TWO NEW INTERPRETATIONS OF MODALITY

James W. GARSON

1. *Topological logic* (1)

It is possible to develop new semantics for modal logic by taking our cue from semantics designed for topological logics. In topological logic, an indexed operator 'T' is added to the notation of propositional logic, along with an infinite set $X = \{x', x'', \ldots\}$ of *index variables*. The formula 'Txp' is read 'p is the case as of x', where the index 'x' might range over dates, positions, coordinates of space-time, or the denotata of indefinite expressions such as 'forty years ago' and 'forty feet from here'. Quantifiers are introduced over the index variables, but there are no individual variables or predicate letters.

Throughout this paper, the letter 'p' ranges over propositional variables, the letter 'x', over index variables, and the letters 'A', 'B', and 'C', over formulas of topological logic. The formulas of topological logic have the forms p, $-$A, $(A \rightarrow B)$, TxA, and $\forall$xA. The other logical constants are introduced by definition in the usual way.

To provide semantics for topological logic, let a *D-model* U be a pair $<D, u>$ consisting of a non-empty set D of *contexts* (dates, places, or what have you), and an *interpretation function* u which assigns a truth value to each proposition in a context. So u is a function from $D \times P$ into the set $\{0,1\}$ of truth values, where P is the set of propositional variables. ($u: D \times P \rightarrow 2$.) A *valuation* v is a function from X into D. ($v: X \rightarrow D$.)

We use letters 'b', 'c', 'd', and 'e' as metavariables over D, and letters 'v' and 'w' as metavariables over valuations. The symbols '$\exists$', '(x)', '$\supset$', '&', '$\sim$', 'iff', and '$\in$' are used in the metalanguage with their usual interpretations. A quantifier which binds a variable in the metalanguage is taken to be restricted to its corresponding set; so, for example, '(d)' abbreviates '$(d)_{\in D}$'.

The predicate $\models_d^{Uv} A$ (read 'the formula A is true on model U at context d on valuation v') is defined recursively as follows:

i. $\models_d^{Uv} p$ iff $u(d, p) = 1$

ii. $\models_d^{Uv} -B$ iff $\sim \models_d^{Uv} B$

iii. $\models_d^{Uv} (B \rightarrow C)$ iff $\models_d^{Uv} B \supset \models_d^{Uv} C$

iv. $\models_d^{Uv} \forall xB$ iff $(w)(w =_x v \supset \models_d^{Uw} B)$, where $w =_x v$ iff w agrees with v for all values other than at x.

A final clause is need governing the truth conditions for formulas of the form 'TxB'. If 'x' is thought of as ranging over dates or spatial coordinates, then we would like TxB to be true in context d just in case B is true in the context denoted by x:

D-v. $\models_d^{Uv} TxB$ iff $\models_{v(x)}^{Uv} B$.

A formula A is *D-valid* iff $(v)(d) \models_d^{Uv} A$ on every D-model U.

However, x may range over the denotata of indefinite expressions (like 'forty feet from here'), where the denotation of x is not taken to be a context but rather a function from contexts to contexts. Then an F-model should be a triple $<D, u, F>$ consisting of a domain D of contexts, and an interpretation function u as before, and a non empty set F of unary functions from D into D. Then a valuation assigns to each variable a member of F. We use letters 'f', 'g', and 'h' as metavariables over F. The sentence 'B is the case as of forty years ago' is true in the context 1970 just in case B is true in the context obtained by applying the function denoted by 'forty years ago' (the function $f(d) = d - 40$) to 1970. So in this case the truth clause for formulas of the form TxB should read:

F-v. $\models_d^{Uv} TxB$ iff $\models_{v(x)(d)}^{Uv} B$, where v(x)(d) is the result of applying the function v(x) to d.

A formula A is S(F)-valid (S(F)-satisfiable) iff $\models_d^{Uv}$ A at every (some) v and d, on every (some) F-model U such that S(F). Let C(F) be the condition that F is the set of constant functions on D. It is not difficult to show that A is D-valid iff A is C(F)-valid.

2. *Semantics for modal logic*

Now let us turn to modal logic. Suppose that D is the set of all possible worlds, i.e. the set of all positions in *logical* space. Then 'TxA' receives the interpretation 'A is the case as of possible world x'. The box may be introduced by definition into the notation of topological logic through the standard definition of necessity:

Def$\Box$: $\Box B =_{Df} \forall x T x B$.

So the formulas of modal logic are taken to be certain closed formulas of topological logic. Semantics for modal logic may be obtained simply by defining as valid the modal formulas of topological logic which are either D-valid, or S(F)-valid.

If we choose the first alternative, the truth condition for modal formulas of the form $\Box$B is

D-vi. $\models_d^{Uv} \Box B$ iff $(e) \models_e^{Uv} B$,

for $\models_d^{Uv} \Box B$ iff $\models_d^{Uv} \forall x T x B$ iff $(w)(w =_x v \supset \models_d^{Uw} TxB)$ iff

$(w)(w =_x v \supset \models_{w(x)}^{Uw} B)$ iff $(e) \models_e^{Uw} B$ iff $(e) \models_e^{Uv} B$, since B is a modal formula, and hence contains no free variables. We may design a semantics specifically for modal formulas using the notion of D-validity, by choosing truth clauses i, ii, iii, and D-vi, deleting mention of valuations. The result is the familiar semantics for S5, hence the D-valid modal formulas are exactly the theorems of S5.

Weaker modal systems may be captured semantically using the notion of S(F)-validity. Then the truth condition for $\Box$B is:

F-vi. $\models_d^{Uv} \Box B$ iff $(f) \models_{f(d)}^{Uv} B$, since $\models_d^{Uv} \Box B$ iff $\models_d^{Uv} \forall x TxB$ iff $(w)(w =_x v \supset \models_d^{Uw} TxB)$ iff $(w)(w =_x v \supset \models_{w(x)(d)}^{Uw} B)$ iff $(f)\models_{f(d)}^{Uw} B$ iff $(f)\models_{f(d)}^{Uv} B$. Semantics for modal formulas alone is obtained by deleting mention of valuations in truth clauses i, ii, iii, and F-vi.

The resulting semantics is a denotational version of the transformation semantics developed by van Fraassen [2], although it was discovered independently. Here we introduce a set F of unary functions (transformations) where Kripke would introduce a binary relation R. Kripke semantics may be defined in a simplified version [3] as follows: A *K-model* is a triple $<D, u, R>$, where D and u are as before and R is a binary relation on D. The truth clauses are i, ii, iii, (with mention of valuations deleted) and

K-vi. $\models_d^U \Box B$ iff $(e)(Rde \supset \models_e^U B)$.

In Kripke semantics each axiom (A) of modal logic corresponds to a condition A(R) on R so that when KS is the system K plus axioms (A_1), ..., (A_n), and S(R) is the conjunction $A_1(R)\&\ldots\&A_n(R)$ of their corresponding conditions on R, then $\models_{KS}A$ iff A is S(R)-valid. (The system K consists of the principles of propositional logic, the necessitation rule: $\vdash A \supset \vdash \Box A$, and the axiom $\Box(A \to B) \to (\Box A \to \Box B)$.)

The same is true of the modal semantics just presented. Each modal axiom (A) corresponds to a condition A(F) on F so that when DS is D plus axioms (A_1), ..., (A_n) and S(F) is $A_1(F)\&\ldots\& A_n(F)$, then $\vdash_{DS}A$ iff A is S(F)-valid. (D is K plus the axiom (D): $\Box A \to \Diamond A$.) A list of some common modal axioms and their corresponding conditions on R and F follows: [4]

(D) $\Box A \to \Diamond A$	D(R) $\exists e Rde$	D(F) $F \neq 0$ [5]
(T) $\Box A \to A$	T(R) Rdd	T(F) $\exists f(f(d) = d)$
(4) $\Box A \to \Box\Box A$	4(R) $Rbc\&Rcd \supset Rbd$	4(F) $\exists f(g(h(d)) = f(d))$
(B) $A \to \Box\Diamond A$	B(R) $Rde \supset Red$	B(F) $\exists f(f(g(d)) = d)$
(E) $\Diamond A \to \Box\Diamond A$	E(R) $Rbc\&Rbd \supset Rcd$	E(F) $\exists f(f(g(d)) = h(d))$.

In general we may produce A(F) from A(R) by the equivalence:

RF Rde iff $\exists f(f(d) = e)$.

The reader may easily verify that the conditions on F in the above list are equivalent to the result of replacing $\exists f(f(d) = e)$ for Rde throughout in the corresponding condition on R.

We now intend to prove the above assertions. We will prove

Theorem 1. A is S(F)-satisfiable iff A is S(R)-satisfiable, where S(F) is the translation of S(R) by RF, and where S(R) entails D(R).

Proof. (left to right) We are given that A is S(F)-satisfiable, so there is a model $U_F = \langle D, u, F \rangle$ and a context d such that S(F) and $\models_d^{U_F} A$. We may produce a Kripke model U_K from U_F such that S(R) and $\models_d^{U_K} A$, by letting $U_K = \langle D, u, R \rangle$, where R is defined by RF: Rde iff $\exists f(f(d) = e)$. Clearly S(R) holds in this model. We prove that $\models_d^{U_K} A$ by showing $\models_d^{U_K} A$ iff $\models_d^{U_F} A$ for all $d \in D$, using an induction on the form of A. The proof is straightforward, and the only case worth reviewing is the one where A has the form $\Box B$. But $\models_d^{U_K} \Box B$ iff $(e)(Rde \supset \models_e^{U_K} B)$ iff $(e)(Rde \supset \models_e^{U_F} B)$ (by the inductive hypothesis) iff $(e)(\exists f(f(d) = e) \supset \models_e^{U_F} B)$ (by RF) iff (f) $\models_{f(d)}^{U_F} B$ (by identity theory) iff $\models_d^{U_F} \Box B$.

(right to left) We are given that A is S(R)-satisfiable, so there is a model $U_K = \langle D, u, R \rangle$ and a context d such that S(R) and $\models_d^{U_K} A$. Now we define U_F from U_K so that S(F) and $\models_d^{U_F} A$, by letting $U_F = \langle D, u, F \rangle$, where F is defined by

Def_F: $f \in F$ iff $f: D \rightarrow D$ & $(d)Rdf(d)$.

Now we show that U_F is such that S(F) by proving RF. (Clearly if S(R) holds and RF holds, then so does S(F), since S(F) results from replacing $\exists f(f(d) = e)$ for Rde in S(R).) The proof of RF follows: Suppose Rde. Let the function g be defined by $g(d) = e$ and $g(c) = b_c$ for $c \neq d$, where b_c is a member of D chosen so that Rcb_c. That such a b_c can be located for each $c \in D$ is a consequence of the assumption that S(R) entails D(R), hence $\exists eRce$. Now $g \in F$, for $Rcg(c)$ for all $c \in D$, because when $c = d$, $g(c) = e$, and we have that Rde, and when $c \neq d$, $g(c) = b_c$, and b_c has been chosen so that Rcb_c. So it follows that $\exists f(f(d) = e)$, namely g. Now suppose $\exists f(f(d) = e)$. Then $Rdf(d)$. But $f(d) = e$, so Rde. With RF in hand we may prove $\models_d^{U_F} A$ exactly as in the previous case.

Theorem 1 amounts to a consistency and completeness proof with respect to the semantics of this section for any modal system DS (as strong as D) for which we know the corresponding condition S(R) on R in Kripke semantics. We may simply use RF to produce the appropriate conditions on F for this system, and then the completeness proof is a corollary of Theorem 1 as follows: We will know that $\vdash_{DS}$ iff A is S(R)-valid. By Theorem 1, A is S(R)-valid iff A is S(F)-valid, hence $\vdash_{DS} A$ iff A is S(F)-valid.

The conditions on F resemble, but are weaker than the following:

T'(F) $\exists f(d)(f(d) = d)$, (F contains the identity function.)
4'(F) $\exists f(d)(g(h(d)) = f(d))$, (F is closed under composition of functions.)
B'(F) $\exists f(d)(f(g(d)) = d)$, (F contains inverse functions.)
E'(F) $\exists f(d)(f(g(d)) = h(d))$.

It turns out that these stronger and more interesting conditions may be substituted for their weaker counterparts, for when this is done, the set of modal formulas defined as valid remains unchanged. (6)

This is immediately apparent in the cases of T'(F) and 4'(F). The proof that A is T(R)-satisfiable iff A is T'(F)-satisfiable may be carried out as in Theorem 1, with the additional remark that since R is reflexive, it follows by Def_F that the identity function f_i is one of the members of F. From $f_i(d) = d$ and Rdd, it follows that Rdf(d). In the case of 4'(F), the proof is the same save that now we must show that given Def_F and the transitivity of R, it follows that F is closed under composition of functions. Let g and h be arbitrary members of F. By Def_F, Rdh(d) and Rh(d)g(h(d)). R is transitive so Rdg(h(d)); hence the function goh such that goh(d) = g(h(d)) is a member of F.

The case of B'(F) involves complications because in order for F to contain inverse functions, F must contain one-one functions only. In this case we may define F in U_F so that f∈F iff f is a one-one function from D to D and (d)Rdf(d). [7] This added restriction leads to difficulty, for we can no longer show that RF, since the function g used to establish this might not be one-one, and so not a member of F. (We have no assurance that $b_c \neq b_d$ for any $c \neq d$.) However, when we know that R is reflexive as well as symmetric, (when the modal system at issue is B or stronger) then we may let g be defined so that g(d) = e and g(c) = c for $c \neq d$. This function is clearly one-one, and since Rcc, it is in F. Given a proof of RF we may show that $\models_d^{U_F} A$ as before. We must also show that S(F), which will require (at least) a demonstration that F contains inverses. But let g be any member of F, and let $\bar{g}$ be its inverse. Then by the definition of F, $R\bar{g}(d)g(\bar{g}(d))$, and by the symmetry of R, $Rg(\bar{g}(d))g(d)$. Since $g(\bar{g}(d)) = d$, we have $Rd\bar{g}(d)$, hence $\bar{g} \in F$.

In cases where we do not know that R is reflexive, but that it is symmetric, we must resort to a more complicated strategy involving two stages. First, for each K-model U_K, we show how to define a new K-model $U'_K = \langle D', u', R' \rangle$ which leaves the set of valid formulas unchanged. This new model will be constructed so that there are enough members of D' to insure (in effect) that $b_c \neq b_d$; then it will be possible to define a one-one function g with which to demonstrate RF. U'_K is defined from U_K as follows. D' is defined recursively from D by stipulating that $D \subseteq D'$ and $d \in D' \supset \{d\} \in D'$. Then we provide an association function a, which maps D' into D as follows:

$a(\mathrm{d}) = \mathrm{d}$, for d∈D, and
$\mathrm{a}(\{\mathrm{d}\}) = \mathrm{b}_{a(\mathrm{d})}$, for d∈D'.

So, for instance {{d}} (for d∈D) is associated with $\mathrm{b}_{\mathrm{b}_\mathrm{d}}$ by a, where $\mathrm{b}_{\mathrm{b}_\mathrm{d}}$ turns out to be the member chosen for b_d such that $\mathrm{Rb}_\mathrm{d}\mathrm{b}_{\mathrm{b}_\mathrm{d}}$.
Now we may define u' and R':

u'(d, p) = u(a(d), p) for d∈D'.
R'de iff Ra(d)a(e) for d, e∈D'.

A thoughtful inspection of this definition should convince one that it is possible to prove $\models_{\mathrm{d}}^{\mathrm{U'}_\mathrm{K}}\mathrm{A}$ iff $\models_{a(\mathrm{d})}^{\mathrm{U}_\mathrm{K}}\mathrm{A}$, for d∈D' by induction on the form of A. As a special case of this we have $\models_{\mathrm{d}}^{\mathrm{U'}_\mathrm{K}}\mathrm{A}$ iff $\models_{\mathrm{d}}^{\mathrm{U}_\mathrm{K}}\mathrm{A}$ for d∈D (since a(d) = d, for d∈D.), which is all we need for the remainder of this proof. Now we define U_F from $\mathrm{U'}_\mathrm{K}$ exactly as we did before. We let $\mathrm{U'}_\mathrm{K}$ and U_F agree on their domains and interpretation functions, and we let F be the set of one-one functions such that (d)Rdf(d). RF is then proven as before save that g is defined so that g(d) = e and g(c) = {c} for c ≠ d. This function is clearly one-one ({c} ≠ {d}, for c ≠ d.), so to show that it is in F, we need only show that R'cg(c) for all c∈D'. When c = d, then g(c) = e, and we have R'de; and when c ≠ d, then g(c) = {c}, hence we must prove that R'c{c} for all c∈D'. But R'c{c} iff Ra(c)a({c}), and $a(\{\mathrm{c}\}) = \mathrm{b}_{a(\mathrm{c})}$. Furthermore $\mathrm{b}_{a(\mathrm{c})}$ has been chosen so that $\mathrm{R}a(\mathrm{c})\mathrm{b}_{a(\mathrm{c})}$, hence R'c{c}. Now that RF is established, we prove that $\models_{\mathrm{d}}^{\mathrm{U'}_\mathrm{K}}\mathrm{A}$ iff $\models_{\mathrm{d}}^{\mathrm{U}_\mathrm{F}}\mathrm{A}$ as before, which together with a previous result yields $\models_{\mathrm{d}}^{\mathrm{U}_\mathrm{K}}\mathrm{A}$ iff $\models_{\mathrm{d}}^{\mathrm{U}_\mathrm{F}}\mathrm{A}$, hence $\models_{\mathrm{d}}^{\mathrm{U}_\mathrm{F}}\mathrm{A}$. Finally we must show that U_F has the property S(F), which entails showing that F contains inverses. But this proof may be supplied exactly as before, once we note that the symmetry of R entails the symmetry of R'.

In the case of E'(F) similar difficulties arise, and they may be disposed of using the same strategies.

So it is possible to provide consistency and completeness proofs on F-semantics for the strong conditions on F for any modal system DS, where S is a selection of the axioms T, 4, B, and E. The proof may be carried out for any such system using the two-stage strategy just outlined. When conditions on R other than B are present, these conditions will remain in force for R' and the corresponding conditions on F will follow without difficulty. As we have seen, for most selections of axioms, there are less complicated methods available.

3. *A group theoretic semantics for modal logic* [8]

Another semantical approach for topological logic has been investigated. [9] It is somewhat less satisfying intuitively, but it yields interesting results for modal logic. An F-model contains a set F of unary functions on D. Instead, let us provide a binary operation o on D, so that a o-model U is a triple $<D, u, o>$, where D and u are as before. Then a valuation assigns to each variable a member of D. ($v: X \rightarrow D$.) The truth clauses remain as in F-semantics save that the clause for formulas of the form TxB reads:

o-v. $\models_{d}^{Uv} TxB$ iff $\models_{dov(x)}^{Uv} B$.

So TxB is taken to be true in the context 1970 just in case B is true in the context one gets by applying o to 1970 and the denotation of x. When x denotes —40 (40 years ago), then 1970o-40 might yield 1930.

If we define the box in topological logic by Def$_{\Box}$, we obtain the following truth condition for modal formulas of the form $\Box B$:

o-vi. $\models_{d}^{U} \Box B$ iff (c) $\models_{doc}^{U} B$.

The predicates S(o)-valid and S(o)-satisfiable are defined on analogy with S(F)-valid and S(F)-satisfiable. Again modal axioms correspond to conditions on o as follows:

D(o) $\exists e \exists c(doc = e)$, But this is guaranteed, as o is an operation.

T(o) $\exists e(doe = d)$,
4(o) $\exists e((boc)od = boe)$,
B(o) $\exists e((cod)oe = c)$,
E(o) $\exists e((cod)oe = cob)$.

The condition A(o) may be obtained from A(R) by the equivalence

Ro: Rde iff $\exists c(doc = e)$,

which we will demonstrate in

Theorem 2. A is S(o)-satisfiable iff A is S(R)-satisfiable, where S(o) is the translation of S(R) by Ro, and S(R) entails D(R).

Proof. (left to right) Let $U_o = \langle D, u, o \rangle$ be a o-model which satisfies S(o), such that $\models_d^{U_o} A$. Let U_K be $\langle D, u, R \rangle$, where R is defined by Ro. Clearly S(R). We prove that $\models_d^{U_K} A$ iff $\models_o^{U_o} A$ by induction on the form of A, the interesting case being when A has the form $\Box B$: $\models_d^{U_K} \Box B$ iff $(e)(Rde \supset \models_e^{U_K} B)$ iff $(e)(\exists c(doc = e) \supset \models_e^{U_o} B)$ iff $(e)(c)(doc = e \supset \models_e^{U_o} B)$ iff $(c) \models_{doc}^{U_o} B$ iff $\models_d^{U_o} \Box B$.

(right to left) Let $U_K = \langle D, u, R \rangle$ be a K-model which satisfies S(R), such that $\models_d^{U_K} A$. Let $U_o = \langle D, u, o \rangle$ where o is defined in terms of R by

$doc = c$, when Rdc, and
$doc = b_d$, when $\sim$Rdc.

(We remember that b_d is chosen so that Rdb_d.)
Now we must prove that S(o) by demonstrating Ro; Suppose Rde, then $doe = e$ and hence $\exists c(doc = e)$. Now suppose $\exists c(doc = e)$. Then either Rdc and $e = c$, or $\sim$Rdc and $e = b_d$. In the first case Rde, and in the second Rdb_d, hence Rde. Now

we may prove $\models^{U}_{d}KA$ iff $\models^{U}_{d}\circ A$ as in the previous case.

The conditions presented for o remind one of the stronger conditions from group theory which follow:

T′(o) $\exists i_{\in D}$ (doi = d & iod = d), (D contains an identity element.)
4′(o) (boc)od = bo(cod), (o is associative.)
B′(o) $\exists$ e(doe = i), (D contains inverse elements.)
E′(o) $\exists$ e(doe = c).

Certain selections of these conditions may be substituted for the corresponding selection of weaker conditions without affecting the set of formulas defined as valid. There is no problem with replacing T(o) with T′(o) as we show in

Theorem 3. A is T(o)-satisfiable iff A is T′(o)-satisfiable. The proof from left to right is trivial. Now suppose that A is T(o)-satisfiable, and let U = <D, u, o> be a model such that T(o), and let d be a context such that $\models^{U}_{d}A$. Now we define a model U′ such that T′(o) and $\models^{U'}_{d}A$, by stipulating that U′ = <D′, u′, ó>, where D′ = D ∪ {i}, for some i not in D, where u′ agrees with u on arguments from D, and u′(i, p) = 1 for all p, and where ó agrees with o for arguments in D, and iód = d and dói = d for all d∈D′. Now U′ satisfies T′(o), for i is an identity element. We must now show that $\models^{U'}_{d}A$, by proving that $(d)_{\in D}(\models^{U}_{d}A$ iff $\models^{U'}_{d}A)$, by induction on the form of A. The interesting case is when A has the form □B: $\models^{U}_{d}\Box B$ iff $(e)_{\in D}$ $\models^{U}_{doe}B$ iff $(e)_{\in D}\models^{U}_{dóe}B$ (since ó agrees with o over D) iff $(e)_{\in D}\models^{U'}_{dóe}$ (by the hypothesis of the induction). All that is needed to complete this case is a demonstration that $(e)_{\in D}$ $\models^{U'}_{dóe}B$ iff $(e)_{\in D'}\models^{U'}_{dóe}B$, since $(e)_{\in D'}\models^{U'}_{dóe}B$ iff $\models^{U'}_{d}\Box B$. The proof from right to left is trivial, and since $D' = D \cup \{i\}$ we need only show $(e)_{\in D} \models^{U'}_{d\acute{o}e} B \supset \models^{U'}_{d\acute{o}i} B$. But dói = d, and by T(o), $\exists e_{\in D}\, d\acute{o}e = d$; hence $(e)_{\in D} \models^{U'}_{d\acute{o}e} B \supset \models^{U'}_{d} B$. As an immediate corrollary of Theorem 3 we have that $\vdash_T A$ iff A is T'(o)-valid.

We turn now to the proof that the axiom (4) corresponds to the associativity of o on this semantics.

Theorem 4. A is 4(o)-satisfiable iff A is 4'(o)-satisfiable. The proof from right to left is again trivial. Let $U = \langle D, u, o \rangle$ be a model such that 4(o) and let d be a context such that $\models^{U}_{d} A$. Let $U' = \langle D', u', \acute{o} \rangle$, where D' is the set of all non-null sequences $d_1 \ldots d_n$ of members of D. To define u', we provide an association function a from D' to D defined as follows:

$a(d) = d$, for $d \in D$; and

$a(de) = a(d)oe$, for $d \in D'$ and $e \in D$.

So, for instance, $a(bcd) = a(bc)od = (boc)od$, for b, c, $d \in D$. Now we let $u'(d, p) = u(a(d), p)$ and we define ó so that dóe = de. Clearly ó is associative, so we need only show that $\models^{U'}_{d} A$, which is a consequence of

L: $\models^{U}_{a(d)} A$ iff $\models^{U'}_{d} A$,

for since $d \in D$, and $a(d) = d$, we have $\models^{U}_{d} A$ iff $\models^{U'}_{d} A$. L is proven by induction on the form of A, the interesting case being when A has the form $\Box B$: $\models^{U}_{a(d)} \Box B$ iff $(e)_{\in D} \models^{U}_{a(d)oe} B$ iff $(e)_{\in D} \models^{U}_{a(de)} B$ (by the definition of a). Now if we can show that $(e)_{\in D} \models^{U}_{a(de)} B$ iff $(e)_{\in D'} \models^{U}_{a(de)} B$, the proof will be complete since $(e)_{\in D'} \models^{U}_{a(de)} B$ iff $(e)_{\in D'} \models^{U'}_{de} B$ (by the hypothesis of the induction) iff $(e)_{\in D'} \models^{U'}_{d\acute{o}e} B$ iff $\models^{U'}_{d} \Box B$. The proof of the missing equivalence is trivial from right to left. Now suppose

$(e)_{\in D}\ \models^{U}_{a(de)}B$. Let c be any sequence $d_1...d_n$ of members of D. Then we must show that $\models^{U}_{a(dc)}B$. Now $a(dc) = (...(a(d)od_1)...od_n)$. By applying the condition 4(o): $\exists e((boc)od = boc)$ to $(...(a(d)od_1)...od_n)$ n times, we may prove that there is a member b of D such that $a(d)ob = (...(a(d)od_1)...od_n)$. Now it follows from $(e)_{\in D}\ \models^{U}_{a(de)}B$ that $(e)_{\in D}\ \models^{U}_{a(d)oe}B$, and so $\models^{U}_{a(d)ob}B$. But $a(d)ob = a(dc)$, hence $\models^{U}_{a(dc)}B$, which is the desired result. As an immediate corollary of Theorem 4, we have a consistency and completess proof for the system D4, for by Theorem 2 we have that A is D(R)&4(R)-satisfiable iff A is D(o)&4(o)-satisfiable. But D(o) is guaranteed by the definition of an o-model, so A is D(o)&4(o)-satisfiable iff A is 4(o)-satisfiable iff A is 4'(o)-satisfiable. So it follows that $\vdash_{D4}A$ iff A is 4'(o)-satisfiable.

We will now provide what amounts to a consistency and completess proof for the system S4 by proving

Theorem 5. A is T(o)&4(o)-satisfiable iff A is T'(o)&4'(o)-satisfiable.

The proof is identical to that for Theorem 4, save that D' is the set of all sequences of members of D, including the null sequence - such that d - = d for d∈D'; and when it comes to proving the missing equivalence, we must let c be any sequence of members of D, including -. When c ≠ -, we procede as before, and when c = -, we have to show that $\models^{U}_{a(dc)}B$ or $\models^{U}_{a(d)}B$. But we know that o satisfies T(o) so $\exists e(a(d)oe = a(d))$. So from $(e)_{\in D}\ \models^{U}_{a(de)}B$, we obtain $\models^{U}_{a(d)}B$, the desired result.

Unfortunately, similar results are not available for the condition B'(o), as the axiom (B) is not B'(o)-valid. ([10]) However, in the presence of both T'(o) and 4'(o), matters improve, so that we will be able to provide a consistency and completeness proof for S5 by demonstrating

Theorem 6. A is T'(o)&4'(o)&B'(o)-valid ⊃ A is D-valid. Proof. Suppose A is T'(o)&4'(o)&B'(o)-valid, i.e. —A is not T'(o)&4'(o)&B'(o)-satisfiable. Suppose for *reductio* that A is not D-valid, so there is a model U = <D, u> and a context d such that $\models^{U}_{d}$— A. Now let us define a model U_o = <D, u, o>, where o is defined as follows: First, we provide a one-one mapping *a* from *Re* (the set of reals) to D. We may assume that D is of the appropriate cardinality without cost, for if D is smaller than *Re*, standard techniques will provide a model with a larger domain on which —A is true at d, and furthermore D need never be of higher cardinality than *Re*, as those who are familiar with Henkin-style completeness proofs for modal logics will see, for the set of all maximally consistent sets is equinumerous with *Re*. Now let us index each member of D by letting d_j be the unique d such that: $a(d) = j$. The operation o is defined by:

$$d_j o d_k = d_{j+k}.$$

This operation has the three properties T'(o), 4'(o), and B'(o), since + does. All that remains is a demonstration that $\models^{U_o}_{d}$— A, for once this is shown, we will know that —A is T'(o)&4'(o)&B'(o)-satisfiable, which completes the *reductio*. $\models^{U_o}_{d}$— A may be proven by showing $\models^{U}_{d}A$ iff $\models^{U_o}_{d}A$ for all d∈D, the interesting case being when A has the form □B. Then $\models^{U}_{d}\Box B$ iff (e) $\models^{U}_{e}B$ iff (e) $\models^{U'}_{e}B$ (by the inductive hypothesis). Once we show that (e) $\models^{U'}_{e}B$ iff (e) $\models^{U'}_{doe}B$, we will have completed the proof of Theorem 6, for (e) $\models^{U'}_{doe}B$ iff $\models^{U'}_{d}\Box B$. The proof from left to right is trivial. Now suppose (e) $\models^{U'}_{doe}B$, then $(k)_{\in Re}$ $\models^{U'}_{d_j o d_k}B$, where $d_j = d$. So $(k)_{\in Re}$ $\models^{U'}_{d_{j+k}}B$. But the set of reals has the property $\exists l_{\in Re}(j + l = m)$ for any j, m ∈*Re*. So

$(k)_{\in Re} \models^{U'}_{d_{j+k}} B$ entails $\models^{U'}_{d_{j+1}} B$, hence $\models^{U'}_{d_m} B$, for any $m \in Re$.

Since every member of D is indexed, it follows that (e) $\models^{U'}_{e} B$.

Now that we have shown Theorem 6, we may show that $\vdash_{S5}A$ iff A is T'(o)&4'(o)&B'(o)-valid. The proof from left to right may be carried out in standard fashion by induction on the form of the proof of A. Now suppose A is T'(o)&4'(o)&B'(o)-valid. Then by Theorem 6, A is D-valid; but we remember that A is D-valid iff $\vdash_{S5}A$.

We will now review the results concerning the strong conditions when using o-semantics:

Modal System	Conditions on o	
T	o has an identity element	(Thm. 3)
D4	o is associative	(Thm. 4)
S4	o is a semi-group	(Thm. 5)
S5	o is a group	(Thm. 6).

The author has verified that modal systems which contain axiom (B) or axiom (E), and which are weaker than S5, are not consistent when the strong conditions are used with o-semantics.

4. *Modality within topological logic*

Once we have defined the box in topological logic via Def□ it becomes possible to capture modal systems axiomatically by choosing the appropriate system of topological logic. We have proven in [3] that $\vdash_{TQ}A$ iff A is D(F)-valid (iff A is D(o)-valid), ([11]) where TQ is the system formed from the principles of quantificational logic plus the rule R: $\vdash A \supset \vdash TxA$ and the axioms:

(— A) Tx — A → —TxA
(A —) — TxA → Tx — A
(A →) Tx(A → B) → (TxA → TxB)
(AQ) $\forall x Ty A \rightarrow Ty \forall x A$, for $x \neq y$.

It follows by Theorem 1 (Theorem 2) that $\vdash_{TQ}A$ iff $\vdash_{D}A$ when A is a formula of modal logic.

Modal systems stronger than D can be captured in topological logic simply by adding the modal axioms not in D to the system TQ. Let S be a selection of modal axioms and TQS and DS, the systems that result from adding S to TQ and D respectively. Then

Theorem 7. $\vdash_{DS}A$ iff $\vdash_{TQS}A$, for any modal formula A. ([12])
The proof is trivial from left to right, for all the principles of D are present in TQ, and axioms in S appear in TQS. The proof from right to left is not trivial, for it might turn out that by using principles and formulas of TQS not available in DS, one might prove a modal formula not provable in DS. Suppose $\vdash_{TQS}A$. It is easy to show the consistency of TQS with respect to F-semantics (or o-semantics), hence $\vdash_{TQS}A \supset A$ is S(F)-valid. But by Theorem 1, A is S(F)-valid $\supset$ A is S(R)-valid, and this, we know, entails that $\vdash_{DS}A$.

University of Pennsylvania James W. GARSON

BIBLIOGRAPHY

[1] J. GARSON, A new interpretation of modality (abstract), *Journal of Symbolic Logic*, vol. 34 (1969), p. 535.
[2] J. GARSON, The completeness of an intensional logic: definite topological logic, forthcoming in *Notre Dame Journal of Formal Logic*.
[3] J. GARSON, Indefinite topological logic, forthcoming in *Journal of Philosophical Logic*.
[4] J. GARSON, *The Logics of Space and Time*, doctoral dissertation, University of Pittsburgh, 1969.
[5] N. RESCHER, Topological logic, *Journal of Symbolic Logic*, vol. 33 (1968), pp. 537-548, with J. Garson.
[6] B. VAN FRAASEN, *Formal Semantics and Logic*, MacMillan, 1971.

NOTES

(1) Topological logics and their semantics are discussed more thoroughly in [2], [3], [4], and [5].

(2) see [6], p. 151 ff.

(3) The semantics here presented eliminates Kripke's G (the actual world). It is easy to show that the two versions are equivalent.

(4) van Fraassen reports virtually the same conditions for (D), (T), (4), and (B) in [6], p. 152. Our results will be a bit more general than his.

(5) The condition D(F) is a consequence of the way we have defined an F-model. There are no difficulties in retracting the stipulation that $F \neq O$ in the semantics for modal logics. When this is done, we obtain a semantical characterization of the theorems of K. It is not difficult to show that $\vdash_K A$ iff (d) $\overset{U}{=}_d A$ on every F-model, where F is possibly empty.

(6) van Fraassen reports similar results in [6], pp. 152-153. Again, our results are a bit more general.

(7) The stipulation that f be a one-one function in the definition of F was suggested to me by Bas van Fraassen.

(8) My thanks to Nuel Belnap, who was extremely helpful during my early research on the results of this section.

(9) It appears in [4], Chapter 5.

(10) In [1], we reported mistakenly that the system B = KTB is such that $\vdash_B A$ iff A is T'(o)&B'(o)-valid.

(11) See Theorem 2 of [3].

(12) This theorem supports claims made on pp. 545-546 of [5].