THE AXIOM OF REGULARITY

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1. This paper is a natural continuation of the work presented in [4]. However, an attempt has been made to make it rather selfcontained.

In [4; § 2] we tried to give some feeling for what the Axiom of Regularity (REG) says and means. For one thing it is equivalent to what we called the Big Bang Axiom which says that if

$$R_0(\beta) = \mathscr{P}(\cup_{\alpha<\beta} R_0(\alpha)) \text{ and } V_0 = \cup_\beta R_0(\beta)$$

then $V = V_0$; here V denotes the class of all objects, with an object being either a set or an urelement. The elements of V_0 we call *regular* sets (and the literature calls well-founded sets).

In [4; § 4] we gave an argument-from-evidence for the truth of REG, at least for classical mathematical structures. Specifically, in essence we argued as follows: consider any classical mathematical objects which conceptually aren't regular sets (e.g., the integers) and formally speaking are those objects satisfying certain special axioms (e.g., Peano's axioms); then it has been found that there are regular sets which satisfy these special axioms.

We could show REG to be true in general if we showed that for any objects introduced formally by special axioms we could find regular sets satisfying these axioms. However, REG is not true in this sense because what if the special axioms specifically call for the existence of non-regular sets (e.g., sets x_1, x_2, ... such that $\ldots \in x_2 \in x_1$). It has been well-known for some time that one could formally add axioms for non-regular sets without introducing any new formal inconsistencies. The point of [4; § 5] was to show how to conceptually construct some non-regular sets in order to make their existence reasonable.

There is strong feeling among most logicians that REG is true though no good intuitive reasons have been given, at least not in the literature other than to say that sets are "obviously" formed only as indicated by the Big Bang Axiom. In § 2 we will prove and after that interpret a theorem which probably formalizes the content of this intuitive feeling.

2. As in [4], let $A(x)$ be the set of all $\in$-ancestors of object x and let $A^*(x) = \{x\} \cup A(x)$. Note that $A(x)$ is the same (viz., $A(x) = \emptyset$) whereas $A^*(x) = \{x\}$ is different for different urelements x. This is one reason we use $A^*(x)$ rather than $A(x)$ to represent the structure of an object x.

A pictorial idea of what an object x "looks like" is given by the notion of a *skeleton of* x which we define as a collection B of points, some of which are connected by directed line segments; the points of B represent x and the $\in$-ancestors of x; for any two points p_1, p_2 in B there will be a line segment directed from p_1 to p_2 iff the object represented by p_1 is an element of the object represented by p_2.

Of course any object will have many possible skeletons. One example of a skeleton of the von-Neumann integer $3 = \{0,1,2\}$ consists of a set $B = \{p_0, p_1, p_2, p_3\}$ and the directed line segments $\overrightarrow{p_0p_1}$, $\overrightarrow{p_0p_2}$, $\overrightarrow{p_0p_3}$, $\overrightarrow{p_1p_2}$, $\overrightarrow{p_1p_3}$, $\overrightarrow{p_2p_3}$.

In [4; p. 274] we defined an *ordered set* as an ordered pair $\langle b, r \rangle$ such that r is a relation on b, i.e., $r \subseteq b \times b$. We can rigorously and abstractly define a *skeleton of an object* x as any ordered set $\langle b, r \rangle$ order isomorphic ($\sim$) to $\langle A^*(x), \in \rangle$, i.e., there exists a $1:1$ correspondence $f : b \leftrightarrow A^*(x)$ such that

$$u\, r\, v \Leftrightarrow (f(u) \in f(v) \text{ for all } u, v \in b.$$

(In terms of our intuitive picture of a skeleton, we have $u\,r\,v$ iff there is a directed line segment from u to v.) Finally, let us define a *regular skeleton* of object x as a skeleton $\langle b, r \rangle$ of x such that b and hence r also are regular sets, i.e., elements of V_0.

THEOREM 2.1. Every object (set or urelement) has at least one regular skeleton. In fact, given any object x, there are as

many different regular skeletons of x as there are ordinals.
PROOF. Let us first find one regular skeleton $\langle b, r\rangle$ of x. Let

$$b = \overline{\overline{A^*(x)}} = \text{cardinality of } A^*(x).$$

Then b is a von-Neumann initial ordinal, cay γ, hence a regular set. By the axiom of choice there is a 1 : 1 correspondence

$$f : \gamma \leftrightarrow A^*(x).$$

Now define ordering r on γ by

$$r = \{ \langle \alpha, \beta \rangle \mid \alpha, \beta \in \gamma \ \& \ f(\alpha) \in f(\beta) \}.$$

Clearly $\langle b, r\rangle \sim \langle A^*(x), \in \rangle$ under f, hence $\langle b, r\rangle$ is a regular skeleton of x.

Having constructed one regular skeleton $\langle b, r\rangle$ of x, we can now make copies of $\langle b, r\rangle$, one copy for each ordinal. In detail, let

$$b_\alpha = b \times \{\alpha\}, \ r_\alpha = \{ \langle \langle u, \alpha\rangle, \langle v, \alpha\rangle\rangle \mid \langle u, v\rangle \in r\}.$$

Clearly $\alpha \neq \beta$ implies $b_\alpha \neq b_\beta$ if $b \neq \varnothing$; and for any x,

$\overline{\overline{b}} = \overline{\overline{A^*(x)}} \geqslant \overline{\overline{\{x\}}} = 1$, hence $b \neq \varnothing$. QED.

One way of looking at set theory is to think of it as a language and mode of thought for representing mathematical relations. (It is not the only one; and for very complicated modern mathematical relations it is not as convenient to use as another one, viz., category theory.) In this case our main concern is that we have enough sets to represent all possible relations. Now 2.1 implies that any relation that can be represented by a set can already be represented by a regular set. Put another way the class of regular sets is just as adequate for representing mathematical relations as is any class of sets. Thus when presenting an axiomatic development of mathematics from a set theoretic viewpoint theorem 2.1 gives a metamathematical proof that one can assume REG without any loss of generality.

3. After the argument just given, together with the well-known result that REG is relatively consistent with the other more standard axioms of set theory, is there any point in studying set theory in general and allowing the possibility of non-regular sets ? We claim that there definitely is.

If our goal is to *use* set theory in order to do mathematics, then 2.1 essentially shows that one can assume REG without loss of generality. However, what if our goal is to *study* set theory in order to understand the notion of a set ?

We know that at the formal level it is safe to assume the existence of non-regular sets; put another way, various formal "non-regular" axioms (from which the existence of non-regular sets can be formally derived) are known to be relatively consistent with the standard axioms of set theory (cf. [1], [2], [3]). The point of [4; § 5] was to show how to conceptually construct some non-regular sets in order to make their existence intuitively reasonable, not just formally acceptable. If the objects of [4; § 5] aren't sets, why not ? If they are sets, why ? In general what is the informal concept of set in our heads which guides us to accept one conceptual construction process as forming sets and another as not ? A foundation study of sets surely should have as one of its goals its explicational of the concept of a set, not just the study of a certain canonical familiy of sets. There may be several acceptable non-equivalent formalizations of the informal concept of a set; in that case, each of these formalizations should be investigated. To deal with such questions and topics one must study sets from a framework more general than the class of regular sets.

4. Our general framework, at least in this paper, will be that of the Generalized Big Bang (GBB) conception of a set, first introduced in [4; § 3]. Say we build sets by stages indexed by the ordinals. At stage β we allow the introduction of objects, called *atoms*, created by some conceptual process P other than just collecting previously formed objects; let $C_P(\beta)$ be the set of all such atoms. The elements of $C_P(\beta)$ could be sets and/or urelements. The function C_P: On $\rightarrow$ V will be called the *creation function for process P*. Also at stage β we form sets made up of objects (i.e., atoms and/or sets) introduced in earlier stages. Let us define the *P-stage function* R_P: On $\rightarrow$ V such that $R_P(\beta)$ denotes the set of all objects introduced (not necessarily for the first time) in stage β. Then we have

$$x \in R_P(\beta) \Leftrightarrow x \in C_P(\beta) \text{ or } x \subseteq \cup_{\alpha<\beta} R_P(\alpha)$$

hence $R_P(\beta) = C_P(\beta) \cup \mathscr{P}(\cup_{\alpha<\beta} R_P(\alpha))$.

Let $V_P = \cup_\beta R_P(\beta)$. Any $x \in V_P$ will be called a *P-object* and

V_P is called the *universe of P-objects* or the *P-universe*; if x is also a set, then we call it a P-set; if x is not a set, it is called a *P-urelement*. Finally we define the *P-rank functions* ϱ_P: $V_P \rightarrow$ On so that $\varrho_P(x)$ equals the first ordinal β such that $x \in R_P(\beta)$: $\varrho_P(x)$ is just the P-stage at which x is first introduced. The GBB axiom with respect to process P would be that $V = V_P$.

A very natural special case of GBB is obtained when we let $C_P(\beta) = \varnothing$ for all β. The corresponding stage function, rank function and universe will be denoted by R_0, ϱ_0 and V_0 respectively. V_0 is of course just the class of regular sets.

5. In [4] no mention was made of how to determine at what stage to introduce a particular atom. In this section we will show that, technically speaking, any stage will do. Put another way, given a class C of atoms, we can break up C into sets $C_P(\beta)$ any way we please so long as $C = \cup_\beta C_P(\beta)$. (We might even put $C = C_P(0)$, $C_P(\beta) = \varnothing$ for $\beta > 0$; but then $C_P(0)$ might be a proper class rather than a set; e.g., consider the case where we want an urelement for each stage.) More generally, we will show that we can even add and/or subtract to a certain extent from a given class of atoms and the resulting universe will remain unchanged. First we need the following result.

LEMMA 5.1 $x \subseteq V_P \Rightarrow x \in V_P$; i.e., any set of P-objects is a P-set.

PROOF. Let β be the least ordinal greater than all the ordinals $\varrho_P(u)$ for $u \in x$. Then $u \in x$ implies $\varrho_P(u) < \beta$, hence $u \in \bigcup_{\alpha<\beta} R_P(\alpha)$. Thus $x \subseteq \bigcup_{\alpha<\beta} R_P(\alpha)$, hence $x \in R_P(\beta)$.

THEOREM 5.2. Let C_1, C_2 be creation functions. For $i = 1, 2$ let

$$R_i(\beta) = C_i(\beta) \cup \mathscr{P}(\bigcup_{\alpha<\beta} R_i(\alpha)) \text{ and } V_i = \bigcup_\beta R_i(\beta).$$

Then $\bigcup_\beta C_2(\beta) \subseteq V_1 \Rightarrow V_2 \subseteq V_1$.

In words, if type-2 atoms are type-1 objects, then all type-2 objects are type-1.

PROOF (by induction). Assume that $R_2(\alpha) \subseteq V_1$ for all $\alpha < \beta$. We have $C_2(\beta) \subseteq V_1$ by hypothesis. We also have

$$\mathscr{P}(\bigcup_{\alpha<\beta} R_2(\alpha)) \subseteq V_1$$

since $x \in \mathscr{P}(\bigcup_{\alpha<\beta} R_2(\alpha))$ implies $x \subseteq \bigcup_{\alpha<\beta} R_2(\alpha) \subseteq V_1$, hence

$x \in V_1$ by 5.1. Thus $R_2(\beta) \subseteq V_1$.

COROLLARY 5.3. $\bigcup_\beta C_2(\beta) \subseteq V_1$ & $\bigcup_\beta C_1(\beta) \subseteq V_2 \Rightarrow V_1 = V_2$.

6. In [4; p. 43] we introduced two candidates, WO* and WBB, for axioms of set theory.

DEFINITION. Consider any $V^* \subseteq V$. We will say that *V^* satisfies WO** if V^* can be well-ordered. We will say *V^* satisfies WBB* iff there exists a function $G : On \to V$ such that (i) $\alpha < \beta \Rightarrow G(\alpha) \subseteq G(\beta)$ and (ii) $V^* = \bigcup_\beta G(\beta)$.

We will now show that GBB, WBB and WO* are all equivalent, assuming the class form of the axiom of choice.

THEOREM 6.1. V^* satisfies GBB $\Leftrightarrow$ V^* satisfies WBB.

Proof of $\Rightarrow$: Let $G(\beta) = \bigcup_{\alpha<\beta} R_P(\alpha)$.

Proof of $\Leftarrow$: Let $C_P(\beta) = G(\beta)$. Then $C_P(\beta) \subseteq V^*$, hence $V_P \subseteq V^*$ by 5.2. Conversely, $x \in V^*$ implies $x \in G(\beta)$ for some β, hence $x \in C_P(\beta) \subseteq V_P$; thus $V^* \subseteq V_P$.

THEOREM 6.2. V* satisfies WBB $\Leftrightarrow$ V* satisfies WO*.

Proof of $\Leftarrow$: trivial.

Proof of $\Rightarrow$: The idea is simple. Let

$$F(\beta) = G(\beta) - \bigcup_{\alpha<\beta} G(\alpha).$$

Then the $F(\beta)$ are mutually disjoint and

$$\bigcup_\beta F(\beta) = \bigcup_\beta G(\beta) = V^*.$$

Well-order (w.o.) each set $F(\beta)$, say $\dashv_\beta$; that this is possible will be shown below. Then w.o. V* by $x \dashv y$ iff

$$\begin{aligned} &\text{(i)} \ x \in F(\alpha) \ \& \ y \in F(\beta) \ \& \ \alpha < \beta \\ \text{or} \quad &\text{(ii)} \ x, y \in F(\beta) \ \& \ x \dashv_\beta y. \end{aligned}$$

To show $\dashv$ is a w.o. of V* consider any $\emptyset \neq K \subseteq V^*$. Let β be the least ordinal such that $K \cap F(\beta) \neq \emptyset$. Then the $\dashv$-first element of K is the $\dashv_\beta$-first element of $K \cap F(\beta)$.

To show the existence of the well-orderings $\dashv_\beta$, let $W(\beta)$ be the set of all w.o. of $F(\beta)$. The set form of the axiom of choice implies that each $W(\beta)$ is non-empty. We must now choose one w.o. $\dashv_\beta$ from each $W(\beta)$. To justify this we note that the class of all $W(\beta)$ is a class of pairwise disjoint sets. Hence by the class form of the axiom of choice (c.f. [4; p. 274]) there exists a choice class C which has exactly one point in common with each $W(\beta)$: such a point is the desired w.o. $\dashv_\beta$ of $F(\beta)$. QED.

7. Consider now the idea of our proof of REG given in § 2. We showed that any one of many regular skeletons could be used as a representation of an object (set or urelement). It would be nice if we had a way of choosing once and for all a representation of each $x \in V$ in such a way that if two objects are related, then so are their chosen representations. E.g., if $x \in A(y)$ or $x \subseteq y$, we would like the regular skeleton representing x to be a subskeleton of the skeleton representing y. If our universe V satisfies GBB, then we can show how to find such a uniform canonical representation.

Let us say that skeleton $\langle b, r\rangle$ is a *subskeleton* of skeleton $\langle c, s\rangle$ (and denoted by $\langle b, r\rangle \sqsubseteq \langle c, s\rangle$) iff $b \subseteq c$ and $r \subseteq s$.

THEOREM 7.1. Consider any GBB universe V_P. Then we can find regular skeletons $\langle b_x, r_x\rangle$ for each object $x \in V_P$ such that

$$x \in A(y) \Rightarrow \langle b_x, r_x\rangle \sqsubseteq \langle_y, r_y\rangle.$$

PROOF. By 6.1 and 6.2 the proper classes V_P and V_0 can be well-ordered, hence both can be put in 1 : 1 correspondence with class On and hence with each other. Let $F : V_P \leftrightarrow V_0$ be such a 1 : 1 correspondence. For any x let

$$b_x = \{F(u) \mid u \in A^*(x)\}$$
$$r_x = \{\langle F(u), F(v)\rangle \mid u, v \in A^*(x) \ \& \ u \in v\}.$$

If $x \in A(y)$, then $A^*(x) \subseteq A^*(y)$, hence $b_x \subseteq b_y$ and $r_x \subseteq r_y$.

8. An obvious restriction we want to place on V_P is that every $\in$-ancestor of a P-atom be an element of V_P: in symbols, $A(C_P(\beta)) \subseteq V_P$. We showed in § 5 that, technically speaking, it doesn't matter at what stage we introduce a P-atom. However, it would be very natural to require that no P-atom be created earlier than the creation or construction of its $\in$-ancestors; also one should leave open the possibility that a set and some of its $\in$-ancestors could be introduced in the same stage, as was done with the simultaneous construction process in [4; § 5]. We will in fact require that an $\in$-ancestor of P-atom x be either a P-atom introduced at same stage as x or some object introduced at an earlier stage: put symbolically,

$$A(C_P(\beta)) \subseteq C_P(\beta) \cup (\cup_{\alpha<\beta} R_P(\alpha)). \quad (*)$$

The next result shows that we can assume C_P satisfies (*) without any loss of generality.

THEOREM 8.1. Assume $A(C_1(\beta)) \subseteq V_1$ for all β. Then we can find a new creation function C_2 such that $V_2 = V_1$ and C_2 satisfies (*).

PROOF. Let $C_2(\beta) = A(C_1(\beta))$. Then $C_2(\beta) \subseteq V_1$ by hypothesis. Also $C_1(\beta) \subseteq A(C_1(\beta)) = C_2(\beta) \subseteq V_2$. Hence $V_2 = V_1$ by 5.3. QED.

Assuming C_P satisfies (*) not only insures that we don't introduce any P-atom before introducing its $\in$-ancestor; it also insures the same result for any P-set, as we now show.

THEOREM 8.2. Assume (*). Then if $y \in V_P$, and $x \in A(y)$, we have (i) $x \in V_P$ and (ii) $\varrho_P(x) \leqslant \varrho_P(y)$.

PROOF. In [4; p. 284] we showed that (even if we allow the existence of infinitely descending $\in$-chains) if $x \in A(y)$, then $x \in x_1 \in \ldots \in x_n = y$ for some finite number of sets $x_1, \ldots, x_n$; hence it is clearly sufficient to prove $x \in y \in V_P \Rightarrow x \in V_P$ & $\varrho_P(x) \leqslant \varrho_P(y)$. Assume $\varrho_P(y) = \beta$, hence $y \in R_P(\beta)$. If $x \in y \in C_P(\beta)$, then $x \in A(C_P(\beta))$, hence $x \in \bigcup_{\alpha<\beta} R_P(\alpha)$ by (*), hence $\varrho_P(x) \leqslant \beta$. If $x \in y \in \mathscr{P}(\bigcup_{\alpha<\beta} R_P(\alpha))$, then clearly $x \in \bigcup_{\alpha<\beta} R_P(\alpha)$, and $\varrho_P(x) < \beta$.

COROLLARY 8.3. Assume (*). Then $y \in R_P(\beta) \Rightarrow y \subseteq \bigcup_{\alpha\leqslant\beta} R_P(\alpha)$ if y is a set.

PROOF. See proof of 8.2.

Thus let us agree on the following formal

DEFINITION 8.4. $V^* \subseteq V$ is called a *GBB universe* iff we can find a creation function $C_P : On \rightarrow V$ (which in turn uniquely determines R_P and V_P) so that we have

$$\text{(i)}\quad R_P(\beta) = C_P(\beta) \cup \mathscr{P}(\bigcup_{\alpha<\beta} R_P(\alpha))$$
$$\text{(ii)}\quad A(C_P(\beta)) \subseteq C_P(\beta) \cup (\bigcup_{\alpha<\beta} R_P(\alpha))$$
$$\text{(iii)}\quad V_P = \bigcup_\beta R_P(\beta) = V^*$$

We say that *V satisfies GBB* iff $V = V_P$ for some GBB universe V_P.

It may happen that $x \subseteq y$, yet $\varrho_P(x) \not\leqslant \varrho_P(y)$; e.g., a subset of $C_P(\beta)$ might possibly not occur until stage $\beta + 1$. If one wants $x \in y$ or $x \subseteq y$ implying $\varrho_P(x) \leqslant \varrho_P(y)$, then one could define

$$R_P(\beta) = C_P(\beta) \cup \mathscr{P}(C_P(\beta) \cup \bigcup_{\alpha<\beta} R_P(\alpha)) \text{ and } V_P = \bigcup_\beta R_P(\beta).$$

However, the improved relation $x \subseteq y \Rightarrow \varrho_P(x) \leqslant \varrho_P(y)$ hardly seems worth the added complication.

The regular stage function R_0 has the property

$$\alpha < \beta \Rightarrow R_0(\alpha) \subseteq R_0(\beta). \quad (^+)$$

In general, the same is not true of R_P since we could have $x \in C_P(\alpha)$, yet $x \notin C_P(\beta)$; then if x is not a set but an urelement, we have $x \notin R_P(\beta)$. However, we do have the following result.

THEOREM 8.5. If $\alpha < \beta$, then any sets in $R_P(\alpha)$ are also in $R_P(\beta)$, even if the sets are atoms.

PROOF. Assume x is a set. Then $x \in R_P(\alpha)$

$\Rightarrow x \subseteq \bigcup_{\lambda \leqslant \alpha} R_P(\lambda)$	by 8.3
$\Rightarrow x \subseteq \bigcup_{\lambda < \beta} R_P(\lambda)$	for any $\beta > a$
$\Rightarrow x \in R_P(\beta)$	QED.

Of course we could easily redefine R_P so that it would satisfy $(^+)$; e.g., let

$$R_P(\beta) = \bigcup_{\alpha \leqslant \beta} C_P(\alpha) \cup \mathscr{P}(\bigcup_{\alpha < \beta} R_P(\alpha)).$$

9. We are now ready to show that a GBB universe is closed under various set theoretic operations. We will show that if $x, y \in V_P$, then $\{x, y\}$, $\bigcup x$, $\mathscr{P}(x)$, and $\mathscr{R}(F \upharpoonright x)$ with function $F \subseteq V_P$ are elements of V_P; also $x \subseteq V_P \Leftrightarrow x \in V_P$ if x is a set. In fact we show the following.

THEOREM 9.1. Any GBB universe satisfies

(i) $x \in R_P(\beta) \Rightarrow x \subseteq \bigcup_{\alpha < \beta} R_P(\alpha)$ if x is a set

(ii) $x \subseteq \bigcup_{\alpha \leqslant \beta} R_P(\alpha) \Rightarrow x \in R_P(\beta)$

(iii) $x \in R_P(\beta)$ & $y \in R_P(\gamma) \Rightarrow \{x, y\} \in R_P(\delta + 1)$ where $\delta = \max\{\beta, \gamma\}$

(iv) $x \in R_P(\beta) \Rightarrow \bigcup x \in R_P(\beta + 1)$

(v) $x \in R_P(\beta) \Rightarrow \mathscr{P}(x) \in R_P(\beta + 2)$

(vi) $x \in V_P$ & $F \subseteq V_P \Rightarrow \mathscr{R}(F \restriction x) \in V_P$

PROOF. Part (i) is just 8.3. Part (ii) is trivial and (iii) is easy. Parts (iv), (v) and (vi) each have two cases. In one case x is a P-urelement; then

$$\bigcup x = \mathscr{P}(x) = \mathscr{R}(F \restriction x) = \varnothing \in R_P(\alpha) \text{ for all } \alpha \geqslant 0 \text{ by 8.5.}$$

This leaves us with the case when x is a P-set. To prove (iv) assume $x \in R_P(\beta)$; then $u \in \bigcup x$

$$\Rightarrow \varrho_P(u) \leqslant \beta \text{ by 8.2}$$

$$\Rightarrow u \in \bigcup_{\alpha \leqslant \beta} R_P(\alpha)$$

$$\Rightarrow \bigcup x \subseteq \bigcup_{\alpha \leqslant \beta} R_P(\alpha)$$

$$\Rightarrow \bigcup x \in R_P(\beta + 1).$$

To prove (v) assume $x \in R_P(\beta)$; then $u \in \mathscr{P}(x)$

$$\Rightarrow u \subseteq x \in R_P(\beta)$$

$$\Rightarrow u \subseteq x \subseteq \bigcup_{\alpha \leqslant \beta} R_P(\alpha) \text{ by part (i)}$$

$$\Rightarrow u \in R_P(\beta + 1);$$

thus $\mathscr{P}(x) \subseteq R_P(\beta + 1)$, hence $\mathscr{P}(x) \in Rp(\beta + 2)$. Finally, to prove (vi) note that function $F \subseteq V_P$ and $u \in x$ insures that $F(u) \in V_P$; let γ be the least ordinal greater than all the ordinals $\varrho_P(F(u))$ for $u \in x$; then

$$u \in x \Rightarrow F(u) \in R_P(\beta) \subseteq \bigcup_{\alpha < \gamma} R_P(\alpha) \quad \text{for some } \beta < \gamma$$

$$\Rightarrow \mathscr{R}(F \restriction x) \subseteq \bigcup_{\alpha < \gamma} R_P(\alpha)$$

$$\Rightarrow \mathscr{R}(F \restriction x) \in R_P(\gamma).$$

10. Let us say K is $\in$*-irreflexive* if $x \notin x$, all $x \in K$ and $\in$*-cycle-free* if $x \notin A(x)$, all $x \in K$; also say K is *almost-*$\in$*-cycle-free* if the only $\in$-cycles in K are of the form $x \in x$, i.e.,

$$x \in x_n \in \ldots \in x_1 \in x \in K \Rightarrow x_i = x \text{ for } 1 \leqslant i \leqslant n.$$

THEOREM 10.1. If C_P is $\in$-irreflexive, $\in$-cycle-free, or almost-$\in$ cycle-free, then so is V_P.

PROOF. We will consider the third case. The other cases have similar or even easier proofs. So assume $R_P(\alpha)$ is almost-$\in$-cycle-free for all $\alpha < \beta$. Now consider any $\in$-cycle $x \in x_n \in \ldots \in x_1 \in x \in R_P(\beta)$. If $x \in C_P(\beta)$, then $x_i = x$, since C_P is almost-$\in$-cycle-free. If $x \subseteq \bigcup_{\alpha<\beta} R_P(\alpha)$, then $x_1 \in R_P(\alpha)$ and $x_1 \in x \in x_n \in \ldots \in x_1$, hence by induction hypothesis $x_i = x_1$ for $2 \leqslant i \leqslant n$ and $x = x_1$.

11. In [4; p. 290] we introduced the STRUCTURE AXIOM which says

$$\langle A^*(x), \in \rangle \sim \langle A^*(y), \in \rangle \Rightarrow x = y, \text{ all } x, y \in V$$

or put more suggestively

$$A^*(x) \sim A^*(y) \Rightarrow x = y, \text{ all } x, y \in V$$

If the universe V of all objects contains at lease one urelement, then V doesn't satisfy STRUCTURE: in detail, if b is any urelement, $A^*(b) = \{b\} \cup A(b) = \{b\} \sim \{\varnothing\} = \{\varnothing\} \cup A(\varnothing) = A^*(\varnothing)$, yet $b \neq \varnothing$. (We might even have a proper class of sets all with the same structure; e.g., this would occur if we introduced some new urelements at each stage β in GBB.) However, we can introduce two related ideas, one if which is applicable to urelements.

DEFINITION Class K is

weakly structural iff $A^*(x) = A^*(y) \Rightarrow x = y$, all $x, y \in K$
(strongly) structural iff $A^*(x) \sim A^*(y) \Rightarrow x = y$, all $x, y \in K$

The notions of weak structurality and $\in$-cycles are closely related as the next two results show.

THEOREM 11.1. If C_P is almost $\in$-cycle-free, then V_P is weakly structural.

PROOF. For any $x, y \in V_P$ we have $A^*(x) = A^*(y)$

$$\Rightarrow \{x\} \cup A(x) = \{y\} \cup A(y)$$
$$\Rightarrow x \in A(y) \text{ and } y \in A(x) \text{ if } x \neq y$$
$$\Rightarrow x \in \ldots \in y \in \ldots \in x \text{ if } x \neq y,$$

hence V_P is not almost-$\in$-cycle-free if $x \neq y$; but V_P is almost-$\in$-cycle-free by 10.1.

THEOREM 11.2. If C_P is weakly structural, then V_P is almost-$\in$-cycle-free.

PROOF. Assume $R_P(\alpha)$ is almost-$\in$-cycle-free for $\alpha < \beta$. It is now sufficient to show that

$$x \in \ldots \in y \in x \in R_P(\beta) \Rightarrow x = y.$$

Case I. $y \in x \in C_P(\beta)$. By 8.4 (ii) we have two possibilities. The first is $y \in C_P(\beta)$; but then $x \in \ldots \in y \in x$

$$\Rightarrow x \in A(y) \text{ and } y \in A(x)$$
$$\Rightarrow A^*(x) \subseteq A^*(y) \text{ and } A^*(y) \subseteq A^*(x)$$
$$\Rightarrow A^*(x) = A^*(y) \text{ where } x, y \in C_P$$
$$\Rightarrow x = y \text{ by weak structurality of } C_P$$

The other possibility is $y \in \bigcup_{\alpha<\beta} R_P(\alpha)$, hence

$$y \in x \in \ldots \in y \in R_P(\alpha) \text{ for some } \alpha < \beta,$$

hence $y = x$ by induction hypothesis that $R_P(\alpha)$ be almost-$\in$-cycle-free.

Case II. $y \in x \in \mathscr{P}(\bigcup_{\alpha<\beta} R_P(\alpha))$. But then $y \in \bigcup_{\alpha<\beta} R_P(\alpha)$, a possibility already considered. QED.

We can now derive several corollaries of 11.1 and 11.2.

COROLLARY 11.3. If C_P is weakly structural, then so is V_P.

COROLLARY 11.4. If C_P is $\in$-irreflexive and weakly structural, then V_P is $\in$-cycle-free and weakly structural.

COROLLARY 11.5. If C_P contains at most urelements, then V_P is $\in$-cycle-free and weakly structural.

PROOF. $x \in C_P \Rightarrow x$ is urelement $\Rightarrow A(x) = \emptyset \Rightarrow x \notin A(x)$; thus C_P is $\in$-cycle-free, hence V_P is $\in$-cycle-free by 10.1 and weakly structural by 11.1.

12. We now wish to investigate under what conditions on C_P we can conclude that V_P is (strongly) structural. Without loss of generality we assume C_P, and hence V_P, contains no urelements, for otherwise structurality fails. Our chief result will be that if C_P is $\in$-irreflexive and structural, then V_P is structural.

LEMMA 12.1. $A^*(x) \overset{f}{\sim} A^*(y) \Rightarrow f(v) = \{f(u) \mid u \in v\}$ for any $v \in A^*(x)$.

PROOF. Since $A^*(x) \overset{f}{\sim} A^*(y)$ we have $u \in v \Leftrightarrow f(u) \in f(v)$, hence $\{f(u) \mid u \in v\} \subseteq f(v)$ .Conversely, assume $w \in f(v)$; then since $v \in A^*(x)$, we have $w \in f(v) \in A^*(y)$, hence $w \in A^*(y)$, hence $w = f(z)$ for some $z \in A^*(x)$, hence $w = f(z) \in f(v)$, hence $z \in v$, hence $w \in \{f(u) \mid u \in v\}$.

THEOREM 12.2. If $A^*(x) \overset{f}{\sim} A^*(y)$ and $f(u) = u$ for all $u \in A^*(x) \cap C_P$, then $f(u) = u$ for all $u \in A^*(x) \cap V_P$.

PROOF. Assume $f(u) = u$ for all $u \in A^*(x) \cap R_P(\alpha)$ and all $\alpha < \beta$. Now we consider any $v \in A^*(x) \cap R_P(\beta)$. If $v \in C_P(\beta)$, then $f(v) = v$ hy hypothesis. If $v \subseteq \bigcup_{\alpha<\beta} R_P(\alpha)$, then

$$u \in v \Rightarrow u \in A^*(x) \cap R_P(\alpha) \text{ for some } \alpha < \beta$$
$$\Rightarrow f(u) = u.$$

Thus $f(v) = \{f(u) \mid u \in v\} = \{u \mid u \in v\} = v$.

COROLLARY 12.3 If $A^*(x) \overset{f}{\sim} A^*(y)$, then $f(u) = u$ for all $u \in A^*(x) \cap V_0$.

LEMMA 12.4 $[A^*(x) \overset{f}{\sim} A^*(y) \text{ and } x \notin A(x)] \Rightarrow f(x) = y$

PROOF. Assume $f(x) \neq y$. But $f(x) \in A^*(y) = \{y\} \cup A(y)$, hence $f(x) \in A(y)$. Since f is 1 : 1 onto, there is a unique $u \in A(x)$ such that $f(u) = y$. It is easy to show that for any $u, v \in A^*(x)$ we have

$$v \in A(u) \Leftrightarrow f(v) \in A(f(u)).$$

Thus from $f(x) \in A(y) = A(f(u))$ we get $x \in A(u)$. But $u \in A(x)$, hence $x \in A(x)$, a contradiction. QED.

In [4; p. 290] we proved that V_0 is structural using the Mostowski-Shepherdson theorem. Using our above results we now give a direct proof of this fact. In fact we prove a slightly stronger result about V_0.

THEOREM 12.5. $[A^*(x) \overset{f}{\sim} A^*(y) \ \& \ x \in V_0] \Rightarrow x = y$. (Note that we have not assumed $y \in V_0$.)

PROOF. We have $f(x) = x$ by 12.3; and since $x \notin A(x)$ by 11.5 we have $f(x) = y$ by 12.4.

COROLLARY 12.6. $C_P = \varnothing \Rightarrow V_P$ is structural; i.e., V_0 is structural.

LEMMA 12.7. If $A^*(x_1) \sim A^*(x_2)$, $u_1 \in A^*(x_1)$, and $u_2 = f(u_1)$, then $A^*(u_1) \overset{g}{\sim} A^*(u_2)$ where $g = f \restriction A^*(u_1)$.

PROOF. Clearly $u_1 \in A^*(x_1)$ implies $A^*(u_1) \subseteq A^*(x_1)$, hence f is defined on all of $A^*(u_1)$. Let $g = f \restriction A^*(u_1)$. Obviously g is 1 : 1 and an $\in$-isomorphism between $A^*(u_1)$ and $\mathscr{R}(g)$. To show g is into $A^*(u_2)$, consider any $v \in A^*(u_1)$. If $v = u_1$, then

$$g(v) = g(u_1) = f(u_1) = u_2 \in A^*(u_2)$$

If $v = v_n \in \ldots \in v_1 \in v_0 = u$ where $n \geqslant 1$, then all $v_i \in A^*(u_1)$, hence

$$g(v) = g(v_n) \in \ldots \in g(v_1) \in g(v_0) = g(u_1) = u_2,$$

hence in summary, $v \in A^*(u_1) \Rightarrow g(v) \in A^*(u_2)$. To show g is onto $A^*(u_2)$, consider any $z \in A^*(u_2)$. Since $u_2 \in A^*(x_2)$ we have $A^*(u_2) \subseteq A^*(x_2)$, hence $z \in A^*(x_2)$, hence $z = f(v)$ for some unique $v \in A^*(x_1)$. Say $z = z_n \in \ldots \in z_1 \in z_0 = u_2$. Then

$$v = f^{-1}(z) = f^{-1}(z_n) \in \ldots \in f^{-1}(z_1) \in f^{-1}(z_0) = f^{-1}(u_2) = u_1,$$

hence $v \in A^*(u_1)$, as desired. QED.

We are finally ready to prove our main theorem of this section.

THEOREM 12.8. If C_P is structural and $\in$-irreflexive, then V_P is structural.

PROOF. Assume there are sets $x_1, x_2 \in V_P$ such that $A^*(x_1) \overset{f}{\sim} A^*(x_2)$ yet $x_1 \neq x_2$. Since C_P is structural and $\in$-irreflexive, V_P is $\in$-cycle-free by 11.4, hence $f(x_1) = x_2$ by 12.4, hence $f(x_1) \neq x_1$. Let β be the first ordinal in the set

$$\{\varrho_P(u) \mid u \in A^*(x_1) \ \&\ f(u) \neq u\}.$$

Let u_1 be some element of $A^*(x_1)$ such that $f(u_1) \neq u_1$ and $\varrho_P(u_1) = \beta$. We have $u_1 \in C_P(\beta)$; for if not, we must have $u_1 \subseteq \bigcup_{\alpha<\beta} R_P(\alpha)$, hence $v \in u_1 \Rightarrow \varrho_P(v) < \beta \Rightarrow f(v) = v$, hence by 12.1

$$f(u_1) = \{f(v) \mid v \in u_1\} = \{v \mid v \in u_1\} = u_1; \text{ contradiction.}$$

Using 12.7 we have $A^*(u_1) \overset{g}{\sim} A^*(u_2)$ where $u_2 = f(u_1) \neq u_1$ and $g = f \restriction A^*(u_1)$. Then $A^*(u_2) \overset{h}{\sim} A^*(u_1)$ where $h = g^{-1}$. Now apply above procedure again to get $v_2 \in A^*(u_2)$ with $h(v_2) \neq v_2$ and $\varrho_P(v_2) = \gamma$ where γ is the least ordinal in the set

$$\{\varrho_P(u) \mid u \in A^*(u_2) \ \&\ h(u) \neq u\}.$$

Show $v_2 \in C_P(\gamma)$ in same way we showed $v_1 \in C_P(\beta)$. Using 12.7 we have $A^*(v_2) \sim A^*(v_1)$ where $v_1 = h(v_2)$. Also $v_1 = h(v_2) \neq v_2$ implies $f(v_1) = g(v_1) = v_2 \neq v_1$, hence $\varrho_P(v_1) \geqslant \beta$ but $v_1 \in A^*(u_1)$ implies $\varrho_P(v_1) \leqslant \varrho_P(u_1) = \beta$; thus $\varrho_P(v_1) = \beta$. Then show $v_1 \in C_P(\beta)$ just as we showed in $u_1 \in C_P(\beta)$. In summary we have

v_1, $v_2 \in C_P$ with $A^*(v_1) \sim A^*(v_2)$, hence $v_1 = v_2$ by structurality of C_P, contradicting $v_1 \neq v_2$. QED.

COROLLARY 12.9. If C_P is structural and $\in$-irreflexive, then V_P is structural and $\in$-cycle-free.

COMMENT. Clearly 12.6 is a corollary of 12.8 as well as of 12.5. We presented 12.5 because it is a stronger result about V_0 then just showing V_0 is structural. Question: does an arbitrary V_P satisfy a similar property; in detail, if C_P is structural and $\in$-irreflexive, can we say

$$[A^*(x) \sim A^*(y) \ \& \ x \in V_P] \Rightarrow x = y$$

without assuming $y \in V_P$?

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