IDEALIZATIONAL LAWS AND EXPLANATION

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The issues of idealization are very rarely discussed in the present-day methodology of sciences, and even if they are taken up, they are treated with lesser precision than are many other methodological problems. Yet, as it seems, idealization is one of the fundamental, if not *the* fundamental, research procedures used in advanced empirical sciences, whether natural (such as physics) or social (such as economics).

The present writer's intention is (1) to suggest definitions of the concepts of idealization, ideal type, and related ones; (2) to answer the question about the purpose of idealization procedures by pointing to the role played by idealization in explanation procedures, which, as it seems, results in a new model of explanation; (3) to illustrate the claim that the model of explanation, as mentioned under (2) above, is being in fact applied in empirical sciences. (1)

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Given a system of empirical knowledge $\mathbf{K}_{\mathbf{L}}$ which has as its subset a set of laws $\mathbf{N}_{\mathbf{L}}$ (where the subscript '**L**' indicates the language in which the statements that are in that set are formulated). It is assumed about $\mathbf{N}_{\mathbf{L}}$ that it consists exclusively of laws which are non-vacuously satisfied in the intended model of the system $\mathbf{K}_{\mathbf{L}}$ of empirical knowledge, i.e., that model of the system for the description of which that system has been formulated. In other words, all those laws which are elements of the set $\mathbf{N}_{\mathbf{L}}$ are non-vacuously satisfied in that domain which a given scientist intends to investigate. Assume that the following statements, logically independent of one another:

$$(1)\quad (x)\ (p_1(x) \neq d_1),$$
$$(x)\ (p_2(x) \neq d_2),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$(x)\ (p_k(x) \neq d_k)$$

are among the consequences of the set of laws. It is assumed that the functional term 'p_i' stands for a function p_i such that the number d_i is an extreme (the greatest or the least) numerical value of that function.

In the set N_L we can single out subsets whose consequences are the statements (1) so that a subset N^{p_i} has as its consequence the statement (x) $(p_i(x) \neq d_i)$. Hence the set N^{p_i} contains statements which are logically inconsistent with the statement (x) $(p_i(x) = d_i)$. We can accordingly single out in N_L the subset of those statements which are consistent with the last-named one. That subset will be symbolized N^{+p_i}; hence $N^{+p_i} = N_L - N^{p_i}$.

Assume that the sets N^{p_1}, N^{p_2}, ..., N^{p_k} are pairwise disjoint, so that they have no elements in common. Since it is true for any sets A,B,C that if B is disjoint with C and $B \cup C \neq A$, then the sets $(A - B)$ and $(A - C)$ are not disjoint, hence it may be assumed that if $N^{p_1} \cup N^{p_2} \cup \ldots \cup N^{p_k} \neq N_L$. Then there are in N_L statements which are in $N^{+p_1} \cap N^{+p_2}$, ..., $N^{+p_1} \cap N^{+p_2} \cap \ldots \cap N^{+p_k}$. Thus the set N_L of laws contains statements which are consistent with the statements: (x) $(p_1(x) = d_1)$; (x) $(p_1(x) = d_1)$ and (x) $(p_2(x) = d_2)$; ...; (x) $(p_1(x) = d_1)$ and (x) $(p_2(x) = d_2)$ and ... and (x) $(p_k(x) = d_k)$.

The following definitions will now be adopted:

$x \in U^{(0)}$ if and only if x satisfies every law in the set N_L and at least some of those laws are satisfied by x non-vacuously; hence, for every i: $p_i(x) \neq d_i$.

$x \in \mathbf{U}^{(1)}$ if and only if x satisfies every law in the set $\mathbf{N}^{+p_1}$ and at least some of those laws are satisfied by x non-vacuously, and moreover $p_1(x) = d_1$; hence for every $i > 1$: $p_i(x) \neq d_i$.

$x \in \mathbf{U}^{(2)}$ if and only if x satisfies every law in the set $\mathbf{N}^{+p_1} \cap \mathbf{N}^{+p_2}$ and at least some of those laws are satisfied by x non-vacuously, and moreover $p_1(x) = d_1$ and $p_2(x) = d_2$; hence, for every $i > 2$: $p_i(x) \neq d_i$.

....

$x \in \mathbf{U}^{(k)}$ if and only if x satisfies every law in the set $\mathbf{N}^{+p_1} \cap \mathbf{N}^{+p_2} \cap \ldots \cap \mathbf{N}^{+p_k}$ and at least some of those laws are satisfied by x non-vacuously, and moreover $p_1(x) = d_1$ and $p_2(x) = d_2$ and ... and $p_k(x) = d_k$.

The set $\mathbf{U}^{(0)}$ is the set individual objects which forms the universe of discourse of the intended model of the system of knowledge $\mathbf{K}_{\mathbf{L}}$. The set $\mathbf{U}^{(0)}$ may consist of physical objects, human beings, factories, etc.

The elements of a set $\mathbf{U}^{(i)}$ will be termed *ideal types of the i-th order* of objects from $\mathbf{U}^{(0)}$ on the strength of the set of laws $\mathbf{N}_{\mathbf{L}}$ and the sequence of open sentences $p_1(x) = d_1, \ldots, p_i(x) = d_i$. These open sentences are *idealizational assumptions*.

Let the set $\mathbf{U}$ be the union of the sets $\mathbf{U}^{(0)}, \mathbf{U}^{(1)}, \ldots, \mathbf{U}^{(k)}$, and let $\mathbf{S}_1$ and $\mathbf{S}_2$ be two ordering relations (i.e., relations which are asymmetrical and transitive) defined in that set. A relation $\mathbf{Q}_j$ (for $j = 1, 2$) is defined as a relation which satisfies the condition: $x\mathbf{Q}_j y$ if and only if $\sim(x\mathbf{S}_j y)$ and $\sim(y\mathbf{S}_j x)$. Assume also that $\mathbf{Q}_j$ is transitive. The relation $\mathbf{Q}_j$ thus defined is an equivalence relation in $\mathbf{U}$, and hence divides that set into a set $\mathbf{A}_j$ of equivalence classes such that $\mathbf{A}_j$ is a partition of $\mathbf{U}$. $\mathbf{A}_j$ is ordered by an ordering relation $\mathbf{C}_j$ which holds between two equivalence classes if and only if every element of one class precedes, under relation $\mathbf{S}_j$, every element of the other class.

Consider now two subsets $\mathbf{L}_1$ and $\mathbf{L}_2$ of the set $\mathbf{R}$ of real numbers, which are ordered by the relation "less than" symbolized $<$. Let $\mathbf{F}$ and $\mathbf{H}$ be two functions defined on the set $\mathbf{U}$ which take as their values elements of the sets $\mathbf{L}_1$ and $\mathbf{L}_2$, respectively. Moreover, for any equivalence class $\mathbf{Z}$ of $\mathbf{A}_1$: if $x \in \mathbf{Z}$ and $y \in \mathbf{Z}$, then $\mathbf{F}(x) = \mathbf{F}(y)$. The same holds for $\mathbf{H}$.

Consider now the domain:

$$\mathbf{D}_0 = <\mathbf{U}^{(0)}, \mathbf{R}; \mathbf{S}_1{}^{(0)}, \mathbf{S}_2{}^{(0)}, \mathbf{L}_1, \mathbf{L}_2, \mathbf{F}^{(0)}, \mathbf{H}^{(0)}>,$$

where

$\mathbf{U}^{(0)}$ — the set of individual objects;
$\mathbf{R}$ — the set of real numbers;
$\mathbf{S}_j{}^{(0)}$ — relation $\mathbf{S}_j$ restricted to $\mathbf{U}^{(0)}$ (i.e., $x\mathbf{S}_j{}^{(0)}y$ if and only if $x\mathbf{S}_jy$ and $x \in \mathbf{U}^{(0)}$ and $y \in \mathbf{U}^{(0)}$);
$\mathbf{F}^{(0)}$ — function $\mathbf{F}$ restricted to $\mathbf{U}^{(0)}$ (i.e., $\mathbf{F}^{(0)}(x) = n$ if and only if $n = \mathbf{F}(x)$ and $x \in \mathbf{U}^{(0)}$);
$\mathbf{H}^{(0)}$ — function $\mathbf{H}$ restricted to $\mathbf{U}^{(0)}$.

The domain $\mathbf{D}_0$ is termed *an empirical domain* if and only if it is a submodel of the intended model of the system of knowledge $\mathbf{K}_{\mathbf{L}}$ (which covers, among other things, the laws from the set $\mathbf{N}_{\mathbf{L}}$), i.e., if $\mathbf{U}^{(0)}$ is a subset of the universe of that intended model.

Consider further the domain:

$$\mathbf{D}_0^{p_1} = <\mathbf{U}^{(1)}, \mathbf{R}; \mathbf{S}_1{}^{(1)}, \mathbf{S}_2{}^{(1)}, \mathbf{L}_1, \mathbf{L}_2, \mathbf{F}^{(1)}, \mathbf{H}^{(1)}>,$$

where

$\mathbf{U}^{(1)}$ — the set of ideal types of the first order (under $\mathbf{N}_{\mathbf{L}}$ and the assumption $p_1(x) = d_1$) of individual objects which are elements of $\mathbf{U}^{(0)}$;

$\mathbf{S}_j{}^{(1)}$ — relation $\mathbf{S}_j$ restricted to $\mathbf{U}^{(1)}$;
$\mathbf{F}^{(1)}$ — function $\mathbf{F}$ restricted to $\mathbf{U}^{(1)}$;
$\mathbf{H}^{(1)}$ — function $\mathbf{H}$ restricted to $\mathbf{U}^{(1)}$.

The domain $\mathbf{D}_0{}^1$ will be termed *a domain of ideal types of the first order* under $\mathbf{N}_{\mathbf{L}}$ and the idealizational assumption $p_1(x) = d_1$ if and only if it is a submodel of the intended model, namely $\mathbf{N}^{+p_1}$, to which the statement (x) $(p_1(x) = d_1)$ is joined.

Domains of ideal types of higher orders, namely $\mathbf{D}_0^{p_1}, p_2, \ldots,$ $\mathbf{D}_0^{p}1, \ldots, p_{k-1}$, are formed analogically until the domain

$\mathbf{D}_0^{p_1}, \ldots, p_k = <\mathbf{U}^{(k)}, \mathbf{R}; \mathbf{S}_1{}^{(k)}, \mathbf{S}_2{}^{(k)}, \mathbf{L}_1, \mathbf{L}_2, \mathbf{F}^{(k)}, \mathbf{H}^{(k)}>$
is reached, where

$\mathbf{U}^{(k)}$ — the set of ideal types of k-th order (under $\mathbf{N}_{\mathbf{L}}$ and the sequence of idealizational assumptions $p_1(x) = d_1, \ldots, p_k(x) = d_k$) of individuals which are elements of $\mathbf{U}^{(0)}$; the other constructs with the index 'k' are restricted to $\mathbf{U}^{(k)}$.

That domain *is* termed the domain of ideal types of the k-th order (under $\mathbf{N}_{\mathbf{L}}$ and the sequence of idealizational assumptions $p_1(x) = d_1, \ldots, p_k(x) = d_k$) if and only if it is a submodel of the intended model of knowledge $\mathbf{N}^{+p_1} \cap \mathbf{N}^{+p_2} \cap \ldots \cap \mathbf{N}^{+p_k}$, to which the statements (x) $(p_1(x) = d_1)$, ..., (x) $(p_k(x) = d_k)$ are joined.

An idealizational law (²), under the knowledge $\mathbf{N}_{\mathbf{L}}$ and the sequence of idealizational assumptions $p_1(x) = d_1, \ldots, p_k(x) = d_k$, is a synthetic statement in the form:

(2) $p_1(x) = d_1 \wedge \ldots \wedge p_k(x) = d_k \rightarrow \mathbf{F}(x) = H(x)$.
Or in the equivalent form:

(3) $\mathbf{F}^{(k)}(x) = H^{(k)}(x)$,
interpreted in the domain of ideal types of the k-th order, $\mathbf{D}_0^{p_1, \ldots, p_k}$.

If an idealizational law is non-vacuously satisfied in that domain, then that domain is termed *an ideal model* of that law.

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From a researcher's point of view, formulation of idealizational laws is not, obviously, a goal in itself. Idealizations are used in order to construct a "model" of those empirical facts in which the researcher is interested, a "model" which is simplified enough to bring out certain simple relationships which are hypothetically stated in the idealizational laws. But such laws are merely points of departure toward a reconstruction of the empirical facts in question. These laws are accordingly subjected to concretization: the idealizational assumptions are removed one by one, which brings the laws closer to facts, and appropriate corrections of the laws, resulting from the removal of those assumptions, are introduced. Concretization thus yields an idealizational law (and, in the limiting case, a law free from any idealizational assumptions, i.e., *a factual law*) interpreted in a domain of ideal types of a lower order (and, in the limiting case, in an empirical domain), but at the same time more complex, i.e., taking into account a greater number of those factors which determine the facts in question. ([3])

These somewhat intuitive formulations can be presented in a more precise form: given an idealizational law in the form

(4) $\mathbf{F}^{(i)}(x) = \mathbf{H}^{(i)}(x)$,

the principle of co-ordination of the term '$\mathbf{F}^{(i-1)}$' with the term '$\mathbf{F}^{(i)}$' is a theorem in the form:

(C) $\mathbf{F}^{(i-1)}(x) = \varphi(\mathbf{F}^{(i)}(x), \Theta(\|p_i(x)| - |d_i\|))$,

such that

1. φ, Θ are definite numerical functions,

2. the variable 'x' in the contexts '$\mathbf{F}^{(i-1)}(x)$' and '$p_i(x)$' ranges over the set $\mathbf{U}^{(i-1)}$ of ideal types of the i-1-th order, and in

the context '$\mathbf{F}^{(i)}(x)$' it ranges over the set $\mathbf{U}^{(i)}$ of ideal types of the i-th order,

3. if $||p_i(x)| - |d_i|| = 0$, i.e., if $p_i(x) = d_i$, then

$$\varphi(\mathbf{F}^{(i)}(x), \Theta(||p_i(x)| - |d_i||)) = \mathbf{F}^{(i)}(x).$$

Nor it is excluded that the function Θ may satisfy the condition:

$$\Theta(||p_i(x)| - |d_i||) = ||p_i(x)| - |d_i||.$$

Let the intuitive concepts which underlie the above definition be explained in the simplest case, when $i = 1$. Then, as the principle of co-ordination indicates, the value of the function $\mathbf{F}$ for real objects (i.e., the value of the function $\mathbf{F}^{(0)}$) depends on the value of that function for the ideal types of the first order (i.e., on the values of the function $\mathbf{F}^{(1)}$) which satisfy the idealizational assumption $p_1(x) = d_1$, and on the degree in which those real objects come close to those ideal types with respect to p_1. That "degree of proximity" can be measured by the absolute difference between the extreme value of the function p_1 (which only ideal types have) and the value of that function for real objects.

Now a *strict direct concretization* of the idealizational law (4), as restricted to the set of factual laws $\mathbf{N}_{\mathbf{L}}$, is, under knowledge $\mathbf{K}_{\mathbf{L}}$, a theorem

(5) $\mathbf{F}^{(k-1)}(x) = \mathbf{G}^{(k-1)}(x)$

(for $k = 1, 2, \ldots$), restricted to $\mathbf{N}_{\mathbf{L}}$, such that knowledge $\mathbf{K}_{\mathbf{L}}$ includes the principle of co-ordination of the term '$\mathbf{F}^{(k-1)}$' with the term '$\mathbf{F}^{(k)}$'. The said principle, taken together with (4), yields (5). If a law $\mathbf{T}_3$ is a direct concretization of a law $\mathbf{T}_2$, and if $\mathbf{T}_2$ is a direct concretization of a law $\mathbf{T}_1$, then $\mathbf{T}_3$ is an *indirect concretization* of $\mathbf{T}_1$. If a sequence of idealizational laws $\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_{m-1}$, such that each law is a strict direct concretization of the preceding law, ends with a factual law $\mathbf{T}_m$, then $\mathbf{T}_m$ is termed *the final concretization* of $\mathbf{T}_1$.

Certain comments will now be added to the above. Note first that when we construct an ideal model of an idealizational law in the form of (3), i.e., when we construct the domain $\mathbf{D}_0^{p_1,\ldots,p_k}$, we proceed in a sense in the reverse direction to that in which we proceed when we concretize that law. This is so because when we construct the domain of ideal types of the first order, $\mathbf{D}_0{}^1$, we in a way "suspend" that part of our initial (factual) knowledge $\mathbf{N}_\mathbf{L}$ which is at variance with the first idealizational assumption, namely $p_1(x) = d_1$. Thus the domain $\mathbf{D}_0^{p_1}$ resembles the factual domain $\mathbf{D}_0$ in all respects described in theorems of knowledge $\mathbf{N}_\mathbf{L}$ and not questionated by the said assumption. It differs, however, from $\mathbf{D}_0$ in that the objects which are in its universe, i.e., objects from the set $\mathbf{U}^{(1)}$, do not have the properties described by statements in the set which is questioned by the said assumption, i.e., statements in the set $\mathbf{N}^{p_1}$. When constructing the domain $\mathbf{D}_0^{p_1,p_2}$ we move still farther away from the domain $\mathbf{D}_0$: the elements of the set $\mathbf{U}^{(2)}$, i.e., ideal types of the second order, have still fewer properties in common with the real objects which are elements of $\mathbf{U}^{(0)}$. They have neither those properties of real objects which are precluded by the idealizational assumption $p_1(x) = d_1$, or those which are precluded by the idealizational assumption $p_2(x) = d_2$. Nevertheless certain common properties do remain, namely those which are described by those statements of the set $\mathbf{N}$ which are not questioned by either of these two idealizational assumptions. The ideal types of the k-th order, i.e., elements of $\mathbf{U}^{(k)}$, have fewest properties in common with real objects; they accordingly bear least analogy (if compared with ideal types of lower orders) to the real objects from the set $\mathbf{U}^{(0)}$. Nevertheless certain common properties do remain even here, namely those which are described by those statements in $\mathbf{N}$ which are not questioned by any of the k idealizational assumptions adopted. Now when concretizing an idealizational law we pass from those objects which least resemble real objects to objects which bear more and more resemblance to the latter as they have an increasing number of properties in common with real objects. While previously, when carrying out the idealization, we were in a position — at the cost of moving away from facts H to formulate simple relationships between the magnitudes we are interested in, now, coming closer to facts (i.e., taking into consideration objects which bear more and more resemblance to real ones), we must make those simple relationships more and more complicated by introducing appropriate corrections. In the limiting case of final concretization we remove all the idealizational assumptions of the idealizational law now subject to concretization and we deduce a factual law from it. The degree of the resemblance the objects studied in the successive stages of concretization bear to real objects increases in the process from a minimum (in the case of the ideal types of the highest order) to a maximum, i.e., identity (when the last stage of concretization has been achieved). The conceptual construction described above accordingly guarantees the preservation of the intuitive idea that when concretizing an idealizational law we speak, in the successive stages of the process, about objects (which, useless to add, are "fictional constructs" as are numbers and other similar concepts) bearing increasing resemblance to real ones.

Idealizational laws are sometimes applied to real objects if the latter come "sufficiently close" to corresponding ideal types, but, strictly speaking, in such a case what is applied is not idealizational laws but certain theorems connected with the latter.

Given an idealizational law (4), a theorem in the form:

$$(k_p^{i-0})\ (x)\ (p_1(x) \leqslant \alpha_1 \wedge \ldots \wedge p_i(x) \leqslant \alpha_i \rightarrow |\mathbf{F}^{(0)}(x) - \mathbf{F}^{(i)}(x)| \leqslant \varepsilon),$$

or in the abbreviated form:

$$\mathbf{F}^{(0)}_{\alpha_1,\ldots,\alpha_i}(x) \underset{\varepsilon}{\approx} \mathbf{F}^{(i)}(x),$$

(where $\alpha_1,\ldots,\alpha_i,\varepsilon$ are numbers, fixed in advance, which indicate the measures of permissible deviations) is termed an approximation principle for the idealizational law (4). In current language, that principle might be formulated thus: if the real objects under consideration have properties that come "sufficiently close" to the extreme properties shared by ideal types, then the real value of the function **F** is "sufficiently close" to the theoretical value of that function, as established by the idealizational law. Obviously, it depends on the researcher's decision what definite value is assigned to those deviation measures. He usually sees to it that they be not "too large". Nor are such numbers always fixed explicitly: in the humanities and the social sciences that approximation principle works only roughly.

Assume now that idealizational law (4) has its approximate concretization ,formulated under an approximation law) in a law in the form:

$$(6)\ (x)\ (p_1(x) \leqslant \alpha_1 \wedge \ldots \wedge p_i(x) \leqslant \alpha_i \rightarrow |\mathbf{F}^{(0)}(x) - \mathbf{H}^{(i)}(x)| \leqslant \varepsilon)$$

or in the equivalent form:

$$(x)\ (\mathbf{F}^{(0)}_{\alpha_1,\ldots,\alpha_i}(x) \underset{\varepsilon}{\approx} \mathbf{H}^{(i)})$$

which follows from (4) and from (k_p^{i-0}).

The concepts introduced above make it possible to formulate a model of explaining empirical facts by means of idealizational laws. The explanation follows the schema:

$$\mathbf{T}^{(k)} \dashv \mathbf{T}^{(k-1)} \dashv \ldots \dashv \mathbf{T}^{(i)} \rightsquigarrow\!\dashv \mathbf{T}^{(0)} \wedge \mathbf{P} \rightarrow \mathbf{E},$$

where $\mathbf{T}^{(k)}$ is an idealizational law with k idealizational assumptions, $\mathbf{T}^{(k-1)}$ is an idealizational law with $k-1$ such assumptions, $\mathbf{T}^{(i)}$ is an idealizational law with i idealizational assumptions, $\mathbf{T}^{(0)}$ is factual law (i.e., a law free from idealizational assumptions), **P** stands for the initial conditions of the factual law in question, **E** is the explanandum, $\dashv$ stands for the relation of strict concretization, and ~⊣, for that of approximate concretization.

The above schema is, obviously, not quite general as it pertains to explanation of observation theorems (or, at least, factual theorems) by means of single laws. It is, however, possible to make it so general that it should cover explanation of idealizational laws by means of sets of idealizational laws.

Note also in this connection that concretization of idealizational laws is a method of testing them. This is so because from an idealizational law treated as the initial law and from the appropriate principle of co-ordination we deduce a less abstract idealizational law (one with fewer idealizational assumptions). The latter, taken together with the appropriate principle of co-ordination, serves us to deduce a still less abstract law, etc. If the final concretization is possible, then we we arrive at a factual law which can be tested in the ordinary way by deducing observational consequences from it and by finding out whether they really are true. If this is so, the entire hierarchy of idealizational laws, including the initial one, is thus confirmed. If not, then at least one element of that hierarchy (not necessarily the initial idealization law: it may be, for instance, one of the principles of co-ordination) is a false statement. If such a final concretization cannot be carried out (and this seems to be an exception-free rule in the social sciences), then after a number of steps consisting in strict concretizations an approximate concretization is carried out as the last step, which does not affect the above argumentation in any way whatever.

Note also that this approach to testing makes it possible to explain the role of experiments in the empirical sciences. It seems that experiments are made in order to test those approximation laws which are not applicable under natural conditions, so that their corresponding idealizational laws are not subject to empirical verification. In other words, an approximate concretization in the form of (6) is deduced from an idealizational law in the form of (4). If no real objects such that satisfy the antecedent of (6) can be found, then the researcher tries to ensure such experimental conditions that certain specified objects should satisfy that antecedent (thus, for instance, to ensure that given objects be "sufficiently black", as it is known in advance that they would never be perfectly black, that the resistance of the air be "sufficiently small", as it is known in advance that physical vacuum is not obtainable, etc.). If such conditions are ensured, then the researcher can find out whether the consequent of (6) is satisfied. If it is not, and if the researcher does not decide to apply less stringent criteria of testing by adopting a weaker principle of approximation, then he must reject law (4) as false. It is useless to add that testing may apply not to a statement in the form of (6), but to a consequence of such a statement. (4)

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It will be demonstrated now that the concepts introduced above make it possible to reconstruct the structure of certain laws of science and the method of explaining empirical facts by means of those laws.

Consider Clapeyron's law, which in handbooks on physics is formulated thus:

(7) $\mathbf{pv} = \mathbf{NT}$,

where $\mathbf{p}$ stands for the pressure of a portion gas, $\mathbf{v}$ stands for its volume, $\mathbf{N}$, for the gas constant, and $\mathbf{T}$, for temperature.

Now, as physicists claim, Theorem (7) is not satisfied for real gases as they occur in nature. In other words, Theorem (7) does not hold in the empirical domain

$$\mathbf{F}_0 = \langle \mathbf{U}^{(0)}, \mathbf{R}; \mathbf{S}_1^{(0)}, \mathbf{S}_2^{(0)}, \mathbf{S}_3^{(0)}, \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \mathbf{N}, \mathbf{p}^{(0)}, \mathbf{v}^{(0)}, \mathbf{T}^{(0)} \rangle,$$

where $\mathbf{U}^{(0)}$ stands for the st of portions of real gases; $\mathbf{R}$, the set of real numbers; $\mathbf{S}_1^{(0)}$, $\mathbf{S}_2^{(0)}$, $\mathbf{S}_3^{(0)}$, the relations, defined on $\mathbf{U}^{(0)}$, of having a lower pressure, smaller volume, and lower temperature, respectively (these relations make it possible to define ordered sets of equivalence classes, $\mathbf{A}_1^{(0)}$, $\mathbf{A}_2^{(0)}$, $\mathbf{A}_3^{(0)}$, of sets of portions of gases having the same pressure, volume, and temperature, respectively); $\mathbf{L}_1$, $\mathbf{L}_2$, $\mathbf{L}_3$, definite subsets of the set $\mathbf{R}$ of real numbers; $\mathbf{N}$, the number 22.4/273, designated in $\mathbf{R}$; $\mathbf{p}^{(0)}$, a function defined on $\mathbf{A}_1{}^{(0)}$ and with values in $\mathbf{L}_1$; $\mathbf{v}^{(0)}$, a function defined on $\mathbf{A}_2{}^{(0)}$ and with values in $\mathbf{L}_2$; $\mathbf{T}^{(0)}$, a function defined on $\mathbf{A}_3{}^{(0)}$ and with values in $\mathbf{L}_3$.

But the phycisists claim at the same time that Theorem (7) does hold for what they term perfect gases, i.e., gases which have the following two properties: their particles are material points, and there are no interactions between those particles, so that the inner pressure of those gases equals zero. Thus the physicists adopt the following two idealizational assumptions:

p_1: $p_w(x) = 0$,
p_2: $v_w(x) = 0$,

i.e., the inner pressure of gas particles x equals zero, and the proper volume of gas particles x equals zero. These assumptions are, obviously, not satisfied by any gas portion in the set $\mathbf{U}^{(0)}$, as the domain $\mathbf{F}_0$ is a submodel of the intended model of physical theory $\mathbf{T}_{\mathbf{L}}$ which includes, *inter alia*, the following two factual laws:

(8) $(x)\ (p_w(x) > 0)$,

(9) $(x)\ (v_w(x) > 0)$,

Thus, by adopting the idealizational assumption p_1 the physicist pass to a system of knowledge $\mathbf{T}^{+p_1}$, i.e., that part of $\mathbf{T}_{\mathbf{L}}$ which does not cover statements which are inconsistent with the statement

(10) $(x)\ (p_w(x) = 0)$.

The intended domain of that system of knowledge with the statement (10) joined to it is the domain:

$$\mathbf{F}_0^{p_1} = <\mathbf{U}^{(1)}, \mathbf{R}; \mathbf{S}_1{}^{(1)}, \mathbf{S}_2{}^{(1)}, \mathbf{S}_3{}^{(1)}, \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \mathbf{N}, \mathbf{p}^{(1)}, \mathbf{v}^{(1)}, \mathbf{T}^{(1)}>,$$

the universe of which are the first-order ideal types of real gases (under the assumption p_1). $\mathbf{U}^{(1)}$ includes those gases which satisfy the idealizational assumption p_1, i.e., which have particles that are material points. Apart from this and from those properties which are incompatible with having particles in the form of material points (such as the compressibility and expansibility of particles, the fact that the particles occupy a part of the container in which the gas is kept, etc.) those ideal types have all the properties of the portions of real gases in $\mathbf{U}^{(0)}$, in particular, definite interactions take place between the particles of those gases (in accordance with (8)). Thus the gas portions in $\mathbf{U}^{(1)}$ differ from the real gases as to the property described in p_1, while they resemble real gases as to the property described in p_2. The other constructs which occur in the domain $\mathbf{F}_0^{p_1}$ are analogous to their counterparts in $\mathbf{F}_0$, with the proviso that they refer to the set $\mathbf{U}^{(1)}$.

Since, as the physicists claim, two conditions are necessary to define a perfect gas for which the Clapeyron equation holds, it is obvious that the domain $\mathbf{F}_0^{p_1}$ also is not a model of Theorem (7). This is why the physicists, by adopting the idealizational assumption p_2, pass to a system of knowledge $\mathbf{T}^{+p_1} \cap \mathbf{T}^{+p_2}$, that is, that part of $\mathbf{T}^{+p_1}$ which does not cover statements incompatible with the statement

(11) $(x)\ (v_w(x) = 0)$.

The domain

$$\mathbf{F}_0^{p_1,p_2} = <\mathbf{U}^{(2)}, \mathbf{R}; \mathbf{S}_1{}^{(2)}, \mathbf{S}_2{}^{(2)}, \mathbf{S}_3{}^{(2)}, \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \mathbf{N}, \mathbf{p}^{(2)}, \mathbf{v}^{(2)}, \mathbf{T}^{(2)}>,$$

the universe of which are second-order ideal types of real gases, i.e., perfect gases (in view of the assumptions p_1 and p_2), is a submodel of the intended model of that knowledge with the statements (10) and (11) joined to it. This is so because the elements of $\mathbf{U}^{(2)}$ are those gases which satisfy the assumptions p_1 and p_2 and also satisfy those statements in $\mathbf{T}_{\mathbf{L}}$ which are compatible with (10) and (11), i.e., those gases which satisfy the statements in the set $\mathbf{T}^{+p_1} \cap \mathbf{T}^{+p_2}$. They do not have those properties of real gases only which are described by statements in the set $\mathbf{T}^{p_1} \cup \mathbf{T}^{p_2}$, that is, those factual laws which are incompatible with (10) or with (11). Theorem (7) is true in that domain, since such is a possible explanation of the physicists' claim that Theorem (7) holds for perfect gases.

The above shows that Clapeyron's law is an idealizational law, whose full formulation is:

$$(12)\ (x)\,(p_w(x) = 0 \wedge v_w(x) = 0 \rightarrow (p(x) \vee (x) = \mathbf{NT}(x))).$$

That law is accordingly satisfied vacuously in the domains $\mathbf{F}_0$ and $\mathbf{F}_0^{p_1}$, and is nonvacuously satisfied in the domain $\mathbf{F}_0^{p_1,p_2}$. Consider now how real facts in the sphere of gases are explained by reference to Clapeyron's law (12).

Law (12) may also be written thus:

$$(13)\ p^{(2)}(x)v^{(2)}(x) = \mathbf{NT}^{(2)}(x).$$

Now physicists do not use Clapeyron's law in direct explanation of empirical facts, but deduce from it van der Waals' equation which they claim to be applicable to real gases. Hence Clapeyron's equation is termed the equation for perfect gases, and van der Waals' equation, the equation for real gases. Now, as handbooks on physics usually inform, van der Waals arrived at his equation by taking into account two differences between the real and the perfect gases. The first of them, namely the dimensions of the particles, is manifested in the fact that the particles move less freely in the container in which a given portion of gas is kept than they would if they were points. The volume left for the free movements of the particles is less than the geometrical volume of the container, the difference being a certain magnitude *b*. Hence when taking into consideration the effect of the proper dimensions of the particle the researcher ought to replace the volume of the molecule *v* by the magnitude $\mathbf{v} - \mathbf{b}$, which yields

(14) $\mathbf{p}(\mathbf{v} - \mathbf{b}) = \mathbf{NT}$. [5]

Note in this connection that fairly complicated calculations show that the said magnitude *b* equals the fourfold proper volume of the particles.

The second difference between the real and the perfect gases consists in the interaction forces between particles. This results in the fact that those particles which are at a certain distance from one another attract one another. Because of these attraction forces a given portion of gas occupies a volume *v* which is less than Clapeyron's law would indicate. It is so as if the gas were under a greater pressure p' than the external pressure exerted by the walls of the container. Hence in formula (14) the external pressure ought to be replaced by the magnitude $\mathbf{p}' = \mathbf{p} + p_w$, which yields

(15) $(\mathbf{p} + \mathbf{p}_w)(\mathbf{v} - \mathbf{b}) = \mathbf{NT}$.

The magnitude $\mathbf{p}_w$ is termed the *inner pressure* of a gas. [6]

This method of deducing van der Waals' law, which was presented above in a rather intuitive way, will now be reconstructed with greater precision.

Clapeyron's law (13) may also be written thus:

$$(16)\quad \frac{\mathbf{NT}^{(2)}(x)}{\mathbf{p}^{(2)}(x)} = \mathbf{v}^{(2)}(x).$$

Now, as physicists claim, Theorem (7) is not satisfied for concerned with the proper volume of the particles, the researcher assumes the following theorem

$$(k^{2-1})\quad \frac{\mathbf{NT}^{(1)}(x)}{\mathbf{p}^{(1)}(x)} = \frac{\mathbf{NT}^{(2)}(x)}{\mathbf{p}^{(2)}(x)} - \mathbf{s}(\mathbf{v}(x)),$$

where $\mathbf{s}(v_w(x))$ may be interpreted as that magnitude by which the volume left for the free movements of the particles is less than the geometrical volume of the container (⁷); in other words, it is the magnitude of the volume occupied by the particles of a gas portion x with a proper volume $\mathbf{v}_w$. It turns out that

$$(17)\quad \mathbf{s}(\mathbf{v}_w(x)) = 4\mathbf{v}_w(x).$$

The magnitude $\mathbf{s}(\mathbf{v}_w(x))$ thus corresponds to the correction *b*, mentioned above. Now, in view of (16) and (17), (k^{2-1}) yields:

$$(18)\quad \mathbf{p}^{(1)}(x)\,(\mathbf{v}^{(2)}(x) - 4\mathbf{v}_w(x)) = \mathbf{NT}^{(1)}(x).$$

Theorem (19) is, as can be seen, a reconstruction of Theorem (14). That theorem has as its ideal model the domain of first-order ideal types, $F_0^{p_1}$. It can also be seen that Theorem (18) is a concretization of Clapeyron's law (16) under the idealizational assumption p_2.

Theorem (18) may also be written thus:

$$(19)\quad \frac{\mathbf{NT}^{(1)}(x)}{\mathbf{v}^{(2)}(x) - 4\mathbf{v}_w(x)} = \mathbf{p}^{(1)}(x).$$

The removal of the idealizational assumption p_1 assumes the adoptation of the following theorem:

$$(k^{1-0})\quad \frac{\mathbf{NT}^{(0)}(x)}{\mathbf{v}^{(2)}(x) - 4\mathbf{v}_w(x)} = \frac{\mathbf{NT}^{(1)}(x)}{\mathbf{v}^{(2)}(x) - 4\mathbf{v}_w(x)} + \mathbf{p}_w(x),$$

which, together with (19), yields:

$$(20)\quad (\mathbf{p}^{(1)}(x) + \mathbf{p}_w(x))\,(\mathbf{v}^{(2)}(x) - 4\mathbf{v}_w(x)) = \mathbf{NT}^{(0)}(x).$$

Now Theorem (20) is precisely a reconstruction of van der Waals' law (15), which law is said to be a factual law, non-vacuously satisfied in the empirical domain $\mathbf{F}_0$.

It can easily be noted that the above analysis leads to the conclusion that van der Waals' law is a concretization of Clapeyron's law. The statements (k^{2-1}) and (k^{1-0}) are co-ordination principles for the terms in question, and Theorem (20) is a logical consequence of Theorem (16), and hence van der Waals' law is a concretization of Clapeyron's law.

Van der Waals' law, being factual in nature, serves directly to explain empirical facts. Hence the full schema of the explanation of the behaviour of real gases, as observed by the physicists, may be reconstructed thus:

$$(16) \dashv (18) \dashv (20) \wedge \mathbf{P} \rightarrow \mathbf{E}.$$

As can be seen, this is a special cases of the schema of explanation as described in the preceding Section, a case in which we can do without an approximate concretization, so that the factual law in question is a concretization *sensu stricto* of idealizational laws (of course, on the assumption that conditions p_1 and p_2 are the only idealizational assumptions in Clapeyron's law).

Leszek Nowak

NOTES

(1) What is claimed in this paper is more extensively presented and substantiated (with examples drawn from the various empirical disciplines) in *The Foundations of the Marxian Methodology of Sciences* by the present writer (in Polish), Warsaw 1971.

(2) The concept of idealizational laws as counterfactual conditional sentences was formulated independently by J. Kmita in *Selected Methodological Issues of Interpretation in the Humanities and the Social Sciences*, (in Polish), Warsaw 1971.

(3) The present writer is indebted to J. Kmita for the basic idea of this concept of concretization, which consists in correcting the theorems subjected to concretization.

([4]) Note that Hempel's model of explanation does not cover explanation by reference to idealizational laws. According to Hempel, a theorem which is idealizational in nature is valid as an instrument of research if it is a special case of a non-idealizational theorem (cf. C. G. HEMPEL, *Problems of Concept and Theory Formation in the Social Sciences*, in: *Science, Language and Human Rights*, Proceedings of the American Philosophical Association, Vol. 1, Philadelphia 1952, pp. 81 ff.). Hence, should an idealizational law **I** imply any statement **E**, then it could be an explanation of **E** on that condition only that a non-idealizational (i.e., factual — in the terminology adopted in the present paper) theorem **F**, of which **I** is a special case, has been proved. But then **E** is explained by **F** which bears out the fact that according to Hempel idealizational theorems do not participate in explanation. But if this is so, then it is difficult to say what purpose they serve in science. A more comprehensive criticism of C. G. Hempel's ideas on idealization and explanation is to be found in *The Problem of Explanation in Carl Marx's 'Capital'*, by the present writer, "Quality and Quantity", vol. V, n° 2.

([5]) The discussion of the problem in the physics of gases is based on a Polish translation of the Russian-language *Course in Physics*, Vol. 1, by S. Frish and A. Timoreva, mainly on the data to be found on pp. 237-8 of the Polish version, Warszawa 1962.

([6]) *Ut supra.*

([7]) *Ut supra.*