ON STANDARD MODELS OF SET THEORIES

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A *logical constant* is one of N ('not'), → ('only if'), ⋏ ('and'), v ('or'), ↔ ('if and only if'), ⋀ ('for any'), ⋁ ('for some'), and I ('is identical with') while the membership predicate ε is the only *non-logical constant*. *Formulas* are assumed to be built up in the usual way from denumerably many *individual variables* by means of the constants and parentheses (which are omitted according to the usual conventions). On the assumptions that each variable and constant is in fact a particular natural number and that parentheses denote the operation of forming the finite sequence of the objects around which the parentheses stand, it is clear that every formula is a set. The *occurrence* and *freedom* of variables is understood in the usual way. A *dummy* is an expression D which is like a sentence except in that there is exactly one F such that, for some distinct variables $x_1 \dots x_n$ and n-place predicate P different from I and ε, $F = x_1 P\, x_2 \dots x_n$ occurs in D. S is *the schema determined by D* just when D is a dummy and S is the set of all formulas which result from replacing the improper atomic subformula $x_1 P\, x_2 \dots x_n$ of D with formulas F such that no variables occuring in D besides x_1 through x_n occur in F. That S is *the predicative schema* determined by D is defined in the same way, but with respect to F in which any quantified variable v is first relativized with a formula $\bigvee w\, v \,\varepsilon\, w$ with w the first variable not occurring in F or D. S is a *schema* just when S is the schema or predicative schema determined by some dummy. A *principle* is a sentence or a schema and a *class theory* is a set of principles one of which is the principle of extensionality. If T is a class theory, then the *basic principles of T* are the members of T and the *axioms* of T are the formulas which are either basic principles of T or members of basic principles of T. A *satisfaction relation* is any relation S for which the usual recursive clauses hold between functions assigning values to the variables and formulas. Of course, a given function either stands in S to a given formula or does not. Given a satisfaction relation S, V_S is the set of all formulas F such that aSF for any a in the domain of S. Notice that V_S can be either the set of all formulas true in some absolute sense or the set of all formulas true in a given model. The *rule of true consequences* is the inference rule which allows the derivation of H from F and G when, for any satisfaction relation S, H is in V_S when F and G are. A formula F is *provable* in a class theory T just when there is a finite sequence ending in F whose terms are either axioms of T or follow from at least one earlier term by the rule of true consequences. Since any logical law is a true consequence of any formula, no logical axioms are needed. A *set theory* is a class theory in which $\bigwedge w \bigvee x\, w \varepsilon x$ is provable (w and x the first two variables). It is assumed that the basic principles of a class theory are *effectively determinable* in that they are finite in number and either sentences nameable in the metalanguage or schemas determined by dummies nameable in the metalanguage.

A *model* is a sequence $M = <U,R>$ where U is a non-empty set and R is a relation whose field is U. M is *isomorphic* with a model $M' = <U',R'>$ just when there is a 1-1 function f from U onto U' such that xRy is equivalent with $f(x)\, R'\, f(y)$ for any x and y. The symbols '$\in$', '$\subseteq$', and '$\subset$' denote the metalinguistice relations of *membership*, *inclusion*, and *proper inclusion* respectively. A model $M = <U,R>$ is *standard* if U is a pure set and $R = \in_U$, the membership relation within U. M is *standard complete* if standard and $x \in y \in U$ implies $x \in U$, and *strongly standard complete* if standard complete and $x \subseteq y \in U$ implies $x \in U$. The *power* of M is, of course, that of U. If all the axioms of a class theory T are satisfied in a model M when ε is interpreted as R and I as the identity relation within U, then M is a *model of T*. A *minimal model* of T is a standard complete model $M = <U,\in_U>$ of T such that, for any standard complete model $M' = <U',\in_{U'}>$ of T, $U \subseteq U'$. A class theory T is *embeddable* in a class theory T' when, for any model $M' = <U',R'>$ of T', there is a $U \subseteq U'$ such that, if R is R' within U, then $<U,R>$ is a model of T which is standard complete if M' is. If T' is not also embeddable in T, then T is *properly embeddable* in T'.

An x is *R-minimal* in S just when S is a set, $x \in S$, R is a relation, and there is no $w \in S$ such that $w R x$. A *founding relation* is a relation R such that there is an x R-minimal in S for any non-empty $S \subseteq$ the field of R. A class theory is *founded* just when T has a model $M = \langle U,R \rangle$ with R a founding relation.

ZF is the set theory whose basic principles are either identical with or determined by one of the following expressions.

Extensionality:	$\bigwedge x \bigwedge y(\bigwedge z(z \varepsilon x \leftrightarrow z \varepsilon y) \rightarrow xIy)$.
Foundation:	$\bigvee x\, xP \rightarrow \bigvee x(xP \wedge N \bigvee y(y \varepsilon x \wedge \bigwedge x(xIy \rightarrow xP)))$.
Union:	$\bigwedge x \bigvee y \bigwedge z(z \varepsilon y \leftrightarrow \bigvee w(w \varepsilon x \wedge z \varepsilon w))$.
Power set:	$\bigwedge x \bigvee y \bigwedge z(z \varepsilon y \leftrightarrow \bigwedge w(w \varepsilon z \rightarrow w \varepsilon x))$.
Infinity:	$\bigvee y(\bigvee z(z \varepsilon y \wedge N \bigvee x\, x \varepsilon z) \wedge \bigwedge z(z \varepsilon y \rightarrow \bigvee w(w \varepsilon y \wedge \bigwedge x(x \varepsilon w \leftrightarrow x \varepsilon z \vee xIz))))$.
Replacement:	$\bigwedge w \bigwedge x \bigwedge y(wQx \wedge \bigwedge x(xIy \rightarrow wQx) \rightarrow xIy) \rightarrow \bigwedge z \bigvee y \bigwedge x(x \varepsilon y \leftrightarrow \bigvee w(w \varepsilon z \wedge wQx))$.

We assume here that w through z are the first four variables and that P and Q are the first non-set-theoretical 1-place and 2-place predicates respectively. ZF is Zermelo-Fraenkel-Skolem set theory. An *extension* of ZF is ZF or a set theory which results from adding sentences and schemas to ZF. If T is a class theory and s is a sentence or schema, then $T + s$ is the theory which results from adding s to T. T^u is *T with ultimate classes* and T^p is *T with predicative ultimate classes* for a set theory T. These are obtained from T by first relativizing any quantified variable v in a basic principle of T which is a sentence with the formula $\bigvee w\, v \varepsilon w$ (w the variable next after v). The same is done with the dummies which determine the schemas of T. Also, the comprehension dummy $\bigvee x \bigwedge y(y \varepsilon x \leftrightarrow \bigvee x\, y \varepsilon x \times yP)$ is added. Finally, the predicative schemas determined by the dummies are formed in the case of T^p. The same is done in the case of T^u except in that the schema determined by the comprehension dummy is formed instead of the predicative schema. The resulting basic principles make up T^{p} and T^{u} respectively. ZF^{p} and ZF^{u} are von Neumann-Bernays-Gödel and von Neumann-Morse class theory. Notice that the form of the principle of replacement with a functional variable is provable in ZF^{p} and ZF^{u} because free variables can occur in the formulas of the predicative replacement schema. The principle of regularity with a general variable follows in the same way from the predicative schema of foundation.

We denote the versions among formulas of the metalinguistic statements on the right-hand side below with those on the left-hand side for a given class theory T.

con_T	T is consistent.
mod_T	T has a standard complete model.
$smod_T$	T has a strongly standard complete model.
$dmod_T$	T has a denumerable standard complete model.
$imod_T$	T has a standard complete model of any infinite power.
ac	Every set can be well-ordered.
AC	Every class can be well-ordered.
c	Every set is constructible (in the sense of Gödel [4]).

The results of the present study are the following.

Theorem 1. If T is a class theory, then the following conditions are equivalent:

(1) T is founded.

(2) T has a countable standard complete model.

(3) T has a standard model.

This is a generalization and strengthening of the theorems of Cohen [1] and [2] that ZF has a standard model if it is founded and a standard complete model if it has a standard model. The proof differs from those of Cohen in that no ordinals are employed. (1)

(1) Long after this study had been completed, the author found that Andrzej Mostowski had employed methods occasionally analogous to those of the study to establish some related results. In «An undecidable arithmetical statement» (*Fund-Math. 36,* 1949) and in the first chapter of *Constructible Sets With Applicaions* (Amsterdam, 1969), a collection of theo-

Theorem 2. If T is a set theory, then
(1) T is consistent just when T^p is.
(2) T is founded just when T^p is.

Theorem 2 is a strengthening and generalization of the theorem of Novak [8], Rosser and Wang [9], Shoenfield [11], and Doets [3] that ZF and ZF^p are equiconsistent. Of course, the proof differs from the earlier ones.

Theorem 3. If T is an extension of ZF, then con_T and con_{T^p} are provable in T^u.

This theorem was asserted to hold for T = ZF in Wang [14]. The assertion was based on a proof that T^u implies con_T if T is the weaker Zermelo set theory by methods somewhat different from the present ones.

With the aid of certain devices from Montague and Vaught [6] and Tarski and Vaught [13], Theorem 3 can be strengthened into

Theorem 4. If T is an extension of ZF, then
(1) $smod_T$ and mod_{T^p} are provable in T^u.
(2) If ac is an axiom to T, then $dmod_T$ and $dmod_{T^p}$ are provable in T^u.
(3) $imod_T$ and $imod_{T^p}$ are provable in $T^u + AC$.

Thus, the slight modification of allowing unrelativized quantifiers in the formulas which determine instances of the schema of comprehension results in an enormous strengthening of a normal class theory T^p.

rems related to Theorem 1 are proved via ordinals. In «Some impredicative definitions in axiomatic set theory» (*Fund. Math.* 37, 1950) and in his book, Mostowski gave truth criteria within versions of ZF^p and ZF^u for nameable formulas an dformulas of ZF respectively by methods similar to those used by the author to give truth criteria for formulas within T^u in the proof of Theorems 3 and 4. Finally, in the third chapter of his book, some results reminiscent of special cases of parts of the consequent of Theorem 4 are established from quite different assumptions by different methods. Since the author's results are either new or quite differently formulated and proved than Mostowski's, it seems worthwhile to publish them. It should also be noted that the present study was completed several months before the appearance of Mostowski's book.

Theorem 5. If T is a founded extension of ZF and $T + c$ is embeddable in T, then there are unique U and U' such that

(1) $<U, \in_U>$ is a denumerable minimal model of T in which is true.
(2) $<U', \in_{U'}>$ is a denumerable minimal model of T^p in which c (and so AC) is true.
(3) $U \in U'$ and so $U \subset U'$.
(4) If $<U'', \in_{U''}>$ is a standard complete model of T^u, then $U' \in$ the union of $U'' \in U''$ and so $U' \subset$ the union of $U'' \subset U''$.

Since ZF + c is embeddable in ZF by reasoning with constructible sets as in Gödel [4], Theorem 5 is a generalization and strengthening of the theorem of Cohen [1] and Mostowski [7] that ZF has a minimal model if it has a standard model. Notice that this theorem complements Theorem 2 in stating that a normal class theory T^p is in a sense stronger than T although equiconsistent with T. More explicitly,

Theorem 6. If T is a founded extension of ZF and $T + c$ is embeddable in T, then T and T^p are properly embeddable in T^p and T^u respectively.

This is a generalization and strengthening of the theorem of Takeuti [12] that, if ZF + c has a standard complete model $M = <U, \in_U>$ (the U of which Takeuti called "Cantor's Absolute"), then the system consisting of the M-truths is syntactically embeddable in another theory.

Suppose now that the assumption of the effective determinability of the basic principles of a class theory are dropped. In this way, even non-recursive sets of principles such as the set of all sentences and schemas true in some model of ZF are class theories. It is significant that Theorem 1, Theorem 2, the statement that the antecedent of Theorem 5 implies (1) through (3) of its consequent, and the statement that that the antecedent of Theorem 6 implies that T is properly embeddable in T^p and T^p is embeddable in T^u all remain provable by exactly the same methods.

We now turn to the proofs. It can be shown that

Lemma 1. If R is a founding relation and G is a term (of the metalanguage) expressing an operation, then there is a unique function f defined on the field of R which assigns G of $\{f(w) : wRx\}$ to any x in in its domain.

This is a recursion principle for founding relations from Montague [5] which will not be proved here. For an idea of how the proof goes without any use of ordinal numbers, the reader is referred to the proofs of theorems 90 and 91 of Schock [10].

Now assume that T is a class theory. If T is founded, T has a countable model $M = <U,R>$ with R a founding relation by means of the version of the Löwenheim-Skolem theorem in which a model with a well-ordered universe is restricted to one of its countable submodels and the axiom of choice of the metalanguage. By letting G express the operation which converts an object into itself in Lemma 1, it follows from that lemma that there is a function f defined on U such that $f(x) = \{f(w) : wRx\}$for any $x \in U$. Clearly, xRy just when $f(x) \in \{f(w) : wRy\} = f(y)$. Also, if $x \in U$, $y \in U$, and $f(x) = f(y)$, then $f(w) \in f(x)$ just when $f(w) \in f(y)$ for $w \in U$ and so wRx just when wRy for $w \in U$. Hence, $x = y$ by means of the axiom of extensionality of T. That is, if U' = the range of f and $M' = <U',\in_{U'}>$, then f is an isomorphism between M and M'. Moreover, if $x \in y = \{f(w) : wRz\} \in U'$, then $x = f(w)$ for some $w \in U$ and $x \in U'$. Thus, M' is a countable standard complete model of T and (2) of Theorem 1 holds. Also, (2) obviously implies (3). If $<U,\in_U>$ is a standard model of T, $\in_U$ is a founding relation by the axiom of regularity of the metalanguage. Consequently, (3) implies (1) and Theorem 1 is established.

Lemma 2. If T is a set theory, $M' = <U',R'>$ is a model of T^p or T^u, $U = \{x$: there is an $m \in U'$ such that $xR'm\}$,R is R' within U, and $M = <U,R>$ then

(1) M is a model of T.
(2) M is standard complete if M' is.
(3) The power of M is not greater than that of M'.

This lemma is obvious.

If U is a set, $A(U)$ is the set of all functions from the variables into U and $a\binom{v}{m}$ is the function which is like a except for assigning m to v for $m \in U$, $a \in A(U)$, and variable v. $a \models_M F$ just when $M = <U,R>$ is a model, $a \in A(U)$, F is a formula, and a satisfies F in M. Similarly, $\models_M F$ just when $M = <U,R>$ is a model, F is a formula, and $a \models_M F$ for any $a \in A(U)$; that is, if M is a model and F is a formula, $\models_M F$ means that F is M-true. An M-*expressible subset* is an s such that $M = <U,R>$ is a model, there are $a \in A(U)$, distinct variables v and w_1 through w_n, a formula F with just v or v through w_n free, and x_1 through x_n in U such that $s = \{m : m \in U \text{ and } a\binom{v\ w_1 \dots w_n}{m\ x_1 \dots x_n} \models_M F\}$. If $M = <U,R>$ is a model, then $U_s(M)$ is the set of all M-expressible subsets. Notice that $\{m : mRx\} \in U_s(M)$ for $x \in U$ since these sets are expressible by any formula $v \varepsilon w$ (v and w distinct variables). $R_s(M)$ is the relation R' defined as follows.

(1) The field of $R' \subseteq U_s(M)$.

(2) If s is in the domain of R', then $s = \{m : mRx\}$ for some $x \in U$.

(3) If $x \in U$ and $y \in U$, then $\{m : mRx\}\, R'\, \{m : mRy\}$ just when xRy.

(4) If $x \in U$, there is no $y \in U$ such that $s = \{m : mRy\}$, and $s = \{m : m \in U \text{ and } a\binom{v\ w_1 \dots w_n}{m\ x_1 \dots x_n} \models_M F\}$ for some $a \in A(U)$, distinct variables v and w_1 through w_n, formula F with just v or v through w_n free, and x_1 through x_n in U, then $\{m : mRx\} R'\, s$ just when $a\binom{v\ w_1 \dots w_n}{x\ x_1 \dots x_n} \models_M F$.

Lemma 3. If T is a set theory, $M = <U,R>$ is a model of T, and $M' = <U_s(M), R_s(M)>$, then

(1) M' is a model of T^p.

(3) M' is finite if M is and equipollent with M if M is infinite.

(2) M' is standard complete if M is.

(4) If $M'' = <U'', \in_{U''}>$ is a standard complete model of T^p

or $T^u,U = \{x$: there is an $m \in U''$ such that $xRm\}$, and $R \subseteq \in_{U''}$, then $U_s(M) \subseteq U''$ and $R_s(M) \subseteq \in_{U''}$.

Assume the antecedent. By *elements*, we mean members s of $U_s(M)$ such that $s = \{m : mRx\}$ for some $x \in U$. Given that F is a formula with only relativized quantifiers, F has no more distinct free variables than $x_1,\ldots,x_m,w_1,\ldots,w_n, a \in A(U_s(M))$, $a(x_1)$ through $a(x_m)$ are elements, $a(w_1)$ through $a(w_n)$ are not elements, and, for any i such that $1 \leqslant i \leqslant n$, $a(w_i) = \{x : x \in U$ and $a'\binom{z_i v_{i_1} \ldots v_{i_k}}{x\, s_{i_1} \ldots s_{i_k}} \models_M F_i\}$ for any $a' \in A(U)$ where F_i is a formula, z_i through v_{i_k} are just the distinct variables free in F_i and occur neither in F nor in F_h if $1 \leqslant h \leqslant i$ and s_{i_1} through s_{i_k} are members of U, construct F^a by altering subformulas of F as follows for variables u:

(1) First unrelativize all the quantifiers in F.

(2) Then replace any formula $w_i \varepsilon u$ for $1 \leqslant i \leqslant n$ with $\text{N}u\text{I}u$.

(3) Then replace any identity $w_i \text{I} u$ or $u \text{I} w_i$ for $1 \leqslant i \leqslant n$ with $\bigwedge z_i(z_i \varepsilon u \leftrightarrow F_i)$ if u is not w_h for h such that $1 \leqslant h \leqslant n$ and with $\bigwedge z_i(F_i \leftrightarrow \bigwedge z_h(z_h \text{I} z_i \rightarrow F_h))$ otherwise for $h \neq i$. If $h = i$, use $\bigwedge z_i(F_i \leftrightarrow F_i)$ instead.

(4) Finally, replace any formula $u \varepsilon w_i$ for $1 \leqslant i \leqslant n$ with $\bigwedge z_i(z_i \text{I} u \rightarrow F_i)$.

Now let $a' \in A(U)$ be of a kind that $a'(v_{i_j}) = s_{i_j}$ for i and j such that $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant k$, and also $a'(x_i) =$ the $x \in U$ such that $a(x_i) = \{m : mRx\}$ for i such that $1 \leqslant i \leqslant m$. By an induction on the complexity of relativized formulas, it can be shown that

(5) $a \models_{M'} F$ just when $a' \models_M F^a$.

This is clear by an inspection of the various cases for atomic F with respect to the definition of $R_s(M)$ together with the facts that F has only relativized quantifiers and $a\binom{v}{e}'(v) =$ the $x \in U$ such that $e = \{m : mRx\}$ for elements e and relevant variable v. Consequently, if F^a is an axiom of T which is in a schema, then its relativized version holds in M' since $a' \models_M F^a$. Also, if v and w are consecutive variables such that w does not occur in F^a and $y = \{x : x \in U$ and $a'\binom{v}{x} \models_M \bigvee w\, v \varepsilon w \wedge F^a\} \in U_s(M)$, $a\binom{v}{s} \models_{M'}$ $\bigvee w\, v \varepsilon w \wedge F$ just when $a'\binom{v}{x} \models_M \bigvee w\, v \varepsilon w \wedge F^a$ for $x \in U$ and $s = \{m : mRx\}$ by (5), and so $a\binom{vw}{sy} \models_{M'} v \varepsilon w \leftrightarrow \bigvee w\, v \varepsilon w \wedge F$ for elements s. But then, if F is a relativized formula,

(6) $\models_{M'} \bigvee w \bigwedge v (v \varepsilon w \leftrightarrow \bigvee w\, v \varepsilon w \wedge F)$

and the axiom schema of comprehension of T^p holds in M'.

If $a(v)$ is an element for any variable v free in F, F^a is simply the result F' of unrelativizing F. Hence, by (5),

(7) $a \models_{M'} F$ just when $a' \models_M F'$

for such an a. In other words, even a formula with free variables which is just about sets of M' according to a holds in M' just when its unrelativized version holds for the corresponding sets of M. But then, if F is a sentence,

(8) $\models_{M'} F$ just when $\models_M F'$

and so the relativized versions of all axioms of T which are sentences hold in M' since the axioms hold in M. Because the axiom of extensionality obviously holds in M', it follows that M' is a model of T^p.

If M is standard complete, $R = \in_U$ and so $x = \{m : mRx\}$ for elements x. Consequently, $x R' y$ implies $x \in y$ for x and y in in $U_s(M)$ and conversely. That is, $R' = \in_{U_s(M)}$. Since every member of a member of $U_s(M)$ is a member of U and U is now included in $U_s(M)$, it follows that M' is standard complete.

If M is finite, M' clearly is so since the set of all subsets of U also is. On the other hand, if M is infinite, then the set of all n-term sequences of members of U is equipollent with M

for any positive integer n. Since there are only denumerably many formulas with n free variables, the number of n-expressible subsets determined by any such sequence is equipollent with M and the union of all those determined by n-term sequences of members of U is as well. Since there are only denumerably many positive integers, it follows that $U_s(M)$ and M' are equipollent with M.

We have now shown that (1)-(3) of the lemma hold. To show that (4) holds, assume its antecedent. It follows that M is standard complete and so M' is standard complete by (2) of the lemma. But then the consequent of (4) must hold if the schema of comprehension of T^p is to hold in M''.

Now assume that T is a set theory. If T is consistent, T has a model by the Gödel-Henkin completeness theorem and so T^p has a model by Lemma 3 and is consistent. On the other hand, if T^p is consistent and so has a model, T has a model by Lemma 2 and is consistent. Finally, by lemmas 2 and 3, T has a countable standard complete model just when T^p does. But then T is founded just when T^p is by Theorem 1 and Theorem 2 is proved.
Lemma 4. If T is a set theory and M is a model of T^u, then M is a model of T^p.

This lemma is obvious.

Assume now that T is an extension of ZF. By Theorem 2, Theorem 3 holds if con_T is provable in T^u. Since our metalanguage is a definitional extension of ZF with the axiom of choice, '$\in$' instead of ε, the logical constants of metalanguage instead of I through $\vee$, and the variables of the metalanguage instead of the variables, we can momentarily assume that just all the metalinguistic versions of the axioms and inference rule of T^u hold in the metalanguage. Thus, the axiom of choice is now only assumed if it is assumed in T^u. As usual, a *set* is a class which is a member of something. We assume that our previous definitions have been given for classes in general rather than just for sets and understand S to be the class of all sets. A *satisfaction function for H* is a function f of the following kind.

(1) H is a formula, the domain of f = the class of all ordered

pairs a,F such that $a \in \mathrm{A}(S)$ and F is a subformula of H, and f assigns 0 or 1.

(2) If a,H is in the domain of f, then, for any variables v and w and formulas F and G,

(a) If $H = \mathrm{v}\ \varepsilon\ \mathrm{w}$, $f(a,H) = 1$ just when $a(\mathrm{v}) \in a(\mathrm{w})$.
(b) If $H = \mathrm{vIw}$, $f(a,H) = 1$ just when $a(\mathrm{v}) = a(\mathrm{w})$.
(c) If $H = \mathrm{N}F$, $f(a,H) = 1 - f(a,F)$.
(d) If $H = F \rightarrow G$, $f(a,H) =$ the smaller of 1 and $(1 - f(a,F)) + f(a,G)$.
(e) If $H = F \curlywedge G$, $f(a,H) =$ the smaller of $f(a,F)$ and $f(a,G)$.
(f) If $H = F \vee G$, $f(a,H) =$ the greater of $f(a,F)$ and $f(a,G)$.
(g) If $H = F \leftrightarrow G$, $f(a,H) = (1 -$ the greater of $f(a,F)$ and $f(a,G)) +$ the smaller of $f(a,F)$ and $f(a,G)$.
(h) If $H = \bigwedge \mathrm{v}F$, $f(a,H) =$ the smallest member of $\{f(a\binom{\mathrm{v}}{m},F) : m \in S\}$.
(i) If $H = \bigvee \mathrm{v}F$, $f(a,H) =$ the greatest member of $\{f(a\binom{\mathrm{v}}{m},F) : m \in S\}$.

It can be shown that

(3) If f and f' are satisfaction functions for H, then $f = f'$.

Assume the antecedent and let $K = \{F : F$ is a subformula of H and $f(a,F) = f'(a,F)$ for any $a \in \mathrm{A}(S)\}$. By an induction on the complexity of subformulas of H, it is clear that every subformula of H is in K and so $f = f'$.

In addition,

(4) If H is a formula, then there is a unique satisfaction function for H.

Assume the antecedent and let $K = \{F : F$ is a subformula of H and there is a unique satisfaction function for $H\}$. Notice that the construction of K requires the schema of comprehension of the metalanguage and involves a bound class variable. Since any atomic subformula of H clearly has a unique satisfaction function by (3), and since any obtained such functions for formulas F and G can be combined and extended by adding appropriate ordered pairs into unique satisfaction functions for subformulas $\mathrm{N}F$, $F \rightarrow G$, and so on up through $\bigvee vF$ of H by (3), it follows from an induction on the complexity of subformulas of H that every subformula of H is in K and so H has a unique satisfaction function.

If $a \in \mathrm{A}(S)$, H is a formula, and f is the satisfaction function for H, then a *satisfies* H just when $f(a,H) = 1$. Clearly, (4) implies that all the usual recursive clauses of satisfaction hold. Consequently, for such a and H,

(5) a satisfies H just when a does not satisfy $\mathrm{N}H$.

A formula F is *true* just when every $a \in \mathrm{A}(S)$ satisfies F. By (5),

(6) For any formula F, either F or $\mathrm{N}F$ is not true.

Given any structural descriptive name N of the metalanguage of a formula with the free variable numerals i_1 through i_n and variable a of the metalanguage, let N^a be the formula of the metalanguage obtained by relativizing any quantified variable numeral i with the structural name $<'\bigvee' ji\ '\varepsilon' j>$ (j the variable numeral next after i), replacing all constants and bound variable numerals with the corresponding constants and variables of the metalanguage, and the numerals i_1 through i_n with the terms $<a'('i_1')'>$ through $<a'('i_n')'>$ of the metalanguage. By an induction on the complexity of structural descriptive names of formulas, it can be shown that the metalinguistic formula

(7) If $a \in \mathrm{A}(S)$, then a satisfies N just when N^a.

is provable for any structural descriptive name N of a formula. Consequently, if N is the name of a sentence and N' is its translation in the way given above into the metalanguage, then:

(8) N is true just when N'.

is also provable in the metalanguage. Now, any axiom of T which is not an instance of a schema of T has a structural descriptive name in the metalanguage. Since there are only finitely many such axioms with names N_1 through N_n, the statement of the metalanguage that they are all true is by (8) equivalent to the conjunction of the metalinguistic axioms N_1' through N_n', a statement which obviously is provable in the metalanguage. Consequently, it is provable in the metalanguage that:

(9) Every axiom of T which is not an instance of a schema of T is true.

Assume now that H is an instance of the schema of foundation. Then H is a formula $\bigvee x F \rightarrow \bigvee x(F \wedge N \bigvee y(y \, \varepsilon \, x \wedge \bigwedge x(x \, I \, y \rightarrow F)))$ where x and y are the second and third variables and F is a formula in which y does not occur. If $a \in A(S)$ and there is an $m \in S$ such that $a\binom{x}{m}$ satisfies F, then there is an $m \in S$ such that $a\binom{x}{m}$ satisfies F and there is no $n \in m$ such that $n \in S$ and $a\binom{x}{n}$ satisfies F by means of the schema of foundation of the metalanguage. That is, a satisfies H and so H is true. By reasoning in the same way for any of the finitely many schemas of T, all the members of each can be shown to be true in the metalanguage. But the conjunction of these statements is equivalent to the statement that any member of any schema of T is true. Consequently, it is provable in the metalanguage that

(10) Every axiom of T which is an instance of a schema of T is true.

Assume now that F and G are true formulas and that H is derivable from F and G by the rule of true consequences. Since the class of all ordered pairs a,F such that $a \in A(S)$ and F is a formula which a satisfies exists by full strength comprehension and so is a satisfaction relation, H is also true. In other words, it is provable in the metalanguage that

(11) The rule of true consequences preserves truth.

Also, by full strength comprehension, both the set of all true formulas and the set of all formulas in every set including the axioms of T and closed under the rule of true consequences exist. From (6) and (9) through (11), it follows that the consistency of T is provable in the metalanguage. Since the metalanguage is assumed to be a version of T^u, this establishes that con_T is provable in T^u and Theorem 3 is proved.

A set is *designated* if there is a formula F with just the variable v free such that m is the unique m for which $a\binom{v}{m}$ satisfies F where $a \in \mathrm{A}(S)$. Since the class of these exists by full strength comprehension and has denumerably many members, it and its union U are sets. Let $M = <U, \in_U>$. If $x \in y \in U, x \in$ the union of a designated set. Since the union of a designated set is designated, it follows that $x \in U$. Similarly, if $x \subseteq y \in U, x \in$ the power set of the union of a designated set. Since the latter set is designated, it follows that $x \in U$ in this case as well. Thus,

(12) M is a strongly standard complete model.

Also, as shown in Montague and Vaught [6], if F is a formula, v and w are distinct variables, and w does not occur in F, then there is a formula G in which w does not occur whose free variables are v and those of F such that

(13) $\bigvee vF \rightarrow \bigvee v(F \wedge \bigvee w(\bigwedge v(G \leftrightarrow vIw) \wedge v \varepsilon w))$ is provable in T.

Intuitively, G is the formula expressing that v is the set of all sets of least rank which satisfy F. Assume now that $a \in \mathrm{A}(U)$ and that v_1 through v_n are all the variables free in $\bigvee vF$. Let H be a formula whose only free variable is w which expresses that w is the union of the range of the function which assigns to any n-term sequence $<m_1 \ldots m_n>$ of members of the designated sets d_1 through d_n of which $a(v_1)$ through $a(v_n)$ are members the set of all sets of least rank s such that $a\binom{vv_1 \ldots v_n}{sm_1 \ldots m_n}$ satisfies F. Such a formula exists because d_1 through d_n are all designated by particular formulas. Also, by means of the schema of replacement and the union axiom among the members of S, there is a unique $u \in S$ such that $a\binom{w}{u}$ satisfies H. Thus, u is designated and so $u \subseteq U$. Also, if $m \in S$ and $a\binom{v}{m}$ satisfies F, then, by (13), there is a member m' of u and so of U such that $a\binom{v}{m'}$ satisfies F. If it is assumed that $a \models_M F$ just when a satisfies F for $a \in A(U)$, it follows that

(14) $a \models_M \bigvee vF$ just when a satisfies $\bigvee vF$.

Consequently, since the class of all formulas which a satisfies exists for $a \in A(S)$ by full strength comprehension, it can be shown by an induction on the formulas that, if $a \in A(S)$ and F is a formula,

(15) $a \models_M F$ just when a satisfies F.

Thus, via (9) through (12),

(16) M is a strongly standard complete model of T.

Consequently, by Lemma 3, if $U' = U_s(M)$ and $M' = <U', \in_{U'}>$, then

(17) M' is a standard complete model of T^p

and (1) of Theorem 4 is established. But then, by (16), (17), and Theorem 1, if the axiom of choice for sets is an axiom of the metalanguage,

(18) T and T^p have denumerable standard complete models.

That is, (2) of Theorem 4 also holds. Finally, assume that the axiom of choice for classes holds in the metalanguage. Let R be a well-ordering for S, let k be any infinite cardinal $\in S$, and let K be a subset of S of the power k. Also, let u be a denumerable sequence such that $u_0 = K$ and $u_{n+1} = \{m$: there are an $a \in A(u_n)$,a variable v, and a formula F such that m is the R-first member of $\{m : m \in S$ and $a\binom{v}{m}$ satisfies $F\}\}$ for a natural number n. Finally, let U be the union of the range of u and let $M'' = <U, \in_U>$. Notice that this construction of M'' requires the schema of comprehension with bound class variables. If n is a natural number and $x \in u_n$, then x is the R-first member of $\{m : m \in S$ and $a\binom{wv}{xm}$ satisfies $vIw\}$ for distinct variables v and w and any $a \in A(u_n)$ and so $x \in u_{n+1}$. Also, each member of the range of u and so U has the power k.

Assume now that F is a formula and $a \models_{M''} F$ just when a satisfies F for any $a \in A(U)$. As Tarski and Vaught showed in [13], it follows that, if $a \in A(U)$ and v is a variable, then

(19) $a \models_{M''} \bigvee vF$ just when a satisfies $\bigvee vF$.

Assume the antecedent. It is clear that the equivalence holds from left to right. Assume then that a satisfies $\bigvee vF$. Let n be the greatest n such that $a(w) \in u_n$ for some variable $w \neq v$ free in F if there is such a w and otherwise 0, and let a' be in $A(u_n)$ and assign $a(w)$ to any $w \neq v$ free in F. Clearly, $a'\binom{v}{m} \models_{M''} F$ just when $a\binom{v}{m} \models_{M''} F$ for any $m \in S$. Now, there is an R-first $m \in u_{n+1} \subseteq U \subseteq S$ such that $a'\binom{v}{m}$ satisfies F and so $a'\binom{v}{m} \models_{M''} F$ and $a\binom{v}{m} \models_{M''} F$. That is, $a \models_{M''} \bigvee vF$ and (19) holds. Consequently, via the existence of the class of formulas F which a satisfies for $a \in \mathrm{A}(S)$ by full strength comprehension, it follows by induction that, if $a \in \mathrm{A}(S)$ and F is a formula,

(20) $a \models_{M''} F$ just when a satisfies F.

But then, by (9) through (11),

(21) M'' is a standard model of T of power k.

Consequently, since $\in_U$ is a founding relation by the schema of foundation of the metalanguage, M'' is isomorphic to a standard complete model M''' by reasoning as in the proof of Theorem 1. But then

(22) M''' is a standard complete model of T of power k.

Consequently, by Lemma 3, if $U'' = \mathrm{U}_s(M''')$ and $N = \langle U'', \in_{U''} \rangle$, then

(23) N is a standard complete model of T^p of power k.

Thus, (3) of Theorem 4 and so the theorem as well are established. It should be observed that the axiom of choice need not be assumed in parts (2) and (3) of the theorem if T is ZF or like ZF in that all the axioms of T hold among the constructible sets of T. This is because the proofs of these parts can then be carried out from $a \in \mathrm{A}(K)$ rather than from $a \in \mathrm{A}(S)$ where K is the well-ordered class of all constructible sets.

In what follows, it is no longer presupposed that the metalanguage is a version of T^u.

Now define the operation L by transfinite recursion on ordinal numbers as follows.

(1) L_0 is the empty set.

(2) If n is an ordinal and $M = \langle L_n, \in_{L_n} \rangle$, then L_{n+1} = the of all M-expressible subsets of L_n.

(3) If n is a limit ordinal, then L_n = the union of $\{L_m : m \in n\}$.

The members of the values of L are, of course, the *constructible sets* of Gödel [4]. Let *ord* and *lev* be formulas with free variables v and v and w respectively which express 'v is an ordinal' and 'v $\in L_w$, w is an ordinal, and both v and w are members' respectively. It can be shown that

Lemma 5. If T is an extension of ZF and $M = \langle U, \in_U \rangle$ is a standard complete model of T or T^p or T^u, then, for any x and y in U and $a \in A(U)$,

(1) $a\binom{v}{x} \models_M ord$ just when x is an ordinal.

(2) $a\binom{vw}{xy} \models_M lev$ just when w is an ordinal, $x \in L_y$, and both x and y $\in$ the union of U.

In other words, both *ord* and *lev* are absolute. If the antecedent is assumed, then (1) holds because M is standard complete and so (2) holds because the formulas expressing the relations and operations involved in the definition of L are themselves absolute.

If M is a model, let $0_M = \{x : a\binom{v}{x} \models_M ord\}$ for arbitrary $a \in A(U)$. Thus, 0_M is the set of ordinals of M. It can be shown that

Lemma 6. If T is an extension of ZF and $M = \langle U, \in_U \rangle$ is a standard complete model of $T + c$, then

(1) 0_M is the smallest ordinal $\notin U$ and a limit ordinal.

(2) If $y \in 0_M$, then $L_y = \{x : a\binom{vw}{xy} \models_M lev\}$ for arbitrary $a \in A(U)$.

(3) $U = L_{0_M}$.

(4) If $U' = L_{0_M+1}$ and $M' = \langle U', \in_{U'} \rangle$, then

(a) M' is a standard complete model of $(T + c)^{p}$ and

(b) $0_M + 1 = 0_{M'}$ = the smallest ordinal $\notin U'$.

Assume the antecedent. Since 0_M includes its union and is well-ordered by $\in$, it is clearly an ordinal and so the smallest one $\notin U$. Since 0_M is not empty and has no greatest ordinal in it, it is also a limit ordinal. By means of the axioms of ZF, all formulas as well as all the usual notions of proof theory and semantics are present in M. Consequently, if $m \in U$ and m is a model, then the set of all m-expressible subsets of the universe of m is also a member of U. But then (2) holds by means of transfinite induction up to 0_M and Lemma 5. In other words, the operation of M corresponding to L is in fact L restricted to 0_M. Consequently, since the axiom of constructibility holds in M, every member of U is a member of L_{0_M} via (1) and (3) holds. By reasoning from (3), the assumptions of the present proof, the fact that L_{0_M+1} = the set of all M-expressible subsets of L_{0_M}, and the antecedent of (4), it is clear by Lemma 3 that (a) of the consequent of (4) holds. Since the only set of ordinals in U' which is not a member of U and includes its union is 0_M and is M-expressible, 0_M is the only ordinal in U' not in U. But then the antecedent of (4) also implies (b) of the consequent, (4) holds, and the lemma is established.

Lemma 7. If T is an extension of ZF and $M' = \langle U', \in_{U'} \rangle$, then M' is a minimal model of T^{p} just in case the following conditions are satisfied:

(1) M' is a standard complete model of T^{p}.

(2) If U = the union of U', then $M = \langle U, \in_U \rangle$ is a minimal model of T.

(3) For any standard complete model $M'' = \langle U'', \in_{U''} \rangle$ of T^{p}, if the union of $U' \subseteq$ the union of U'', then $U' \subseteq U''$.

Assume the antecedent. If the left side of the equivalence holds, then (1) follows from the definition of minimality. Assume the antecedent of (2). By Lemma 2, $M = \langle U, \in_U \rangle$ is a standard complete model of T. If $N = \langle V, \in_V \rangle$ is a standard complete model of T and $V' = U_s(N)$, then $N' = \langle V', \in_{V'} \rangle$ is a standard complete model of T^{p} by Lemma 3.

From the definition of minimality, it follows that $U' \subseteq V'$ and so $U \subseteq$ the union of $V' = V$. That is, M is a minimal model of T and (2) holds. For (3), assume that $M'' = <U'', \in_{U''}>$ is a standard complete model of T^p. From the definition of minimality, it follows immediately that $U' \subseteq U''$ and (3) holds. So assume instead that (1) through (3) hold. To establish that M' is a minimal model of T^p, it is by (1) and (3) sufficient to show that, if $N' = <V', \in_{V'}>$ is a standard complete model of T^p, $U =$ the union of U', and $V =$ the union of V', then $U \subseteq V$. By Lemma 2, $M = <U, \in_U>$ and $N = <V, \in_V>$ are both standard complete models of T. But M is minimal by (2) and so $U \subseteq V$. Thus, M' is minimal and the lemma is proved.

Now assume the antecedent of Theorem 5. If T and T^p have minimal models, it is clear that they are unique. By Theorem 1, T has a denumerable standard complete model and so $T + c$ does as well by embeddability. Consequently, the least ordinal m such that $M = < L_m, \in_{L_m} >$ is a standard complete model of $T + c$ exists by Lemma 6. If $M' = <U, \in_U>$ is a denumerable standard complete model of T, then there is a $V \subseteq U$ such that $N = <V, \in_V>$ is a standard complete model of $T + c$ by embeddability and so $L_m \subseteq L_{0_N} = V \subseteq U$ by Lemma 6. Consequently, L_m is denumerable and M is a denumerable minimal model of T. Again by Lemma 6, it follows that $M'' = <L_{m+1}, \in_{L_{m+1}}>$ is a standard complete model of $(T + c)^p$. If $M' = <U, \in_U>$ is a standard complete model of T^p such that $L_m =$ the union of $L_{m+1} \subseteq$ the union of U, then $L_m \subseteq U$ and so $L_{m+1} \subseteq U$ and M'' is denumerable by Lemma 3. Consequently, since M is a minimal model of T, M'' is a denumerable minimal model of T^p by Lemma 7. Since $L_m \subseteq L_{m+1}$ and $L_m \in L_{m+1}$ while $L_m \notin L_m$, $L_m \subset L_{m+1}$. Finally, assume that $N = <U'', \in_{U''}>$ is a standard complete model of T^u. By Theorem 4 and the absoluteness of being a standard complete model of T^p among standard complete models of T^u, U'' has a standard complete model of T^p as a member of a member. Since the minimal model of T^p and so L_{m+1} are then members of the union of U'' by the completeness of N and the union of $U'' \in U''$ since $\bigvee w \bigwedge v(v \,\varepsilon\, w \leftrightarrow \bigvee w\, v \,\varepsilon\, w \curlywedge vIv)$ is an axiom of T^u, $L_{m+1} \subset$ the union of $U'' \subset U''$ by the axiom of regularity of the metalanguage. Thus, (1) through (4) of Theorem 5 are established and the theorem is proved. Theorem 6 is an immediate consequence of Lemma 2, Lemma 4, and Theorem 5.

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