A NATURAL DEDUCTION VARIANT OF SYSTEMS T, S4, S5, AND THE BROWERIAN SYSTEM

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Natural deduction systems of propositional logic are fairly widely known and used. Modal extensions of these systems have been developed by several authors, notably F. B. Fitch (1), and J. M. Anderson and H. W. Johnstone (2). Both of these systems are equivalent to Lewis' S4 (3). More recently, G. E. Hughes and M. J. Cresswell (4) have formulated an extension encompassing Fey's system T (5), as well as Lewis' S4 and S5.

These and other natural deduction systems of modal propositional logic, though well known, have not been widely adopted. One reason for this is, I think, that the Strict Reiteration method, introduced first by Fitch and subsequently borrowed by Hughes-Cresswell, requires a repetition of formulas which results in awkwardly long proofs. Moreover, difficulties in devising proof strategies sometime result from the requirement that certain sequences of lines be grouped together. The Anderson-Johnstone system, though it departs from Fitch in several respects, employs reiteration and is subject to the same objection.

The systems of the present paper borrow features of both the Hughes-Cresswell and Anderson-Johnstone methods, but circumvents the Strict Reiteration device. The latter is replaced by a device that I shall call "*tagging*" which was suggested by Quine's device for marking variables introduced by existential instantiation. (6)

The base of the following modal systems is a set of natural deduction rules for propositional logic. The elementary valid argument forms devised by I. M. Copi (7) are adopted, though any similar set of rules not calling for reiteration would be equally acceptable.

'It is necessary that p' will be written '$\Box p$'; 'it is possible that p' will be written '$\Diamond p$'. To the list of elementary argument forms for propositional logic will be added the following elementary rules of modal logic. M1 and M2 are rules of inference; M3 through M5 are rules of replacement.

M1. Necessity Elimination (N.E.)

$$\Box p \therefore p$$

M2. Possibility Introduction (P.I.)

$$p \therefore \Diamond p$$

M3. Definition of Possibility (Pos.)

$$\Box p \text{ iff } \sim\Diamond\sim p$$
$$\Box\sim p \text{ iff } \sim\Diamond p$$
$$\sim\Box p \text{ iff } \Diamond\sim p$$
$$\sim\Box\sim p \text{ iff } \Diamond p$$

M4. Definition of Strict Implication (S. Impl.)

$$\Box(p \supset q) \text{ iff } p ⥽ q$$

M5. Definition of Strict Equivalence (S. Equiv.)

$$\Box(p \equiv q) \text{ iff } p ⥼⥽ q$$
$$(p ⥽ q) \cdot (q ⥽ p) \text{ iff } p ⥼⥽ q$$

Since some of these rules may be derived from others, not all are required. But this is also true of Copi's elementary valid argument forms.

Copi's Strengthened Rule of Conditional Proof may be employed in constructing modal proofs. However, the set of elementary valid forms plus M1 - M5 will not permit all valid inferences of any system of this paper even where supplimented by the method of Conditional Proof. An additional device is called for, and it will be referred to as "*The Method of Tagging*".

We will indicate that a certain line in a formal proof is tagged by placing an asterisk next to the number of that line, and will note the justification for the tag to the right of the justification for introducing that line. The tagging rules for

system T are as follows:

Rule T: A line validly inferred by N.E. is tagged.

Rule I.T. (Inherited Tag):

inferred from tagged premisses alone by any elementary argument form or M1 - M5 (but not N.I.) is tagged.

Rule T.C.P. (Tagged Conditional Proof): The discharge line of a proof employing the Strengthened Rule of Conditional Proof is tagged provided that every line within the scope of the assumption either (i) is the assumption line itself, or (ii) is validly inferred from premisses all of which lie within the scope of the assumption, or (iii) is tagged.

An additional rule of inference similar to M1 and M2 is now called for.

Rule N.I (Necessity Introduction): $\alpha \therefore \Box\alpha$ (where α is a tagged premiss)

The following simple proof of '$\Box p ⥽ (p ⥽ q)$', which is a theorem of system T, should suffice to illustrate the procedure.

→ 1. $\Box p$		
2.* p	1,	N.E., T
3.* $P \vee \sim q$	2,	Addition, I.T.
4.* $\sim q \vee p$	3,	Commutation, I.T.
5.* $q \supset p$	4,	Implication, I.T.
6. $\Box(q \supset p)$	5,	N.I.
7. $q ⥽ p$	6,	S. Impl.
8.* $\Box p \supset (q ⥽ p)$	1 - 7,	Conditional Proof, T.C.P.
9. $\Box[\Box p \supset (q ⥽ p)]$	8,	N.I.
10. $\Box p ⥽ (q ⥽ p)$	9,	S. Impl.

The Browerian System is obtained by adding to the rules of T the following rule of inference.

MB: $p \therefore \Box\Diamond p$

The tagging rules remain unchanged.

S4 requires an additional tagging rule, namely,

Rule S4: Lines of forms $\Box\alpha$ and $\sim\Diamond\alpha$ are tagged.

Rule MB may not be used in proofs within system S4, nor may Rule S4 be employed in proofs in the Browerian system. System S5 is the result of using both MB and Rule S4 together with rules of system T. A simpler formulation of S5 would, however, replace both MB and Rule S4 by the following.

Rule S5: Lines of forms $\Diamond\alpha$ and $\sim\Box\alpha$ are tagged.

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NOTES

(1) *Symbolic Logic*, New York: Ronald Press Co., 1952.

(2) *Natural Deduction*, Belmont, Calif.: Wadsworth Publishing Co., 1962.

(3) C. I. Lewis and C. H. Langford, *Symbolic Logic*, New York: Century Co., 1932.

(4) *An Introduction to Modal Logic*, London: Methuen and Co., 1968.

(5) Robert Feys (ed. by Joseph Dopp), *Modal Logics*, Louvain: E. Nauwelaerts, 1965.

(6) W. V. O. Quine, *Methods of Logic*, New York: Henry Holt and Co., rev. 1959.

(7) I. M. Copi, *Symbolic Logic*, New York: Macmillan Co., 3rd ed. 1967.