SET THEORY CAN PROVIDE FOR ALL ACCESSIBLE ORDINALS

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A (strongly) *inaccessible ordinal* is an uncountable ordinal x such that the cardinals of the power set and union of any less powerful subset of x are also smaller than x. An ordinal is *accessible* if it is smaller than any inaccessible ordinal. Let ZF and ZF^p be the class theories of Zermelo, Fraenkel, and Skolem and of von Neumann, Bernays, and Gödel respectively (1). From an intuitive point of view, it seems that the axioms of ZF and ZF^p provide for the existence of all accessible ordinals. This intuition is not entirely clear, but, since the set of ordinals of any standard complete model of ZF and ZF^p is an ordinal of the metalanguage, it can be understood to mean the conjunction of the following two model-theoretic statements.

A. The set of ordinals of any standard complete model of ZF is an inaccessible ordinal.

B. The set of ordinals of any standard complete model of ZF^p is the successor of an inaccessible ordinal.

In Montague and Vaught [2], it was shown that the existence of an inaccessible ordinal implies the existence of a strongly standard complete model of ZF whose set of ordinals is less than inaccessible and so the falsity of *A*. The condition can be reduced to that of the existence of a standard model of ZF by means of the strong Löwenheim-Skolem Theorem. Cohen and Mostowski applied this procedure in [1] and [3] to prove

(1) For a description of the axioms of ZF and ZF^p and definitions of the metamathematical terms of the present study, the reader is referred to Schock [4].

that the existence of a standard model of ZF implies the existence of a denumerable minimal model of ZF, and so of a standard complete model of ZF whose set of ordinals is denumerable. With the aid of the same procedure, it was shown in Schock [4] that the assumption that ZF is founded implies the existence of denumerable minimal models of both ZF and ZF^p and so the falsity of both *A* and *B*. One is therefore tempted to conclude that intuition has once more gone astray and doubt both *A* and *B*. (2) But such a conclusion is premature. In the proofs of the falsity of *A* and *B* referred to above, the assumption made is essetial to the proof of the existence of a standard complete model in which not all accessible ordinals are attained. In fact,

Theorem 1. The following conditions are equivalent.

(1) ZF is founded

(2) ZF^p is founded.

(3) *A* is false.

(4) *B* is false.

If (1) holds, (2) does by Theorem 2 of Schock [4]. If (2) holds, (1) does and *A* is false by Theorem 5 of Schock [4]. By Lemma 3 of Schock [4], (3) implies (4). Finally, if (4) holds, ZF^p and so ZF is founded by Theorems 1 and 2 of Schock [4].

By means of Theorem 1, a closely related equiconsistency theorem can be proved. Let ZF + a be the result of adding the axiom a to ZF, and similarly for ZF^p. Also, let *cn* and *inc* be the formulas of ZF^p about sets which assert the conjunction of *A* and *B* and the existence of an inaccessible ordinal respectively, let *fd* and fd^p be such formulas which assert the foundedness of ZF and ZF^p respectively, and et *mod* and mod^p be such formulas which assert the existence of standard complete models of ZF and ZF^p respectively. Finally, assume that N is 'not'. Then,

(2) In [2], Montague and Vaught in fact doubted what appears to be *A* weakened to strongly standard complete models, but not *B* westened similarly since it is provable.

Theorem 2. If T is ZF or ZF^p, then the following theories are equiconsistant.

(1) T

(2) $T + {}_{N}inc$

(3) $T + {}_{N}mod + {}_{N}mod^p$

(4) $T + {}_{N}fd + {}_{N}fd^p$

(5) $T + cn$.

Assume first that T is ZF. It was shown in Sheperdson [5] that (1) and (2) are equiconsistant since the existence of an inaccessible ordinal implies the existence of a standard complete model of ZF^p and so of ZF by Lemma 2 of Schock [4]. Consequently, if (3) is consistent, so is (2). On the other hand, if (3) is inconsistent, ZF implies that ZF has a model and so its own consistency. But then ZF is inconsistent by Gödel's theorem on the unprovability of consistency and (2) is as well. That is, (2) and (3) are equiconsistent. Also, by Theorem 1 of Schock [4], the foundedness of ZF or ZF^p is equivalent to the existence of a standard complete model of either. Since these proofs and the proof of Theorem 1 can be given within ZF, it is clear from Theorem 1 that (3) through (5) are equiconsistent. if T is ZF. From Theorem 2 of Schock [4], it follows that (1) through (5) are also equiconsistent if T is ZF^p.

Consequently, it is possible to assume without contradiction in ZF or ZF^p that all accessible ordinals are provided for by either theory, but only at the cost of denying that there is an inaccessible ordinal, that either theory has a standard complete model, and that either theory is founded. Thus, the assumption does not seem to be very plausible. On the other hand, notice that all these implausible assumptions hold in the minimal models of ZF and ZF^p.

REFERENCES

[1] COHEN, P., A minimal model for set theory, *Bull. Amer. Math. Soc.* 69, 1963.

[2] MONTAGUE R. and VAUGHT R., Natural models of set theories, *Fund. Math.* 47, 1959.

[3] MOSTOWSKI, A., On models of Zermelo-Fraenkel set theory satisfying the axiom of constructibility, *Studiae Logico-Mathematica et Philosophica*, Helsinki, 1965.

[4] SCHOCK, R., On standard models of set theories, *Logique et Analyse* 63, 1973.

[5] SHEPERDSON, J., Inner models for set theory III, *J. Symb. Logic* 18, 1953.