THE EXAMINATION PARADOX AND FORMAL PREDICTION

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§ 1. *Introduction*

1.1. In treating paradoxes which are formulated in common language, one is sometimes tempted to go too far, by making a mathematical model that demonstrates in a correct way the fundamental assertion of the paradox or its negation. But this is not always relevant, because in several cases the point is to find a logical formulation of the non-formal reasoning given by the paradox, and to show — as exactly as possible — where it is incorrect or paradoxical. For example, we do not say anything relevant to the paradox of Achilles and the tortoise by defining a certain mathematical series and thus demonstrating that Achilles reaches the tortoise in a finite time: the point here is to follow step by step Zeno's argument and to show where it is logically wrong. Perhaps in doing it we are led to make more rigorous or more explicit Zeno's argument, but in any case the logical devices employed must respect the "spirit" of the original reasoning. I emphasize this by saying that *the informal reasoning is a substantial part of the paradox itself.*

1.2. The examination paradox is also a good example of this situation. It has two parts: an announcement and an informal reasoning; the conclusion of the latter seems paradoxical. Again, the point here is to follow step by step the informal argument and to show where it is wrong or where it is truly paradoxical; it is evident that in doing so we must introduce some technical devices, but it is an important requirement that these devices must be coherent with the spirit of the original argument. I shall call this *the coherence requirement.*

1.3. The startpoint is thus the informal formulation of the paradox. This runs as follows:

A teacher makes the following announcement to his pupils: "There will be a test on one of the next n days, and you will not know in advance which day" (1).

The paradox consists in this announcement and the following informal reasoning:

Let $d_1, \ldots, d_n$, be the days referred to in the announcement. The test cannot occur on d_n because, knowing that it has not occurred on $d_1, \ldots, d_{n-1}$, the student can deduce from this and the announcement that the test must occur on d_n; this reasoning taking place on d_{n-1}, it contradicts the second part of the announcement. The test cannot occur on d_{n-1} because, knowing that it has not occurred on $d_1, \ldots, d_{n-2}$, the students can deduce from this, from the announcement and from the fact that d_n was ruled out as a test day, that the examination must occur on d_{n-1}; this reasoning taking place on d_{n-2}, it contradicts the the second part of the announcement. The rest of the argument proceeds by induction, showing that the test cannot occur on any of the days $d_1, \ldots, d_n$. This conclusion contradicts the first part of the announcement, then this is self-contradictory.

1.4. To fulfill what I have called in 1.2 the coherence requirement, one must take into consideration both the announcement and the "proof" outlined in 1.3. The announcement seems paradoxical *because* there is such a proof: it follows that any plausible interpretation of the announcement must take into account the implicit interpretations just contained in the informal proof.

In this manner we see that the paradoxical effect is logical in character; the informal proof seems to demonstrate that the announcement is self-contradictory, while this logical status is not intuitively evident. Because of this observation I think that it is out of question to introduce any epistemological or pragmatical consideration; the main argument of the paradox has nothing to do with philosophical problems such as "rational behaviour", "ideal knower", and so on, but it proceeds in a strictly logical way.

According with the use the informal proof makes of the expression "to know in advance", this has a sense similar to that of "to predict" in ordinary scientific work. In both cases the point is to establish a proposition about the future as a consequence of factual knowledge and theoretical principles. (For a more detailed treatment of "to predict", see § 7). This point of view is clearly indicated by the informal proof sketched in 1.3. In fact, when (following that "proof") we conclude that day d_n is to be ruled out, we arrive at this conclusion because it is demonstrated that on day d_{n-1} a *prediction* would follow from empirical data (no test during $d_1, \ldots, d_{n-1}$) and a theoretical principle (the announcement). And the manner in which this prediction *follows* has clearly the character of a logical deduction. I do not claim that the expression "to know in advance" always entails a reference to logical deduction; I say that in *this problem* this is the only interpretation coherent with *the text* of the paradox. If we understand "to know in advance" in a different way we can perhaps solve another interesting problem, but not *this* problem. With a different interpretation, the informal proof given in 1.3 is not relevant, and then there is nothing to explain, the paradox emerging just because the informal proof is supposed to be relevant.

The conclusion is the following:

"To explain" or "to solve" the paradox signifies to give the announcement an interpretation such that the informal proof be still relevant, and to decide — within the framework of this interpretation — whether the informal proof is correct or not and where does "the cause" of the paradoxical effect lie.

§ 2. *The Quine-Binkley argument*

2.1. A somewhat pragmatical and modal interpretation was given by W. V. Quine (2) and in a certain sense it was later improved and completed by Robert Binkley (3), who says also that "this places my own discussion in what might be called a Quinean tradition of skepticism with regard to authorities".

Let me place myself out of this tradition, though I accept with pleasure Quine's great authority.

The "skepticism with regard to authorities" consists in taking into account from the beginning the possibility that the teacher's announcement will not be fulfilled.

The first observation I make regarding this position is that it is far from clear: what sense may be attached to the expression "the teacher's announcement will not be fulfilled" ? It is evident that, to know whether the announcement will be fulfilled or not, we must know in advance what the announcement does signify. This observation becomes plausible by considering that the expression "to know in advance", which occurs in the announcement, has been given many different interpretations by different authors. Suppose that we choose an interpretation according to which the announcement is logically self-contradictory. As it is well known, such an interpretation has been actually adopted. In this case the speculation about the *possibility* that the announcement will not be fulfilled becomes less interesting: it is plausible to accept that a self-contradictory announcement *cannot* be fulfilled. This argument shows the necessity of a logical investigation prior to any pragmatical consideration.

2.2. The disparity of opinions (and, consequently, the need for a previous semantical and logical interpretation) begins at the beginning. Quine's version is the following:

"A puzzle that has had some currency from 1943 onward is concerned with a man who was sentenced on Sunday to be hanged on one of the following seven noons, and to be kept in ignorance, until the morning of the fatal day, as to just which noon it would be. By a faulty argument the man persuaded himself that the sentence could not be executed, only to discover his error upon the arrival of the hangman at 11:55 the following Thursday morning". (4)

In the last sentence — "By a faulty argument..." — there is a *non sequitur*: from "K persuaded himself that the sentence could not be executed" and from "the arrival of the hangman took place at 11:55 the following Thursday morning", the conclusion "K's argument was erroneous" does not follow. (The man is called K by Quine, perhaps under a certain kafkian influence). To see this we may remember the interpretation (actually plausible) according to which the announcement is a self-contradiction: in this case it is obvious that K's argument persuading him that the sentence could not be executed is not erroneous, even if the hangman arrives at 11:55 the following Thursday morning, and even if his arrival constitutes a sort of surprise for K. This situation is treated in 6.1 and 7.8. Thus we see again that a semantical and logical interpretation of "to know in advance" is prior to any statement about the fulfilment of the decree.

2.3. To see clearly that Quine's argument neither solves nor explains the paradox, we may give the announcement a more precise form, as for example:

"There will be a test on one of the next *n* days, and it will be impossible to deduce in advance which day, taking this announcement as a premise".

In this form Quine's argument vanishes, but the paradox remains because the text of the informal proof sketched in 1.3 is still relevant. We see then that Quine's argument only *seems* to solve a problem, but actually it is based on a lack of precision in the meaning of "to know in advance".

2.4. Quine himself states the paradox — within his interpretation — into its sharpest form, calling attention to the one-day case:

"There will be a test to-morrow but you won't know to-day on which day the test will occur". (This particular formulation is mine).

According to a logical interpretation in terms of "to deduce" (like (1) or even some weaker form), this one-day announcement is obviously self-contradictory. In Quine's interpretation, however, this announcement has the following form:

"*p*, but you do not know that *p*".

Let us denote by J the operator whose intuitive meaning is

"The students know that", and let e be the empirical assertion: "An examination will occur to-morrow". Then, the teacher's one-day announcement becomes, in this Quinean interpretation:

$$e \,.\, \sim Je \tag{2}$$

This is just the version given by R. Binkley [5], who demonstrates, using a modal logic and the notion of ideal knower, that this announcement is, in form (2), *paradoxical* in the sense that the students cannot believe it though it may be true.

But this is an entirely different problem. The Quine-Binkley argument is interesting in itself, but do not solve at all the examination paradox such as it stands. This non-relevance can be sensed by reflecting that, if we accept form (2) of the announcement, the informal proof sketched in 1.3 is not relevant at all. This contradicts the condition established at the end of 1.4. There is incompatibility between the "spirit" of the Quine-Binkley interpretation and that of the paradox's own text.

2.5. On the other hand, in the usual version of the paradox the announcement itself seems normal while the result of the informal proof seems paradoxical; in the Quine-Binkley interpretation the situation is just the opposite: the announcement itself, in form (2), seems paradoxical, while the result of the reasoning, i.e., that (2) is incredible for the students, seems perfectly normal. It is in fact plausible that the students would not believe a strange announcement such as (2).

2.6. To sum up, we may reject the Quine-Binkley interpretation of *this* paradox in virtue of five reasons:

(a) Before considering from the beginning as a possibility that the announcement will not be fulfilled, it is necessary to clarify the sense of "to know in advance". (See 2.1).

(b) In Quine's version from "K persuaded himself that the sentence could not be executed" and "the arrival of the hangman took place at 11:55 the following Thursday morning", the conclusion "K's argument was erroneous" does not follow. (See 2.2, and also 3.3).

(c) If we give the announcement the more precise form (1), the paradox remains in exactly the same terms, but the Quine-Binkley interpretation does not apply. (See 2.3).

(d) If we accept the Quine-Binkley interpretation, which leads to the form (2) in the one-day case, the paradox does not remain in the same terms but we are faced with *another* problem. Condition at the end of 1.4 is not fulfilled. (See 2.4).

(e) In the usual form of the paradox, the announcement seems normal and the conclusion seems paradoxical, while in the Quine-Binkley interpretation the announcement seems paradoxical and the conclusion seems normal. (See 2.5).

§ 3. *Interpretation in terms of "to deduce"*

3.1. In this paragraph I follow the excellent account made by J. Bennett in his review of several papers in The Journal of Symbolic Logic (*log. cit.*).

Let us adopt the following abbreviations (Bennett):

A: "A test occurs on one of the days $d_1, \ldots, d_n$";

NT_i: "No test occurs on any of the days $d_1, \ldots, d_i$";

A_1: "There is no i such that (A, $NT_i \vdash$ a test occurs on d_{i+1})";

A_2: "There is no i such that (NT_i is true and (A, $NT_i \vdash$ a test occurs on d_{i+1}))";

A_3: "There is no i such that (NT_i is true and (A, NT_i, $A_3 \vdash$ a test occurs on d_{i+1}))".

As Bennet remarks, A_1 is logically false, since it fails for $i = n - 1$. A_2 is true if a test occurs not later than d_{n-1}, otherwise false. A_3 is not logically false, but it is inconsistent with A, as can be shown by induction.

The two most plausible versions of the announcement in terms of deducibility are the conjunctions $A.A_2$, and $A.A_3$. Both fulfills the condition stated at the end of 1.4: in the case of $A.A_2$ the informal proof in 1.3 is relevant but wrong, and in the case of $A.A_3$ the informal proof is relevant and right.

As a pragmatical teacher's announcement, the most plausible is $A.A_2$, and with respect to the informal proof of 1.3, the most adequate is $A.A_3$. In this point lies perhaps the paradoxical effect of this puzzle: the announcement seems at first sight to have the meaning of $A.A_2$, while the "proof" interprets it as $A.A_3$.

3.2. In the one-day case, A_2 and A_3 are equivalent, and both contradicts A: the announcement is self-contradictory with strong evidence.

Thus, the interpretation in terms of "to deduce" not only *solve* the paradox, but illuminates also the possible psychological sources of "paradoxical effects". We shall see now another of these effects which deserves mention.

3.3. From a *pragmatical* point of view, the interpretation of the announcement as $A.A_3$ leads to a situation similar to that faced by the kafkian protagonist in Quine's version. In fact, K remarks that the announcement is self-contradictory, and from this remark he does not conclude that the hangman will not arrive, but simply that the announcement is void. Now K knows also that there are two possibilities essentially important for him: to be hanged and not to be hanged. This second act is doubtless relevant with respect to K, but it is not relevant at all with respect to the solution of the examination paradox, which is supposed to be achieved in the first act.

§ 4. *A set-theoretic approach*

4.1. Let us replace days $d_1, \ldots, d_n$, by the natural numbers $1, \ldots, n$, and consider the set

$$E = \{1, \ldots, n\}.$$

Define a *situation* as an ordered n-tuple $(a_1, \ldots, a_n)$ such that exactly one of its members is 1 and the others are 0. In the intuitive interpretation, the situation that has 1 in the ith place means that the examination occurs on day d_i. Let U be the set of all situations and let u_i (for any $i \in$ E) be that element of U which has 1 in the ith place. For $i \in$E and $i \neq 1$, and for x$\in$U, let x$\sim u_i$ mean that x coincides with u_i in places preceding i (i.e., x has zeros from place 1 to place $i - 1$). For $i = 1$, we take the relation x$\sim u_1$ as valid for all x$\in$U by definition. I shall define now axiomatically the concept of a *solution set.*

Definition D1. A *solution set* is any set S such that the following axioms hold:

(i) S$\subset$U.

(ii) For every $i \in$E, if the condition

$$\text{C: } (\text{x} \in \text{S and x} \sim u_i) \vdash \text{x} = u_i$$

is satisfied, then $u_i \notin$S.

In these terms, the announcement is interpreted as:

There exists a non void solution set. (3)

4.2. *Plausibility of this interpretation.* The mathematical concept of a solution set stands for the intuitive idea of set of situations that fulfil teacher's announcement. With this interpretation in mind, axiom (i) corresponds obviously to the first part of the announcement, i.e. proposition A of 3.1. I shall show that axiom (ii) is a plausible interpretation of the second part of the announcement, taking into account what I have called the coherence requirement (1.2).

I have established in 1.4. that the expression "to know in advance" must be interpreted — regarding *this problem* — in terms of logical deduction. Thus the paradox deals with deductions of the form

$$\text{P} \vdash \text{a test occurs on } d_i$$

where P stands for the premises. The second part of the announcement states that such deductions are not possible, and here lies the predictability (or rather the unpredictability) character of the examination. The point is, of course, to clearly set up the premises P.

It is evident that one of the premises must be NT_{i-1} (see 3.1). The informal proof starts with the following argument: day d_n cannot be a testday because we have

$$A, NT_{n-1} \vdash \text{a test occurs on } d_n \quad (4)$$

and thus the test would be predictable. Then d_n is ruled out. In our terminology the above implication (4) reads:

$$(x \in U \text{ and } x \sim u_n) \vdash x = u_n,$$

and the conclusion consisting in ruling out d_n is expressed by $u_n \notin S$. Thus, in a first approach, one is tempted to set up the following axiom:

(ii') For every $i \in E$, if the condition

$$C': (x \in U \text{ and } x \sim u_i) \vdash x = u_i$$

is satisfied, then $u_i \notin S$.

But this formulation is not sufficient to interprete the informal proof because, in the second step of this, one makes use of the following argument: if $x \sim u_{n-1}$ and *if we suppose that* x *is not ruled out*, we must have $x \neq u_n$, then we can *deduce* $x = u_{n-1}$; and because of this argument we rule out u_{n-1}. The scheme of this reasoning is:

Because the condition

$$(x \text{ is not ruled out and } x \sim u_{n-1}) \vdash x = u_{n-1}$$

is satisfied, we rule out u_{n-1}.

But "ruled out" has been expressed above by "$\notin$ S"; then, "not ruled out" is expressed by "$\in$ S"; generalizing for every $i \in$E we obtain axiom (ii) of Definition D1. It is easy to see that this interpretation holds good also in the extreme cases, $i = n$ and $i = 1$.

4.3. Now, an elementary argument based on induction proves that, if A is a solution set, A is necessarily void. This shows that there is no solution, or that the conditions imposed to the set A are contradictory. Otherwise stated, the announcement in the form (3) is self-contradictory taking into account that it is an assertion about solution sets, and that it is possible to consider Definition 1 as included in (3).

4.4. Definition D1 and proposition (3) seem to be the set-theoretic counterpart of the logical product $A.A_3$ presented in 3.1. But I said also in 3.1 that the most plausible versions of the announcement in terms of deducibility are the conjunctions $A.A_2$ and $A.A_3$. It is then natural to call for a set-theoretic counterpart of $A.A_2$. This is done as follows:

Definition D2. A *solution* set is any set S such that the following axioms hold:

(i') $S \subset U$.

(ii') For every $i \in E$, if the condition

$$C': (x \in U \text{ and } x \sim u_i) \vdash x = u_i$$

is satisfied, then $u_i \notin S$.

In these terms, the announcement is interpreted also as (3).

We have then the following correspondence:

$$A.A_2 \leftrightarrow D_2.(3)$$

$$A.A_3 \leftrightarrow D_1.(3)$$

Observe now that axioms (i') and (ii') do not lead to contradiction and there is at least one non void solution set, so the announcement in the form (3) is true.

From the anecdotic point of view (3) has not the form of what is usually called an announcement; for this purpose we may replace (3) by

I shall choose an element of a non void solution set.

4.5. Although I have remarked a correspondence between $A.A_3$ and $D_1.(3)$, the self-referring nature of A_3 makes it somewhat unsatisfactory; thus, the set-theoretic version $D_1.(3)$ has the slight advantage of avoiding self-reference.

§ 5. *A probabilistic approach*

5.1. I shall now give another interpretation of the paradox, which is closely similar to the set-theoretic one just given, but which uses the language of probabilities.

The probabilistic interpretation of the announcement is the following:

There will be a test on one of the next n days but, on the day before, the probability that the test would occur in the morrow will be different from 1. (5)

To arrive at a formal version of (5), let *a situation* be, as in 4.1, an ordered n-tuple such that exactly one of its members is 1 and the others are 0, and let U be the set of all situations, and E the set of natural numbers from 1 to *n*. Define the elements u_i and the relation $x \sim u_i$ as in 4.1. Now, take U as a space of possibilities and let w be the constant *weight function* defined on U. The formal version of (5) is based on the following definition, which makes use of *conditional probabilities.*

Definition D3. A *solution set* is any set S such that the following axioms hold:

(i'') $S \subset U$

(ii'') For every $i \in E$, if

$$\Pr(x = u_i \mid x \in S \,.\, x \sim u_i) = 1,$$

then $u_i \notin S$.

With this new interpretation of *solution set*, the formal version of announcement (5) is still (3) or, as remarked at the end of 4.4:

I shall choose an element of a non void solution set. (6)

It is now easy to demonstrate by induction that, according to definition D3, every solution set is void. The announcement (in forms (3) or (6)) is again self-contradictory.

5.2. The conjunction D_3.(3) corresponds in an intuitive way to both the conjunction $A.A_3$ and the conjunction D_1.(3) (see 4.4). I shall now complete the other correspondence remarked in 4.4.

Definition D4. A *solution set* is any set *S such that the following axioms hold:*

(i''') $S \subset U$.

(ii''') For every $i \in E$, if $\Pr(x = u_i \mid x \sim u_i) = 1$, then $u_i \notin S$.

Then the announcement is again (3) or (6), and in this case it is not self-contradictory and there is no paradox at all.

We have thus the following correspondence, which completes that given in 4.4:

$$A.A_2 \leftrightarrow D_2.(3) \leftrightarrow D_4.(3)$$

$$A.A_3 \leftrightarrow D_1.(3) \leftrightarrow D_3.(3)$$

§ 6. *Cargile's argument*

6.1. James Cargile [6] gives an interesting probabilistic interpretation of the paradox, though I cannot share his epistemological presuppositions. The core of Cargile's epistemological view (also maintained by other authors) is the following:

"But the teacher's announcement is one that will be true if he gives a test on a day such that the students do not know by the night before which day the test will be, and false otherwise". (7)

I believe that this is a circular argument, because the preceding condition is introduced to clarify an announcement whose difficulty lies in the notion of "to know in advance", but the condition itself makes use of this notion which calls for a clarification. This point is similar to 2.6 (a), established as an objection to the Quine-Binkley argument.

To see this better, we may ask which is the meaning of the expression *"do not know by the night before"*, contained in Cargile's epistemological condition, and thus we are at the beginning of the paradox; to take an example, if our answer is that the expression *"do not know by the night before"* signifies something like the logical interpretation given by A_3 in 3.1 above, then Cargile's epistemological condition applies perfectly well to $A.A_3$. In fact, in this case it is actually impossible *to give a test on a day such that the students do not know by the night before which day the test will be*. It seems to me that when Cargile affirms that the announcement in the form $A.A_3$ do not fulfil his epistemological condition, he gives to the expression "to know in advance" two different meanings, one in the announcement and the other in the condition. It is not suprising then to find that the first do not fulfil the second.

To put my objection in a sharper form I consider two cases:

(a) If the expression "do not know by the night before" has different meanings in the announcement and in the condition, then the latter is arbitrary and hence a non plausible epistemological condition.

(b) If the expression "do not know by the night before" has the same meaning in the announcement and in the condition, then the latter is obvious at this level and resembles Tarski's truth condition.

6.2. In the same paper Cargile writes:

The students' argument is thus not so much a matter of formal deduction as it is a matter of having good reasons for claiming to know.

This leads, in fact, to a new situation, because this proposition contains in a certain sense a sort of definition of the expression "to know in advance". But, unfortunately, this does not destroy the informal proof sketched in 1.3 because the students may have the opinion that *formal deducibility from true premises* is a good reason for claiming to know. But Cargile's position becomes clearer when he says:

"*It is rather a matter of showing that the students would know on Thursday if there were going to be a test on Friday. And this involves that Thursday night the students are going to be in reasonable good intellectual condition and that they have good reason to believe that the tacher will keep his word about giving a test some day of the week*".

The last expression, "*to believe that the teacher will keep his word about giving a test some day of the week*", reveals that — in Cargile's interpretation — the teacher's announcement is formed by two very different parts:

A: There will be a test on one of the next *n* days;

B: You won't know in advance which day.

Cargile seems to think that only part A must be taken for granted; part B is something like "a good wish". With this interpretation in mind, the teacher's announcement must take a form such as

> *There will be a test on one of the next n days and I shall try to give it in such a way that you won't know in advance which day.*

In this form, the logical interpretation in terms of deducibility does not apply, but also the informal proof in 1.3 becomes irrelevant. There is no paradox at all. In particular, condition at the end of 1.4 is not fulfilled. The paradox is replaced by a game played by the teacher and the students: this is in fact what Cargile do in his paper.

This game may be (and actually it is) an interesting problem, but one must confess that proposition (7) differs strongly from the teacher's announcement as it is posed in ordinary versions of the paradox (for example, in 1.3). Then Cargile's paper deals with *another* problem. As this *other* problem has nevertheless some points in common with the very problem of the examination paradox, I shall discuss now these points.

6.3. Cargile gives an acute analysis of this game in terms of decision theory. He examines mainly the situation created on Wednesday evening, i.e., on the evening of the day we have called d_{n-2}. By an argument based on desirability and probability matrices he arrives at the following conclusion:

So a Thursday d_{n-1} test will be *dictated* by Bayesian considerations to our ideally rational teacher if and only if the probability p of such a test's being a surprise is nonzero". (*loc. cit.* p. 557).

This probability p is obviously the key of the argument. It may be treated as an intuitive and *a priori* probability (as Cargile himself remarks) but — in some other interpretation — it may be treated as a perfectly determined number. For example, if we choose an interpretation like that given in 5.1, we may think that this probability p is nonzero if and only if

$$\Pr(x = u_{n-1} \mid x \neq u_i \text{ for all } i \neq n - 1) \neq 1.$$

This equality is false, then $p = 0$.

But we may think also that p is nonzero if and only if

$$\Pr(x = u_{n-1} \mid x \neq u_i \text{ for all } i < n - 1) \neq 1.$$

This is true, then $p \neq 0$.

The anecdotic conditions supposed irrelevant (such as that the students are not drunk, the teacher is not stupid, and so on) *the only relevant data available to estimate the probability p* *are the logical interpretations of the announcement itself.*

But the only way I can see that a logical interpretation of the announcement could supply any concrete datum which might allow us to *calculate* probability p, is by choosing some version in terms of formal deducibility, or any of its equivalents in terms of set-theoretic or proababilistic languages (as made above for calculate $p = 0$ in the first case and $p \neq 0$ in the second).

Then, it seems to me that the following conclusion holds:

Even in the game-interpretation proposed by Cargile, if we wish to arrive at some concrete result, the problem is always a matter of choice between different logical interpretations of the announcement in terms of formal deducibility or equivalent set-theoretic or probabilistic versions.

§ 7. *Formal predictions*

7.1. It is well known that the word "prediction" has received several interpretations in the philosophical literature. I shall not discuss this point here, but I restrict myself to what may be called "formal prediction". Hereafter a *scientific system* will be a system constitued by an axiomatic calculus and an empirical interpretation of this. It is not my purpose to discuss here the problem of formal calculus and its empirical interpretations, so the expressions used above regarding these matters must be understood in a commonly accepted scientific sense. Let S be a scientific system and E a class of propositions expressing empirical data observed at time t_o, and let P be a proposition about empirical facts occuring at a time t.

For $t_o \leqq t_1 < t$ I say that P is *a formal prediction with respect to the system* S, *the empirical data* E *and the time* t_1, if P is logically deducible from S and E. (Time t_1 may be called *the date of the prediction*).

Time t_1 is interpreted as the instant at which the prediction is performed.

It is now natural to say that, with reference to the same elements, P *is formally unpredictable with respect to the system* S, *the empirical data* E *and the time* t_1, if P is not logically deducible from S and E.

Thus, with respect to the same elements, "not formally predictable" and "formally unpredictable" are equivalents. (Remark that these definitions refer to the proposition P, and not to a fact denoted by it).

These concepts are elementary and perhaps trivial, but they suffice to draw an important remark:

The adjectif *unpredictable* applied to P has not an absolute meaning, but it presupposes a (explicit or implicit) reference to a scientific system S, a set E of empirical data and a time t_1.

7.2. I shall now try to analyse the examination paradox within the framework of a schematic theory of formal prediction (as outlined in 7.1).

The teacher's announcement is then a rather complicate statement involving the concept of formal prediction. It may be rephrased as a conjunction F.G, the latter referred to a scientific system S defined below:

F: *There will be a test on one and only one of the days* $d_1, \ldots, d_n$.

G: *If the test occurs on* d_i, *the proposition "The test occurs on* d_i*" is formally unpredictable with respect to the three following elements: (a) The system* S *defined below; (b) The amount* E *of empirical data available until* d_{i-1} *and consisting in the proposition* "NT_{i-1}"; *(c) the date* d_{i-1} *(the date of the prediction).*

The system S consists exactly in the following:

(S_1) Ordinary logic.

(S_2) Propositions F and G above.

7.3. In this manner not only the self-referring character of the announcement is evident, but also we see that the formal unpredictability of the proposition "The test occurs on d_i" is referred to a system which includes the formal unpredictability of the same proposition. It must be remarked that this is a very irregular procedure in scientific work. To see it, let me perform a meteorological version of the same paradox.

7.4. A meteorologist affirms the conjunction F'.G', where:
F': *There will be a rain on one and only one of the days* $d_1, \ldots, d_n$.
G': *If the rain occurs on* d_i, *the proposition "the rain occurs on* d_i*" is formally unpredictable with respect to the three following elements: (a) The system* S' *defined below; (b) the amount* E' *of empirical data available until* d_{i-1}; *(c) the date* d_{i-1}.

The system S' consists exactly in the following:

(S'_1) Ordinary logic.
(S'_2) Proposition F' above.

We do not include G' in (S'_2) because it is ridiculous for a meteorologist to predict an unpredictable rain whose unpredictability is a consequence of (among other premises) this same unpredictability.

We see that in the form just adopted, there is no "rain paradox".

7.5. Let us consider now the question of *the origin* of the paradox. In his quoted review in The Journal of Symbolic Logic, Jonathan Bennett says that Shaw believed that "the origin of the paradox lies in the self-referring nature of [A_3]", and then he comments: "Shaw himself shows that a suitable conjunction of non-self-referring sentences can, like A_3, be shown by induction on *n* to contradict A. Is A_3's self-reference the *origin of* the contradiction, then, in any sense except that A_3 contributes to the contradiction and is self-referring ?"

Bennett's question is relevant, and I shall try to give an answer. As Bennett remarks, Shaw himself provides an example demonstrating that self-reference is not a necessary element of the paradox, and I have show the same feature in §§ 4 and 5 by means of the set-theoretic and the probabilistic interpretations of the paradox. Nevertheless, it is intuitively evident that in all cases there is a kind of circularity. I hope that this point has been clarified in 7.3: the circularity consists in the unusual fact that *the formal unpredictability of a proposition is referred to a system which includes the formal unpredictability of the same proposition.* It may be said that this observation holds only for the formulation in terms of prediction stated in 7.2, but not for the set theoretic or the probabilistic versions.

7.6. Let us examine briefly these two. The only difference between definition D1 (see 4.1) and definition D2 (see 4.4) lies in conditions C, C', included in axioms (ii) and (ii') respectively: the first part of C's antecedent is "$x \in S$" whereas the first part of C' 's antecedent is "$x \in U$", and there is no other difference. It is then plausible to say that the replacement of "U" by "S" is the origin of the paradox *in this set-theoretic version.* But observe that, from an intuitive point of view, formula "$x \in S$" has — among others — the connotation of "the examination on the day indicated by the member 1 of the ordered n-tuple x, is unpredictable"; and this is accepted as a premise in a reasoning leading ultimately to the conclusion "$u_i \notin S$", which has the intuitive meaning of "examination on d_i is predictable". Briefly: unpredictability is allowed as a premise to obtain predictability or unpredictability as a conclusion. A similar interpretation holds in the case of the probabilistic approach (§ 5), allowing that a condition in a conditional probability may be understood as a *premise.*

7.7. To sum up, we have the following conclusions which — I hope — explain *the origin* of the paradox:

(a) In all interpretations presenting the announcement as self-contradictory, this self-contradiction seems to arise *because unpredictability is accepted as a premise to deduce predictability or unpredictability.* (This assertion is formulated, evidently, in a non-formal language).

(b) The psychological effect of the paradox (the only way in which there is a paradox at all) lies in the fact that the announcement (informally established) seems to have the meaning of the top row in diagram at the end of 5.2, whereas the informal proof adopts the interpretation given by the bottom row.

7.8. J. Bennett says also in his review that Shaw remarked that the announcement, although logically false, looks at though it could be satisfied, but he [Shaw] ventures no explanation of this.

I believe that conclusion (b) in 7.7 above gives a psychological explanation of this point. I shall insist however on the necessity of an interpretation before asking whether the announcement can be satisfied or not. If we choose one of the strong interpretations (bottom row in diagram of 5.2), the announcement cannot be fulfilled: even if the test occurs in fact on day $d_k (1 < k \leq n)$ and even if it constitutes a sort of surprise, it remains true that the announcement (in the strong interpretation) has not been fulfilled. At any rate, this curious situation may convince someone that the strong interpretations are not adequate as logical equivalents of the informal announcement. In this case, one must choose the weak interpretation (top row in 5.2) and conclude that the informal proof given in 1.3 is relevant but incorrect.

7.9. Finally, conclusion (a) of 7.7 above clarifies also the "ridiculness' of the induction in the informal proof (see, for example, J. Cargile: *loc. cit.* p. 552): this induction seems ridiculous just because the treatment of unpredictability as detailed in 7.7 (a) differs strongly from its use in ordinary scientific work, as it was remarked at the end of 7.3.

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(1) As stated by Jonathan BENNETT in his review of several papers in: *The Journal of Symbolic Logic*, Vol. 30, 1, March 1965: 101-102.

(2) W. V. QUINE: "On a so-called paradox" (*Mind*, Vol. 62, January 1953), reprinted as "On a supposed antinomy" in Quine's "*The ways of paradox, and other essays*" (Random House, New York, 1966).

(3) R. BINKLEY: "The surprise examination in modal logic" (*The Journal of Philosophy*, Vol. LXV, N° 5, March 7, 1968).

(4) QUINE: *loc. cit.*

(5) BINKLEY: *loc. cit.*

(6) J. CARGILE: "The surprise test paradox", *The Journal of Philosophy*, Vol. LXIV, Number 18, September 21, 1967.

(7) CARGILE: *loc. cit.*