) would in effect nullify that system's basic motivation. Not so with the present formulation. And the reason essentially is that free predicate variables and predicate constants may represent impredicative content even though bound predicate variables range only over predicative properties and relations. That content, as noted in the introduction, need not be construed categorially.

Compare in this regard the nominalist who eschews all bound predicate variables and who interprets free predicate variables as schema letters. Predicate constants for such a nominalist do not represent categorial content but are instead a special type of syncategorematic sign. (Cf. [5]) Yet the substitution rule, (S), barring bound predicate variables, is valid for this nominalist. Indeed, for him the categorial content represented by quantification over properties and relations is replaced by a formal *content* immanent to the logistic system and represented by a metalinguistic quantification over predicates and (open) wffs. The (metalinguistic) substitution rule is in effect the nominalists' (weaker) replacement of the categorial principle of universal instantiation of wffs for generalized predicate variables.

Analogously the (metalinguistic) rule (S) for the present formulation of predicative second order logic amounts to a (weaker) replacement of the categorial principle (UI) of standard second order logic. Impredicative contexts, i.e., wffs containing essential occurrences of bound predicate variables, are allowed here a logistic efficacy which is not null but which need not be interpreted categorially.

Before concluding this section, we shall need for our proof of completeness two further derivable rules regarding generalization. Where ζ is a term and π is a predicate variable or constant neither of which occur in any wff in Γ, then:

(a) if $\Gamma \vdash \varphi[{}_{\zeta}]$, then $\Gamma \vdash \bigwedge \alpha\varphi$;

(b) if $\Gamma \vdash \varphi[{}^{\sigma}_{\pi}]$, then $\Gamma \vdash \bigwedge \sigma\varphi$.

These rules are proved in the usual manner which we shall omit repeating here. Similarly, we omit proof of the deduction theorem which is easily seen to hold by the usual argument.

§ 5. *Soundness and Completeness:*

Proof of the soundness of our axioms relative to the proposed semantics is straightforward. For, by definition, (CP!) is true in every normal model and is therefore valid. The remaining axioms, (A1)-(A9), are easily seen to be true (by definition of satisfaction) in every model (in the language of which they are wffs), whether normal or not, and, accordingly (A1)-(A9) are valid. ((A6)-(A7) require noting that an assignment's satisfaction of a wff depends on the variables free in that wff and no others.) Finally, since modus ponens preserves validity, every theorem is therefore valid.

Before proceeding to our proof of completeness, it is noteworthy to point out that each instance of (CP!) is *vacuously true* in a model with no predicative properties and relations, and that therefore such models are normal. [22] Moreover, since normalcy is not required for the remaining axioms or for modus ponens, the following wff:

(Null!$_n$) $\sim \bigvee !\pi \bigwedge \alpha_1 \ldots \bigwedge \alpha_n[\pi(\alpha_1,\ldots,\alpha_n) \leftrightarrow \sigma(\alpha_1,\ldots,\alpha_n)]$

is therefore consistent, where n is a positive integer and π, σ are distinct n-place predicate variables. The free occurrence of σ in (Null!$_n$) indicates a schematic or metalinguistic quantification over all n-ary relations (representable by wffs of the system) to the effect that none of them are predicative. That (Null!$_n$) is consistent here is as it should be, though of course in Church's formulation (Null!$_n$) is not consistent. That formulation, however, represents a constructive theory of predication, while the present system represents a realist theory of predication along Fregean lines. And for the latter type of framework, whether or not there exists a predicative n-ary relation for any given positive integer n is clearly a contingent matter. (Experience, of course, may *verify* that for certain positive integers, e.g., 1 or 2, there exist predicative properties or relations of that degree.)

Completeness Theorem: Every consistent set of wffs is simultaneously satisfiable in some normal model.

Proof: Suppose Γ is a consistent set of wffs of some language L, and let λ be the least infinite ordinal equinumerous with or greated than L. As usual, we supplement L with new individual constants $\zeta_0,\ldots,\zeta_\mu,\ldots(\mu\in\lambda)$ and, for each $n\in\omega-\{0\}$, new n-place predicate constants $\pi^n_0,\ldots,\pi^n_\mu,\ldots(\mu\in\lambda)$. Call the resulting language L'.

Let $\Sigma_1,\ldots,\Sigma_\mu,\ldots(\mu\in\lambda)$ be an ordering of the set of wffs of L' of the form $\bigvee\alpha\varphi$ of $\bigvee!\sigma\varphi$. By ordinal recursion, we define the chain $\Gamma_0,\ldots,\Gamma_\mu,\ldots(\mu\in\lambda)$ as follows: (1) $\Gamma_0 = \Gamma$; (2) if $\bigcup_{\mu\in\nu}\Gamma_\mu\cup\{\Sigma\nu\}$ is not consistent, then $\Gamma_\nu = \bigcup_{\mu\in\nu}\Gamma_\mu$; (3) if $\bigcup_{\mu\in\nu}\Gamma_\mu\cup\{\Sigma_\nu\}$ is consistent, then:

(a) if $\Sigma_\nu = \bigvee\alpha\varphi$, for some wff φ of L′ and some individual variable α, then $\Gamma_\nu = \bigcup_{\mu\in\nu}\Gamma_\mu \cup \{\varphi[{}^{\alpha}_{\zeta_\iota}]\}$, where ι is the least ordinal such that ζ_ι does not occur in any member of $\bigcup_{\mu\in\nu}\Gamma_\mu \cup \{\Sigma_\nu\}$;

(b) if $\Sigma_\nu = \bigvee!\sigma\varphi$ for some wff φ of L′ and some n-place predicate variable σ, then $\Gamma_\nu = \bigcup_{\mu\in\nu}\Gamma_\mu \cup \{\varphi[{}^{\sigma}_{\pi^n_\iota}], \bigvee!\varrho\bigwedge\alpha_1\ldots\bigwedge\alpha_n[\varrho(\alpha_1, \ldots,\alpha_n) \leftrightarrow \pi^n_\iota(\alpha_1, \ldots,\alpha_n)]\}$, where ϱ is the first n-place predicate variable different from σ, $\alpha_1,\ldots,\alpha_n$ are the first n individual variables, and ι is the least ordinal such that π^n_ι does not occur in any member of $\bigcup_{\mu\in\nu}\Gamma_\mu \cup \{\Sigma_\nu\}$.

By induction, we show first that each Γ_ν is consistent, for $\nu\in\lambda$. If $\bigcup_{\mu\in\nu}\Gamma_\mu \cup \{\Sigma_\nu\}$ is not consistent, then Γ_ν is consistent by definition and the inductive hypothesis. Suppose then that $\bigcup_{\mu\in\nu}\Gamma_\mu \cup \{\Sigma_\nu\}$ is consistent but that Γ_ν is not consistent. Then, in case $\Sigma_\nu = \bigvee\alpha\varphi$, by the deduction theorem

$$\bigcup_{\mu\in\nu}\Gamma_\mu \vdash \sim\varphi[{}^{\alpha}_{\zeta_\iota}]$$

and therefore, by generalization on a term which does not occur in any wff in $\bigcup_{\mu\in\nu}\Gamma_\mu$,

$$\bigcup_{\mu\in\nu}\Gamma_\mu \vdash \bigwedge\alpha\sim\varphi$$

which is impossible since then $\bigcup_{\mu\in\nu}\Gamma_\mu \cup \{\Sigma_\nu\}$ is not consistent. On the other hand, if $\Sigma_\nu = \bigvee!\sigma\varphi$, then by the deduction theorem $\bigcup_{\mu\in\nu}\Gamma_\mu \vdash \bigvee!\varrho\bigwedge\alpha_1\ldots\bigwedge\alpha_n[\varrho(\alpha_1,\ldots,\alpha_n) \leftrightarrow \pi^n_\iota(\alpha,\ldots,\alpha_n)] \rightarrow \sim\varphi[{}^{\sigma}_{\pi^n_\iota}]$

and therefore by generalization on a predicate constant not occurring in any wff in $\bigcup_{\mu\in\nu}\Gamma_\mu$, (A5) and (CP!),

$$\bigcup_{\mu\in\nu}\Gamma_\mu \vdash \bigwedge !\sigma \sim\varphi$$

which is also impossible by assumption. Accordingly, not only is each Γ_ν consistent but $\bigcup_{\mu\in\nu}\Gamma_\mu$ is too. Hence, by Lindenbaum's lemma we conclude that there is a maximally consistent set K of wffs of L' such that $\bigcup_{\mu\in\nu}\Gamma_\mu \subseteq K$.

We now proceed to construct the appropriate model based upon K. Where η is a term of L', let $[\eta] = \{\xi\colon \xi$ is a term of L' and $\eta \equiv \xi \in K\}$. If $\eta \equiv \xi \in K$, then, since identity is an (impredicative) equivalence relation $[\eta] = [\xi]$. Also, note that by (A8) and the way K was constructed $[\eta] = [\zeta_\iota]$, for some $\iota\in\lambda$. Accordingly, $\{[\eta]\colon \eta$ is a term of L'$\} = \{[\zeta_\iota]\colon \iota\in\lambda\}$, which we shall identify with A, the domain or universe of discourse. Now let R be that function with the set of individual and predicate constants of L' as domain and which is such that $R(\eta) = [\eta]$, for each individual constant η of L', and, for each n-place predicate constant ϱ of L', $R(\varrho) = \{<[\eta_1],\ldots,[\eta_n]>\colon \varrho(\eta_1,\ldots,\eta_n)\in K\}$. Also, let $<X_n>_{n\in\omega-\{0\}}$ be the ω-indexed family such that $X_n = \{R(\varrho)\colon \varrho$ is an n-place predicate constant of L' and for some n-place predicate variable σ and distinct individual variables $\alpha_1,\ldots,\alpha_n$, $\bigvee !\sigma \bigwedge\alpha_1\ldots\bigwedge\alpha_n[\sigma(\alpha_1,\ldots,\alpha_n) \leftrightarrow \varrho(\alpha_1,\ldots,\alpha_n)]\in K\}$. Finally, let $\mathfrak{a}$ be that function with the set of variables as domain and which is such that $\mathfrak{a}(\alpha) = [\alpha]$, for each individual variable α, and for each n-place predicate variable σ, $\mathfrak{a}(\sigma) = \{<[\eta_1],\ldots,[\eta_n]>\colon \sigma(\eta_1,\ldots,\eta_n)\in K\}$. We set $\mathfrak{U} = <A, R, <X_n>_{n\in\omega-\{0\}}>$ and note that by definition $\mathfrak{U}$ is a model suited to L' and that $\mathfrak{a}$ is an assignment in $\mathfrak{U}$. Note too that for each term η of L', $ext(\eta, \mathfrak{U}, \mathfrak{a}) = [\eta]$, and for each n-place predicate variable or constant ϱ, $ext(\varrho, \mathfrak{U}, \mathfrak{a}) = \{<[\eta_1],\ldots,[\eta_n]>\colon \varrho(\eta_1,\ldots,\eta_n)\in K\}$.

We now show by induction that if φ is a wff of L', then $\mathfrak{a}$

satisfies φ in $\mathfrak{A}$ iff $\varphi \in K$. The induction is on the number of logical constants occurring in φ. If φ is an identity $\eta \equiv \xi$, note that $\mathfrak{a}$ satisfies $\eta \equiv \xi$ in $\mathfrak{A}$ iff $\mathrm{ext}(\eta, \mathfrak{A}, \mathfrak{a}) = \mathrm{ext}(\xi, \mathfrak{A}, \mathfrak{a})$, i.e., iff $[\eta] = [\xi]$ and hence iff $\xi \equiv \eta \in K$. If φ is an atomic wff of the form $\varrho(\eta_1,\ldots,\eta_n)$, then $\mathfrak{a}$ satisfies $\varrho(\eta_1,\ldots,\eta_n)$ in $\mathfrak{A}$ iff $<\mathrm{ext}(\eta_1, \mathfrak{A}, \mathfrak{a}), \ldots, \mathrm{ext}(\eta_n, \mathfrak{A}, \mathfrak{a})> \in \mathrm{ext}(\varrho, \mathfrak{A},)$, i.e., iff $\varrho(\eta_1,\ldots,\eta_n) \in K$. If φ is a negation or conditional, the conclusion follows from the inductive hypothesis. Suppose then that φ is of the form $\bigwedge \alpha \psi$ and that $\mathfrak{a}$ satisfies $\bigwedge \alpha \psi$ in $\mathfrak{A}$. If $\bigwedge \alpha \psi \notin K$, then, since K is maximally consistent, $\sim \bigwedge \alpha \psi \in K$, and therefore from the way that K was constructed, $\sim \psi[^{\alpha}_{\zeta_\iota}] \in K$, for some $\iota \in \lambda$. But then $\psi[^{\alpha}_{\zeta_\iota}] \notin K$ and therefore, by the inductive hypothesis $\mathfrak{a}$ does not satisfy $\psi[^{\alpha}_{\zeta_\iota}]$ in $\mathfrak{A}$, from which it follows by the semantic lemma of § 3 that $\mathfrak{a}(^{\alpha}_{\mathrm{ext}(\zeta_\iota, \mathfrak{A}, \mathfrak{a})})$ does not satisfy φ in $\mathfrak{A}$, which is impossible by assumption. Conversely, if $\bigwedge \alpha \psi \in K$ and $\mathfrak{a}$ does not satisfy $\bigwedge \alpha \psi$ in $\mathfrak{A}$, then for some $\iota \in \lambda$, $\mathfrak{a}(^{\alpha}_{\mathrm{ext}(\zeta_\iota, \mathfrak{A}, \mathfrak{a})})$ does not satisfy ψ in $\mathfrak{A}$; and therefore, by the same semantic lemma, $\mathfrak{a}$ does not satisfy $\psi[^{\alpha}_{\zeta_\iota}]$ in $\mathfrak{A}$. But then, by the inductive hypothesis, $\psi[^{\alpha}_{\zeta_\iota}] \notin K$, which is impossible by assumption, (UI-1) and the fact that K is maximally consistent. Finally, suppose φ is of the form $\bigwedge !\sigma\psi$, for some n-place predicate variable σ, and that $\mathfrak{a}$ satisfies $\bigwedge !\sigma\psi$ in $\mathfrak{A}$ but that $\bigwedge !\sigma\psi \notin K$. Then $\bigvee !\sigma \sim \psi \in K$, and therefore, from the way K was constructed, $\sim \psi[^{\sigma}_{\pi^n_\iota}] \in K$ and $\bigvee !\varrho \bigwedge \alpha_1 \ldots \bigwedge \alpha_n [\varrho(\alpha_1,\ldots,\alpha_n) \leftrightarrow \overset{n}{\pi}_\iota(\alpha_1,\ldots,\alpha_n)] \in K$, for some $\iota \in \lambda$. But then, by definition of $\mathfrak{A}$, $\mathrm{ext}(\overset{n}{\pi}_\iota, \mathfrak{A}, \mathfrak{a}) \in X_n$, and, accordingly,

$\mathfrak{a}\binom{\sigma}{ext(\pi^n_\iota,\, \mathfrak{A},\, \mathfrak{a})}$ satisfies ψ in $\mathfrak{A}$, by assumption. Therefore, by the semantic lemma of § 3, $\mathfrak{a}$ satisfies $\psi[\frac{\sigma}{\pi^n_\iota}]$ in $\mathfrak{A}$, and hence, by the inductive hypothesis, $\psi[\frac{\sigma}{\pi^n_\iota}] \in K$, which is impossible since K would then be inconsistent. For the converse direction, suppose $\bigwedge !\sigma\psi \in K$ but that $\mathfrak{a}$ does not satisfy $\bigwedge !\sigma\psi$ in $\mathfrak{A}$. Then there is an n-place predicate constant ϱ such that $R(\varrho) \in X_n$ and $\mathfrak{a}\binom{\sigma}{ext(\varrho,\, \mathfrak{A},\, \mathfrak{a})}$ does not satisfy ψ in $\mathfrak{A}$, that is, by the same semantic lemma, $\mathfrak{a}$ does not satisfy $\psi[\frac{\sigma}{\varrho}]$ in $\mathfrak{a}$. Therefore, by the inductive hypothesis, $\psi[\frac{\sigma}{\varrho}] \notin K$, which is impossible since $\psi[\frac{\sigma}{\varrho}] \in K$ by (UI!) and the fact that $R(\varrho) \in X_n$, i.e., the fact that $\bigvee !\sigma \bigwedge \alpha_1 \ldots \bigwedge \alpha_n [\sigma(\alpha_1, \ldots, \alpha_n) \leftrightarrow \varrho(\alpha_1, \ldots, \alpha_n)] \in K$. This concludes the inductive argument on the wffs of L'.

Since K is maximally consistent every instance of (CP!) is in K, and therefore every instance of (CP!) is true in $\mathfrak{A}$. Consequently, $\mathfrak{A}$ is normal. Since $\Gamma = \Gamma_0 \subseteq \bigcup_{\mu \in \lambda} \Gamma_\mu \subseteq K$, Γ is simultaneously satisfiable by $\mathfrak{a}$ in $\mathfrak{A}$. (Q.E.D.)

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FOOTNOTES

(1) The author was partially supported by NSF Grant GS 38677.

(2) Cf. COCCHIARELLA [2] for a discussion of the difference between the two forms of Russell's paradox and especially for an analysis and resolution (without resorting to type theory) of the paradox of predication. The present paper, as well as COCCHIARELLA [3], can be viewed as disarming the paradox of membership.

(3) Frege, however, did take predicates (or open wffs) to be "arguments" of quantifiers which he called second and third level "concepts". But quantifiers can be commuted, iterated and occur within the scope of one another and must not therefore be confused with predicates (or open wffs) for which commutation, iteration and scope inclusion are syntactically inapplicable. In addition, concepts do not "fall under" (but are said to "fall within") second and third level "concepts", so that they certainly are not "arguments" of the latter in this sense. Second and third level "concepts" are really logical or syncategorematical "concepts" and do not correspond to the concepts (propositional functions) of any order in Russell's ramified hierarchy. (The syncategorematic expressions of a logistic system are understood here to be those expressions of the system that are not permissible substituends of any *bound* variable of any syntactic type — though in some systems they might be substituends of free (schema) variables.)

(4) We should distinguish at least two kinds of content that expressions of a formal system might have. The first is generally called *descriptive*, but historically has been called *categorial*, which we prefer here since even without (applied) descriptive constants the content is still *indicated by bound variables.* (Hence our reference to categorial content.) The second is generally called *logical* or *formal*, or, traditionally, *syncategorematic* and is understood to be immanent to the logistic system in question. This latter content is usually said to be null or non-existent because it is not *denoted* or *designated* by corresponding constants, or, equivalently, because it is not indicated by any type of *bound* variable. (It may however be «indicated» in a secondary sense by free or schematic variables, and therefore also by constants that are substituends of these free or schematic variables.) This rather standard view is untenable, however; for if the corresponding or associated expressions have logistic efficacy in the system, that fact can be accounted for only in terms of their representing content of some sort. On the other hand, because of its immanency, this content need not be therefore accorded categorial existence, i.e., it need not be indicated by bound variables. Our point here, however, is that categorial existence is not the only philosophically viable notion of existence. In ramified type theory (without the axiom of reducibility), impredicativity has neither categorial nor syncategorial existence. In the new predicative second order

logic, impredicativity has syncategorial but not categorial existence. In standard second order logic, impredicativity has categorial existence.

(5) A perspicuous representation of this logistic efficacy is the rule (S) of substitution of wffs for free (schematic) predicate variables or constants occurring in theorems. (Cf. § 4 below for a description and derivation of (S).) This rule, though derivable in the predicative second order logic formulated here, is not derivable in Church formulation. Indeed, its addition there as a new rule results in standard, and not predicative, second order logic. This is not the case in the new formulation given here.

(6) The principle of universal instantiation, (UI), of wffs — those containing as well as those not containing bound predicate variables — for a generalized predicate variable is now both formulabale and valid when the generalized predicate variable is bound by the standard quantifier. This principle implies the weaker rule (S) and therefore contains, and goes beyond, the logistic efficacy of that rule.

(7) Cf. Cocchiarella [3], §§5-6, for a discussion and development of this richer framework.

(8) By an *essential bound occurrence* we mean that, other than by rewriting, the bound predicate variable in question cannot be eliminated from φ without resulting in some non-equivalent wff, that is, φ is not provably equivalent in the system to a wff which contains no bound occurrences of predicate variables.

(9) Instead of (CP), the same posit can be made by allowing φ, with respect to the distinct variables $\alpha_1,\ldots,\alpha_n$ (as argument indicators), to be a substituend in the principle of universal instantiation of a wff for a generalized predicate variable:

$$\text{(UI)} \qquad \bigwedge\pi\psi \rightarrow \check{S}\Phi\pi(\alpha_1,\ldots,\alpha_n)\psi$$

It is known that (UI) and (CP) are equivalent axiomatic alternatives for standard second order logic (relative of course to the remaining axioms).

(10) Note that φ is a substituend of such a predicate variable (occurring in φ itself) iff (CP) holds regarding φ. Cf. the preceding footnote.

(11) This dependency is at least twofold: it presupposes on the one hand the notion of an essential occurrence of a bound predicate variable and, secondly, the use of (CP) or, equivalently, (UI) both to posit the existence of impredicative properties and relations represented by (open) wffs containing bound predicate variables and to warrant the substitutability of wffs for the bound predicate variables they contain.

(12) Frege's special "set theoretical" axiom, viz. his basic law (V):

$$\text{(V)} \qquad \bigwedge\pi\bigwedge\sigma(\{\alpha:\ \pi(\alpha)\}=\{\alpha:\ \sigma(\alpha)\}\leftrightarrow\bigwedge\alpha[\pi(\alpha)\leftrightarrow\sigma(\alpha)]$$

is easily seen to be derivable from our existential posit of a membership relation satisfying extensionality and (Conv). Conversely, our existential posit follows from (V) and the following *impredicative* instance of (CP):

$$\bigvee \in \bigwedge \alpha \bigwedge \beta(\alpha \in \beta \leftrightarrow \bigvee \pi[\beta \equiv \{\gamma : \pi(\gamma)\} \wedge \pi(\alpha)])$$

(We follow standard practice throughout and, for example, write '$\alpha \in \beta$' instead of '$\in(\alpha,\beta)$'.) Note that the posited membership relation is only impredicatively specifiable.

([13]) Since membership is existentially posited, we view '$\in$' here as an instantiated 2-place predicate variable fulfilling the existential posit.

([14]) Stipulating that $\alpha \notin \alpha$ is a permissible substituend of a generalized 1-place predicate variable (and therefore represents a value of that variable) is equivalent to the above posit that nonself-membership is a property. This is but a special case of the equivalence of (CP) and (UI) cited earlier.

([15]) Extensionality also involves essential occurences of bound predicate variables:

$$\text{(Ext)} \quad \bigwedge \pi \bigwedge \sigma(\bigwedge \alpha[\alpha \in \{\beta : \pi(\beta)\} \leftrightarrow \alpha \in \{\beta : \sigma(\beta)\}] \rightarrow \{\beta : \pi(\beta)\} \equiv \{\beta : \sigma(\beta)\})$$

([16]) This fact is of course all the more obvious if we were to take Frege's basic law (V) as our initial posit. For then, as noted earlier, membership can be posited only by an impredicative instance of (CP); and without membership so posited, (Conv) is unavailable for Russell's argument.

([17]) Cf. [3], for further discussion of this issue.

([18]) Throughout this paper, 'semantics' is used only in the sense of model theory.

([19]) Since (CP!) contains no descriptive constants, relativization to the wffs of a model's language is unnecessary in the definition of normalcy.

([20]) We assume throughout that $\alpha_1,\ldots,\alpha_n$ are pairwise distinct individual variables.

([21]) Since free predicate variables have only predicative properties or relations as values in Church's system, $\bigwedge !\pi\psi \rightarrow \psi$ is a theorem of that formulation. Applying (S) to this theorem results in (UI), the general instantiation law for all wffs, those containing as well as those not containing bound predicate variables, and (UI), as already pointed out, is equivalent to (CP).

([22]) Note that each instance of (CP!) must begin with a non-null universal quantifier prefix on the predicate variables occurring free in the comprehending wff. The prefix must be non-null since no predicate variable can be bound by quantifiers in the comprehending wff; nor can $\equiv$ or any predicate constant occur therein.