WEAK AND STRONG COMPLETENESS IN SENTENTIAL LOGICS

John Corcoran

Abstract

The purpose of this paper is to give a method which weakens deriving power of a sentential logic while leaving proving power unchanged. In particular if the method is applied to a strongly complete logic the resultant will not have strong completeness but it will retain weak completeness. Moreover, because of the nature of the method two added facts obtain. First, weak completeness of the resultant is always an obvious corollary to weak completeness of the operand, so that no new completeness proof need be constructed. Second, failure of strong completeness in the resultant is always obvious, so no multi-valued matrices need be constructed. The method applies to all sentential logics formulated using only axiom schemes and schematically statable rules. These results generalize and simplify a result of Hiz (*Journal of Symbolic Logic*, 24 (1959), pp. 193-202).

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Let L be the sentential language based on the countably infinite set C of sentence constants and the connectives $\sim$ and $\supset$ (using parentheses for punctuation). Assuming that $\supset$ and $\sim$ have the usual truth-functional meanings, let T be the set of tautologies and let A be the set of (tautologously) valid arguments, i.e., A is the set of ordered pairs (P,c) where c is a tautologous consequence of the set P of sentences. By a *standard consequence system* (SCS) we mean a pair (S,R) where

S is a finite set of tautologous *axiom schemes* and *R* is a finite set of valid *rule schemes*, i.e., *R* contains a finite number of scheme sequences, each of finite length $\geqslant 2$ and such that the last member of a sentence sequence which is an instance of a scheme sequence in *R* is a tautologous consequence of the preceding members. A *derivation* of (P,c) in (*S*, *R*) is a finite sentence sequence s ending with c and such that each subsequent member is either (1) in P, (2) an instance of a member of *S* or (3) the last member of an instance of a member of *R* whose preceding members are earlier members of s. (If P is empty, s is a *proof of c*). If all tautologies are provable (*S*,*R*) is *weakly complete*, and if all valid arguments are derivable (*S*,*R*) is *strongly complete.*

Prior to publication of [2], evidently, all known standard consequence systems which were weakly complete were also strongly complete. (1) In any case the main force of [2] is to construct a standard consequence system, here called *H*, which is weakly complete but not strongly complete. (2) *H* has two axiom schemes and three ternary rule schemes. From [2] it is not clear how *H* was arrived at. (3) Secondly, the fact that *H* is not strongly complete emerges as an isolated and surprising result. Moreover, weak completeness of *H* had to be proved *ab initio* without benefit of previously known results.

The purpose of this paper is to give a method which weakens the deriving power of a SCS while leaving the proving power unchanged. In particular, if (*S*,*R*) is strongly complete then (*S*,*R*∗), the resultant of the method, will not be strongly complete but it will be weakly complete. Moreover, because of the nature of the method, weak completeness of (*S*,*R*∗) is always a corollary to weak completeness of (*S*,*R*), so if (*S*,*R*) is known to be weakly complete no new proof is necessary, *and* failure of strong completeness in (*S*,*R**) is always obvious so no multi-valued matrices need be constructed.

1. *The Method*

The possibility of weakening deriving power while leaving proving power untouched is obvious once one notices that in constructing proofs in a SCS the full strength of the rule schemes is generally not used. The rules of the system are, in effect, relations on L whereas, since all lines in all proofs are tautologies, all applications of the rules in proofs are applications of their restrictions to T. Thus if the union of the restrictions to T of the rules in *R* is the same as that of the rules in *R*∗ then all proofs in (*S*,*R*) are proofs in (*S*,*R*∗) and, therefore, if (*S*,*R*) is strongly complete then (*S*,*R*∗) is weakly complete but not necessarily strongly complete.

Consider the Church system ([1], p. 149), here called *P*, which has three axiom schemes and the single rule scheme, modus ponens. The sentence sequence i(i⊃j)j is an instance of modus ponens which is not used in any proof (the single sentence constant i is not a tautology). If we form *P*' by replacing modus ponens by I(I⊃ ∼J)∼J and I(I⊃(J⊃K))(J⊃K) then the restrictions to T of the rules of the two systems will be the same and weak completeness of *P*' is a corollary to weak completeness of *P*. To see that *P*' is not strongly complete notice that the only way of getting a sentence consisting of a single sentence constant into a derivation is as a premise. Thus, e.g., if the only premises in a derivation are i and (i⊃j) then j cannot occur (alone) at all. Again, if the only premise in a derivation is ∼∼i then i cannot occur at all.

Now let *P*'' be formed from *P* by replacing modus ponens by (I⊃J)((I⊃J)⊃K)K and ∼I(∼I⊃J)J. Again the restrictions to T of the rules of the two systems are the same so *P*'' is weakly complete. To see that *P*'' is not strongly complete notice that there is no way to use a single sentence constant in a derivation once it has occurred. In particular where P-C is the result of deleting all sentence constants from P, for d not in P if d is derivable from P then d is also derivable from P-C. Thus if ∼∼i is derivable from i then ∼∼i is provable and if j is derivable from i and (i⊃j) then j is derivable from (i⊃j).

If (*S*,*R*) is a SCS and *R* contains one or more rules having a single schematic letter as conclusion then (*S*,*R*'), *the conclusion-weakening of* (*S*,*R*), is got by replacing each scheme r in *R* by two schemes as follows: r_1 and r_2 are got by substituting (J⊃K) and ∼J respectively for the single schematic letter J in r, where of course K does not occur in r at all. (*S*,*R*''), *the premise-weakening of* (*S*,*R*), is defined similarly except that a scheme r in *R* which has n single schematic letters as premises is replaced by 2^n schemes not having single letters as premises. It is obvious that if (*S*,*R*) is strongly complete its weakenings, *if defined*, are weakly complete but not strongly complete. It remains to show that for all strongly complete (*S*,*R*) at least one weakening exists.

Actually, for each strongly complete SCS, both weakenings exist, i.e., each strongly complete SCS has a rule scheme having a single schematic letter as a premise and a rule scheme having a single schematic letter as conclusion. Seeing this amounts to seeing that the reasoning about the examples *P*' and *P*'' is general. Explicitly, if a SCS has no rule with a single schematic letter for conclusion then no derivation from a set P devoid of single sentence letters can have a single sentence letter anywhere in it. For example no derivation from ~~i can have contain i. Again, if a SCS has no rule with a single schematic letter as a premise then no derivation from a set P of single sentence letters can contain anything except tautologies and sentences in P, for example no derivation from i can contain ~~i. (4)

2. *Admissible, Derivable and Derivable by a Scheme*

Let r be a means of specification of a relation on L and let [r] be the relation specified by r. There are several distinctions which one may be led to make in consideration of the effect of adding r to a SCS. If r were not a rule scheme then the new system would not be an SCS but it may be equivalent to an SCS in one or more of several senses. Assume that derivations and proofs are defined for arbitrary pairs ([S], [R]) where [S] is a set of sentences and [R] is a set of sentence sequences (not necessarily of the same length). Let ([*S*],[*R*&r]) be the result of including r in the specification of the rules of (*S*,*R*). In [2], r is called *admissible* if the same sentences are provable in ([*S*],[*R*&r]) as in (*S*,*R*). (5) We call r *strongly admissible* if the same arguments are derivable in ([*S*],[*R*&r]) as in (*S*,*R*). The rule of substitution is admissible in any weakly complete SCS but it is not strongly admissible in any SCS. It seems to be standard to call r *derivable* if there is an effective method according to which from a given proof of the premises of the rule it is always possible to obtain a proof of its conclusion ([1], p. 83). In other words, r is derivable if there is a recursive function f mapping the set of finite sequences of L into itself so that if $p_1p_2\ldots p_nc$ is in [r] then for all sequences s if s is a proof containing $p_1,p_2,\ldots,$ and p_n then $sf(p_1p_2\ldots p_nc)$ is a proof of c. There are trivial cases where r can be derivable but not recursive. For example, let r be a non-recursive set of sequences of tautologies and g be a recursive function defined on L so that if c is a tautology then g(c) is a proof of c in a weakly complete system (S,R). For all $p_1p_2\ldots p_nc$ let $f(p_1p_2\ldots p_nc) = g(c)$. Since we are working in a framework including derivations as well as proofs we use *weakly derivable* in the sense of derivable above and we define *strongly derivable* as the obvious generalization. Finally, there is an especially important, more restrictive sense of derivability which needs defining. It may happen that for a given r there is a sequence D of schemes ending with the schematic letter C and containing no schematic letters except those occurring in the n schemes $P_1,P_2,\ldots$ and P_n; where [r] restricted to T is $[P_1P_2\ldots P_nC]$ restricted to T; and such that when E is a "proof scheme" containing P_1, $P_2,\ldots$ P_n as members then ED is a proof scheme (of C). In this case r is *weakly derivable by a scheme.* Analogously we define *strongly derivable by a scheme.* In case it is not possible to derive a rule by means of one proof scheme it may be possible to do so using only finitely many. We assume that *derivability by schemes* are defined to cover the latter cases. The following implications are obvious.

Strongly derivable by schemes	→	Weakly derivable by schemes	
↓		↓	
Strongly derivable	→	Weakly derivable	
↓		↓	
Strongly admissible	→	Weakly admissible	

Moreover, since in any logic the set of proofs is decidable and, for each finite P, the set of derivations from P is decidable, it follows that all weakly admissible rules are weakly derivable and that all strongly admissible rules are strongly derivable. Thus contrary to impressions got from [2] (p. 194) modus ponens is weakly derivable in *H* and the reason that modus ponens is not strongly derivable in *H* is the fact (seen to be trivial as above) that it is not even strongly admissible in *H*. A possibly interesting question is whether modus ponens is weakly derivable in *H* by means of schemes.

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NOTES

(1) This fact may have lead to the opinion, called "unjustified" in [2], that "... a complete system of tautologies constitutes an adequate characterization of valid inferences of the sentential kind..."

(2) In *H* the rule of modus ponens is not strongly admissible (see below). Indeed, if it were then the strong completeness of *H* would have followed. It is obvious that a weakly complete SCS in which modus ponens is strongly admissible is also strongly complete. The above quoted opinion is "corrected" by appending "... in a SCS with modus ponens...". There are, however, obvious ways of interpreting it so that it is correct as quoted.

(3) By converting a Boolean algebra in a natural way Leblanc [3] constructed a sentential logic which is weakly complete but not strongly complete. In [3] (p. 558) reference is made to two other publications which treat weakly complete, strongly incomplete systems.

(4) This insight renders transparent failure of strong completeness in *H* of [2].

(5) This strange terminology is standard in the literature. It would be more sensible to call such rules "theorem-preserving".

REFERENCES

[1] CHURCH, Alonzo *Introduction to Mathematical Logic*, Princeton (1956).

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