ALGEBRAIC ANALYSIS OF ENTAILMENT I

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In [1]-[4], the authors have developed a semantical analysis of Ackermann-Anderson-Belnap style systems of entailment similar to the well-known analyses of Lewis style strict implication due to Kripke, Hintikka, Lemmon, and others. The present paper uses these semantic insights — in particular those of [3] — to develop a general algebraic analysis of entailment logics. Such an analysis has already been furnished by Dunn in [5] for the system **R** of relevant implication, who interpreted the system **R** in a certain class of partially ordered algebraic structures, namely the DeMorgan monoids (1). A similar analysis, as we report, will do for entailment logics generally. This present analysis, as it remarkably turns out, is strongly reminiscent of the very differently motivated connections drawn between the theory of combinators and certain theorems of intuitionist logic by H. B. Curry in [6] and [7]. The present paper will analyze chiefly negation-free entailment logics, which are the most natural algebraically; some remarks, however, will be inserted to show where the enterprise tends when negation too is added. Our key algebraic notion will be that of an *Ackermann groupoid*, defined below, which serves to explicate algebraically the minimal relevant logic **B**+ of [3] and which comes on the addition of postulates to explicate also more familiar relevant, modal, and intuitionist logics, such as **T**+, **E**+, **R**+, **S4**+, and the intuitionist sentential calculus **J**.

I

1. *Church monoids and Dunn monoids.*

Dunn-style algebraizations were offered in [8] of the pure calculus $\mathbf{R}_I$ of relevant implication (identical with Church's weak theory of implication) and of the positive fragment **R**+

of the system **R** of relevant implication. Church monoids corresponded to $\mathbf{R}_I$ and Dunn monoids to **R+**. These definitions are recalled.

A *Church monoid* is a structure $\mathbf{C} = \langle C, \leqslant, o, \rightarrow, 1\rangle$, where C is a set, $1 \in C$, $\leqslant$ is a binary relation on C, and o, $\rightarrow$ are binary operations on C, such that the following postulates hold for all a, b, c in C.

P1. C is a partially ordered groupoid under $\leqslant$ and o; i.e., $\leqslant$ is reflexive, transitive, and antisymmetric; moreover, if $a \leqslant b$ then $a \circ c \leqslant b \circ c$ and $c \circ a \leqslant c \circ b$.

P2. C is a commutative monoid under o; i.e., for a, b in C, $1 \circ a = a$; furthermore o is commutative and associative.

P3. C is residuated with respect to $\rightarrow$; i.e., for all a, b, c in C, $a \circ b \leqslant c$ iff $a \leqslant b \rightarrow c$.

P4. C is semi-idempotent; i.e., $a \leqslant a \circ a$, for all a in C.

Correspondingly, a *Dunn monoid* is a structure $\mathbf{D} = \langle D, o, \rightarrow, \wedge, \vee, 1\rangle$, where with respect to the partial order $\leqslant$ defined for all a, b by $a \leqslant b$ iff $a \vee b = b$, P1-P4 hold and moreover.

P5. D is a distributive lattice with respect to $\wedge$, $\vee$, $\leqslant$; furthermore, D is lattice-ordered; i.e., for a,b,c, in D, $a \circ (b \vee c) = (a \circ b) \vee (a \circ c)$ and $(b \vee c) \circ a = (b \circ a) \vee (c \circ a)$.

Evidently there are four principal constituents which go into the making of Church and Dunn monoids. They are (i) the identity 1, which may be interpreted as a logically true proposition which implies all other logically true propositions; (ii) the residual $\rightarrow$, to be interpreted as propositional implication; (iii) the binary operation o, which may be interpreted as propositional consistency and defined by $a \circ b =_{df} -(a \rightarrow -b)$ in the presence of an operation corresponding to the negation of **R**; (iv) the partial (or lattice) order $\leqslant$, including the monotonic replacement properties imposed under P1 and P5 [2]; where $\wedge$ and $\vee$ are present, they correspond of course to (truth-functional) conjunction and disjunction respectively.

We note that $\wedge$ and $\vee$ enter explicitly only under point (iv) of the last paragraph. To put the matter somewhat differently, conjunction and disjunction are algebraically pretty trivial for relevant logics; to accomodate them, one strengthens P1 to P5 — imposing tighter conditions on the *order* of what one admits as models but leaving untouched the basic insights about implication carried in P2-P4. This corresponds well to what was intended by Anderson and Belnap in constructing the logics **E** and **R** in particular, who wished to express their new relevance-regarding insights about implication without upsetting the purely truth-functional logic of &, $\vee$, and —; and indeed the relative triviality of the move from P1 to P5 made possible the proof in [8] that the accomodation of truth-functional logic in a natural way in **R** conversely does not upset the basic implicational insights of Church's $\mathbf{R}_I$. So we may take & and $\vee$ (and, in another sense, —, given the conservative extension results of [9]) as inessential to the basic insights which relevant logics are trying to capture and in the algebraic expression of these insights. So our guiding principle in the search for a general algebraic analysis of relevant logics — not simply one tailored to the particular character of the system **R** of relevant implication — will be to leave P1 and P5 alone, letting the latter in particular accomodate underlying intuitions about ordinary conjunction and disjunction.

2. *Ackerman groupoids.*

In accordance with the plan suggested in the concluding remarks of the last section, we can accomodate & and $\vee$ in a fairly automatic fashion. The operation o is another story; indeed, troubles over o are the root difficulties which it is our present purpose to overcome. To see what these difficulties are, we remark that o is triply motivated, in the case — e.g. — of Church monoids. First, there is a formal algebraic motivation springing from the theory of residuation; in propositional algebra, as Curry points out, $\rightarrow$ functions as a kind of division; in o, Dunn found for **R** the corresponding multiplication, tied to $\rightarrow$ by the residuation postulate P3 [3]. Second, there is an evident syntactic interpretation of o, utilized by Dunn in his Gentzen system [10] for **R**+; indeed, P3 *looks* like a deduction theorem, which one might put by saying that the *sequence* of formulas A, B *yields* C if and only if A yields $B \rightarrow C$. We make here no claims to know what a proposition is, but passing to the kind of talk which algebraic analysis facilitates, a natural claim is that for propositions a, b, a o b is the conjunction of all propositions *relevantly* entailed by a, b. Since a feature of a relevance-regarding entailment is, according to the motivating remarks of [10], that all the premises of a deduction should actually be *used* in that deduction, what is relevantly entailed by b alone need not be so entailed by a o b. Thus $a \circ b \leqslant b$ is no law for the algebras of relevant logics, though $a \wedge b \leqslant b$ is a law. But otherwise, o functions in relevant logics as a *relevant*, or intensional, conjunction, to be sharply distinguished from ordinary conjunction (4). Finally, o has, as remarked under (iii) above, a straightforward interpretation in the propositional algebra of the system **R** as propositional consistency.

It is the final part of the triple motivation which breeds difficulty in extending the algebraic analysis of [5] to relevant logics weaker than **R**. The point is that it must be viewed as a happy formal accident that, in **R**, relevant conjunction and consistency coincide. Thus it is a wild goose chase — variously pursued by Dunn and by Meyer (e.g., in [11]) — to attempt to offer algebraic counterparts of weaker relevant logics by taking off from the operation of propositional consistency. (One gets postulates, of course, but nothing is revealed.) Accordingly, to get interesting algebraic counterparts of relevant logics in general it is necessary to take o as a new primitive, interpreted not as consistency but as relevant conjunction. There is no *a priori* objection to doing so; on the contrary, the contrast between o and $\wedge$ makes a good hook on which to hang the distinction between relevant and deductive consequence which one wants to make on motivational grounds in any event. The problem rather lies in showing that the axioms and rules one wants for o on the syntactical level, and the corresponding postulates on the algebraic level, are conservative, in that they do not disturb underlying implicational insights (5).

Shockingly, particularly in view of the last footnote, not only do correct views about o prove conservative for positive relevant logics in general, but they offer the most succinct way of differentiating formally among such logics. The attempt to build algebraic counterparts of relevant logics got by varying axioms provides as well the cheapest formal way to characterize these logics.

Let us return to our postulates for a Church monoid. Where are the underlying insights, given that we are bound and determined to interpret o as relevant conjunction ? Evidently these insights appear principally in P1 and P3. In the first place, $\leqslant$ gets naturally interpreted as the *relation* of entailment between propositions. (In this guise it must be separated from the *operation* $\rightarrow$; the difference is that $a \leqslant b$ either holds or it doesn't, but it is not itself an element of the algebra; for elements a, b, on the other hand, $a \rightarrow b$ is of course another element of the algebra.) One would hardly want to deny that, as a relation, entailment partially orders propositions, which is the first part of P1. One could deny the second part of P1, which may be viewed as asserting in somewhat stronger fashion the transitivity of the *relation* of entailment — namely, that certain monotonic replacement properties hold relative to o (and, through P3, to $\rightarrow$). This would make the algebra less pretty, which is perhaps reason enough to reject such denial; a more compelling reason, in light of motivation, is that if a entails b we should *expect* to get from a, c jointly whatever we might get from b,c jointly, and from c,a jointly whatever we might get from c,b jointly. So P1, for the systems of this paper, is firm; so also is P3, since o was motivated (as relevant conjunction) by the desire to have a relevant deduction theorem of which P3 is the algebraic analogue.

This leaves us P2 and P4 to play with. P2 is almost completely **R**-specific; while it may be nice to build commutativity and associativity into o — they aren't, after all, bad properties for a conjunction to have, even a relevant one — there are good reasons not to have such properties, too. In the first place **E**, anyway, is supposed to be a *modal* logic in the sense of Lewis, and since Lewis arguments have been advanced that, when one's implication is strict, $A \rightarrow (B \rightarrow C)$ does not necessarily come to the same thing as $B \rightarrow (A \rightarrow C)$. By our motivating remarks, a o b does not accordingly come to the same thing for the algebra of **E** as b o a. So commutativity goes, in the general case. Associativity is even shakier; it goes, too (6). Finally, to turn to P4, there is no general reason why we should be stuck with that strange semi-idempotence condition; it goes (7). Truth-functional conjunction and disjunction are not our present concern, since we intend to add them trivially; the instrument for so doing is P5, which we accordingly leave untouched for roughly the reasons we left P1 untouched.

Only the identity 1 has thus far escaped comment. Like o, 1 keeps turning up (8). It is wanted in the present context for two reasons. First, its presence enables us to relate entailment-as-a-relation with the operation $\rightarrow$, since by P3 and P2, $1 \leqslant a \rightarrow b$ iff $a = 1 \circ a \leqslant b$; this gets us past the annoying problem, pointed out by Dunn in [5], that an algebraic analogue of *modus ponens* is otherwise hard to come by. Second, 1 is of particular use in formulating those systems — e.g., **E** — which have an explicit Lewis-style modality. A warning is in order, however; commutativity makes 1 a two-sided identity when algebraizing **R**. But though $1 \circ a = a$ is in general well-motivated, $a \circ 1 = a$ is not; defining Na following Ackermann as $1 \rightarrow a$, from the latter equality the fallacy of modality $a \leqslant Na$ follows. Thus in general 1 is to be a *left identity*, but not a right identity. Similarly, $\rightarrow$ is a *left residual* when defined by P3; to be also a right residual $\rightarrow$ will have to satisfy also $b \circ a \leqslant c$ iff $a \leqslant b \rightarrow c$, which in the absence of commutativity in general it will not (9).

We arrive at length at the generalizations we have been seeking. We call the most basic relevant algebras *Ackermann groupoids* (10). A structure $\mathbf{G} = \langle G, \leqslant, \circ, \rightarrow, 1 \rangle$ is an *Ackerman* groupoid provided that

(1) G is a partially ordered groupoid under $\leqslant$ and o; i.e., P1 holds.

(2) $1 \circ a = a$ for all a in G.

(3) G is residuated with respect to $\rightarrow$ (more accurately, *left-residuated*); i.e., P3 holds.

Like Church monoids, Ackermann groupoids are introduced for the specific purpose of explicating pure implicational calculi; when we want to emphasize this point, we shall call them *implicational Ackermann groupoids*. Similarly explicating positive relevant logics in general are *positive Ackermann groupoids*, which are structures $\mathbf{G} = \langle G, o, \rightarrow, \wedge, \vee, 1 \rangle$ satisfying (2)-(3) just above and strengthening (1) to

(4) G is a distributive lattice with respect to $\wedge$, $\vee$, and $\leqslant$ defined as before; i.e., P5 holds.

An *interpretation* of a sentential logic **L** in an Ackermann groupoid is a function I defined on all formulas of **L** and such that, whenever the corresponding connectives and constants are present in the language of **L**, the following hold, for all formulas A and B:

(i) $I(A \rightarrow B) = I(A) \rightarrow I(B)$;
(ii) $I(A \circ B) = I(A) \circ I(B)$;
(iii) $I(t) = 1$
(iv) $I(A \,\&\, B) = I(A) \wedge I(B)$
(v) $I(A \vee B) = I(A) \vee I(B)$

Under (iv) and (v), we presuppose a *positive* Ackermann groupoid. A formula A of **L** is *true* on an interpretation I in an Ackermann groupoid iff $1 \leqslant I(A)$ and is otherwise *false* on I; A is *valid* in **G** iff A is true on all interpretations therein.

3. *Ackermann groupoids algebraize* **B+**

In this section we prove algebraic consistency and completeness results for the minimal positive relevant logic **B+** introduced in [3]. We recall that a *positive model structure* (+ *m.s.*) is a triple $\langle O, K, R \rangle$, where K is a set, $O \in K$, and R is a ternary relation on K, such that the following definitions and postulates hold for all a, b, c, d in K and with quantifiers ranging over K.

d1. $a < b$ =df R0ab
d2. R^2abcd =df $\exists x$(Rabx & Rxcd)
d3. $R^2a(bc)d$ =df $\exists x$(Rbcx and Raxd)
p1. $a < a$
p2. $a < b$ & Rbcd imply Racd
p3. $b < c$ & Racd imply Rabd
p4. $c < d$ & Rabc imply Rabd

(We note that the antecedents of p2-p4 may be succinctly expressed by $R^2 0acd$, $R^2a(0b)d$, and $R^2 0(ab)d$ respectively; we confess, too, that we were postulate-chopping in [3], so that p3-p4 were not assumed there at full strength. Thus, despite our use of "recall" above, the postulates have been slightly strengthened, for a + *m.s.*; since all postulates hold in canonical model structures, this does not increase the stock of valid formulas; moreover, it more correctly expresses the underlying semantical insights on which **B+** rests — essentially, that R is monotone decreasing in its first two arguments and monotone increasing in its third relative to the quasi-order $<$.)

We recall also that if $<O, K, R>$ is a + *m.s.* and **SL+** is a sentential language, a *valuation* v of **SL+** in $<O, K, R>$ is a function which assigns a truth-value to each sentential variable p in **SL+** at each point a of K, subject to the restriction

(a) $a < b$ and $v(p,a) = T$ imply $v(p,b) = T$.

I is the *interpretation* associated with v in $<O, K, R>$ provided that I is a function from **SL+** $\times$ K to $\{T, F\}$ satisfying the following conditions, for all p in **SL+**, A,B in **SL+**, and a in K (when listed connectives are in **SL+**).

(i) $I(p,a) = v(p,a)$
(ii) $I(A \& B,a) = T$ iff $I(A) = I(B) = T$
(iii) $I(A \vee B,a) = F$ iff $I(A) = I(B) = F$
(iv) $I(A \rightarrow B,a) = T$ iff, for all b,c in K, if both Rabc and $I(A,b) = T$ then $I(B,c) = T$.
(v) $I(A \circ B,a) = T$ iff, there exist x,y in K such that Rxya and $I(A,x) = T$ and $I(B,y) = T$.
(vi) $I(t,a) = T$ iff $0 < a$.

A formula A is *true* on a valuation v (or on associated I) iff I(A,a) = T; A is *verified* on v (or I) iff I(A,0) = T; A is valid in a + *m.s.* iff A is verified on all valuations therein.

B+ was formulated in [3] without o; the recursive condition (v) comes from [1]. Without o, **B**+ may be formulated with the following axiom schemes and rules (11):

A1. $A \to A$
A2. $A \& B \to A$
A3. $A \& B \to B$
A4. $(A \to B) \& (A \to C) \to (A \to B \& C)$
A5. $A \to A \vee B$
A6. $B \to A \vee B$
A7. $(A \to C) \& (B \to C) \to (A \vee B \to C)$
A8. $A \& (B \vee C) \to A \& B \vee A \& C$
A9. t
A10. $t \to (A \to A)$
R1. A and $A \to B \Rightarrow B$ (*modus ponens*)
R2. A and $B \Rightarrow A \& B$ (adjunction)
R3. $A \to B \Rightarrow (B \to C) \to (A \to C)$ (suffixing)
R4. $B \to C \Rightarrow (A \to B) \to (A \to C)$ (prefixing)

A basic result of [3] was
Lemma 1. A formula A is derivable in **B**+ iff A is valid in all + *m.s.*, if o does not occur in A.
Proof in [3].

The exception is made in lemma 1 simply because it did not occur to us to include o in [3]. Using ⇔ to indicate a two-sided rule (either side to be inferred from the other), we may extend lemma 1 simply by incorporating in **B**+ the rule

R5. $(A \circ B) \to C \Leftrightarrow A \to (B \to C)$ (residuation)

Lemma 2. If **B**+ is formulated with A1-A10, R1-R5, a formula A is derivable in **B**+ iff A is valid in all + *m.s.*
Proof. The proof is as in [3], making the necessary adjustments to accomodate R5 and the semantic condition (v).

Lemma 2 is, if one pleases, a conservative extension result for **B**+ relative to the addition of o. It also makes trivial the main business of this section, showing that Ackermann groupoids algebraize **B**+.

Theorem 1. A is a theorem of **B**+ iff A is valid in all positive Ackermann groupoids.

Proof. We show first that an arbitrary theorem of **B**+ is true on any interpretation I in any positive Ackermann groupoid **G** = $<G, o, \rightarrow, \wedge, \vee, 1>$. Suppose first that A is an axiom. Then A is of the form $B \rightarrow C$, and we must show [12] $1 \leqslant I(A) = I(B) \rightarrow I(C)$; i.e., by (3) of the last section, $1 \, o \, I(B) \leqslant I(C)$ i.e., since 1 is a left identity by (2), $I(B) \leqslant I(C)$. This is trivial by the fact that **G** is a distributive lattice except in the cases A4, A7, A10; the first two are dual so we examine only A7. Given the residuation postulate 3, evidently it suffices to show, for all a,b,c in **G**, that $(a \rightarrow c) \wedge (b \rightarrow c) \, o \, (a \vee b) \leqslant c$. But o distributes over $\vee$ by (4), which by lattice and order properties reduces the question to whether $(a \rightarrow c) \, o \, a \leqslant c$, and similarly with b for a; but this is an immediate consequence of (3), ending the verification of A7.A10, since $I(t) = 1$, reduces similarly to $1 \, o \, a \leqslant a$ by residuation, an evident truth by (2). This ends verification of the axioms. To verify the rules, we assume their premises true on arbitrary I and show the same for their conclusions. For R1, assume $1 \leqslant a$ and $1 \leqslant a \rightarrow b$; as above, $a \leqslant b$, whence $1 \leqslant b$, which is what is wanted. R2 is trivial by lattice properties. To verify R3, we may assume $a \leqslant b$; we must show $b \rightarrow c \leqslant a \rightarrow c$; i.e., by (3), $(b \rightarrow c) \, o \, a \leqslant c$. But $(b \rightarrow c) \, o \, b \leqslant c$, whence on assumption the desired conclusion follows from the order postulate included in (1), and lattice properties. R4 is similar. (Note, however, that to get *both* R3 and R4 we must have both $coa \leqslant cob$ and $aoc \leqslant boc$ whenever $a \leqslant b$; or, what implies this conclusion, that o is as stated in P5 both left- and right-distributive over $\vee$.) This ends the proof of semantic consistency of **B**+ relative to the algebraic interpretation.

Conversely, suppose A is not a theorem of **B**+. There is, by lemma 2, a + *m.s.* $<O,K,R>$ and an interpretation I on which $I(A,0) = F$. A subset J of K is a *strike* provided that, whenever $a \in J$ and $a < b$, $b \in J$. It is readily verified that the set of all strikes — call it S(K) — determined by $<0,K,R>$ is a positive Ackermann groupoid $<S(K),o,\rightarrow,\wedge,\vee,1>$, given the following definitions for all J_1,J_2 in S(K):

$$J_1 \circ J_2 = \{c\colon c \in K \text{ and for some } a \in J_1,\ b \in J_2,\ Rabc\}$$
$$J_1 \rightarrow J_2 = \bigcup \{J\colon J \in S(K) \text{ and } J \circ J_1 \subseteq J_2\}$$
$$J_1 \wedge J_2 = J_1 \cap J_2$$
$$J_1 \vee J_2 = J_1 \cup J_2$$
$$1 = \{a\colon a \in K \text{ and } 0 < a\}$$

Define now an interpretation I which refutes the non-theorem A by setting, for each sentential variable p, $I(p) = \{a\colon a \in K$ and $v(p,a) = T\}$. The restriction on what counts as a valuation assures that this will be a strike, and one proves on inductive hypothesis that, for each formula B of **B+**, I(B) is the member of S(K) consisting precisely of those elements of K at which B is true on the valuation v; since in particular the non-theorem A is falsified on v at 0, $0 \notin I(A)$ and so $1 \nleq I(A)$. This ends the proof of theorem 1.

Theorem 1, in a sense, is arrived at by cheating, since we might have shown in a direct manner (by forming the Lindenbaum algebra of **B+** and showing that it is an Ackermann groupoid) that B+ is complete relative to the algebraic interpretation. Thus the proof of completeness actually chosen rests on something else — specifically, a kind of Stone theorem, which shows that every Ackermann groupoid can be represented as a sub-groupoid of S(K) for some + *m.s.* $<0,K,R>$. Moreover, given a positive Ackermann groupoid **G**, a + *m.s.* $<0,K,R>$ can be constructed in a natural way where K is the set of prime filters of **G**. Similar moves were made in [1], and we can derive here too the conclusion that every Ackermann groupoid may be embedded in a complete Ackermann groupoid; as in [1], this means that sentential quantifiers may be added conservatively to **B+** if desired.

B+ came with & and $\vee$; let $\mathbf{B}_\mathrm{I}$ be the system determined by A1, A9-A10, R1, R3-R5. Then

Theorem 2. A is a theorem of $\mathbf{B}_I$ iff A is valid in all implicational Ackermann groupoids.

Proof. Like theorem 1. A slightly sticky point concerns the question whether **B**+ is a conservative extension of $\mathbf{B}_I$; adjustment of the completeness proof of [3] answers this question with a straightforward "Yes," after which there are no difficulties in the proof of theorem 2.

$\mathbf{B}_I$ contains t and o, to accord with the algebra. It needn't, however, as the following satisfying theorem, in support of the claim that $\mathbf{B}_I$ is really minimal, makes plain.

Theorem 3. Let A be a formula of **B**+ in which no connective or constant but $\rightarrow$ occurs. Then A is a theorem of **B**+, or $\mathbf{B}_I$, iff A is of the form $B \rightarrow B$.

Proof. We form a + *m.s.* as follows: Let K be the set of formulas containing all sentential variables and closed under $\rightarrow$; i.e., K is the set of pure implicational formulas. Let P(K) be the power set of K. Let 0 be the set of all formulas of the form $B \rightarrow B$. For all a,b,c in P(K), let Rabc iff whenever $B \rightarrow C$ is in a and B is in b, C is in c, for all formulas B,C in K. Verify that $<0,P(K),R>$ is a + *m.s.* (This rests on the rather trivial fact that R0ab iff $a \subseteq b$, for all a,b in P(K).) Set $v(p,a) = T$ iff $p \in a$, for all sentential variables p and sets a of implicational formulas in P(K), and show then for any formula A in K, and for each a in P(K), that A is true on v at a iff $A \in a$. Since in particular 0 contains just the identities, only identities are verified on this interpretation, ending the proof of the theorem.

This concludes our investigation of **B**+; we turn in the sequel to its extensions.

II

4. *Every positive relevant logic has a natural algebraic counterpart.*

In [3], we got extensions of **B**+ by keeping the axioms and rules of **B**+ and choosing additional axioms from among the following:

B1. $A \& (A \rightarrow B) \rightarrow B$
B2. $(A \rightarrow B) \& (B \rightarrow C) \rightarrow (A \rightarrow C)$
B3. $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$
B4. $(B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)$
B5. $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$
B6. $(t \rightarrow A) \rightarrow A$
B7. $A \rightarrow ((A \rightarrow B) \rightarrow B)$
B8. $A \rightarrow (B \rightarrow B)$

Axioms whose addition we might have contemplated, but didn't, might include

B9. $A \rightarrow (B \rightarrow A)$
B10. $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
B11. $(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$
B12. $A \rightarrow (A \rightarrow A)$

Corresponding semantical postulates for B1-B8 were given in [3] as follows:

q1. Raaa
q2. Rabc $\Rightarrow$ $R^2a(ab)c$
q3. $R^2abcd \Rightarrow R^2b(ac)d$
q4. $R^2abcd \Rightarrow R^2a(bc)d$
q5. Rabc $\Rightarrow$ R^2abbc
q6. Ra0a
q7. Rabc $\Rightarrow$ Rbac
q8. R00a

Corresponding to B9-B12, we might have chosen

q9. Rabc $\Rightarrow$R0ac
q10. $R^2abcd \Rightarrow$ Ex(R^2acxd and Rbcx)
q11. $R^2abcd \Rightarrow R^2acbd$
q12. Rabc $\Rightarrow$ R0ac or R0bc

Corresponding algebraic postulates are as follows:

r1.	$a \leqslant a \circ a$	W_*
r2.	$a \circ b \leqslant a \circ (a \circ b)$	
r3.	$(a \circ b) \circ c \leqslant b \circ (a \circ c)$	B'
r4.	$(a \circ b) \circ c \leqslant a \circ (b \circ c)$	B
r5.	$a \circ b \leqslant (a \circ b) \circ b$	W
r6.	$a \leqslant a \circ 1$	
r7.	$a \circ b \leqslant b \circ a$	C_*
r8.	$a \circ b \leqslant b$	K'
r9.	$a \circ b \leqslant a$	K
r10.	$(a \circ b) \circ c \leqslant (a \circ c) \circ (b \circ c)$	S
r11.	$(a \circ b) \circ c \leqslant (a \circ c) \circ b$	C
r12.	$a \circ a \leqslant a$	

Capital letters written after algebraic postulates are combinators in the sense of Curry; the point in each case is that the cited combinator, applied to a left-hand side of an inequality in question, will produce what stands on the right-hand side. We note moreover that combinators attached to algebraic postulates are in general exactly those attached by Curry to the corresponding implicational formulas as their functional character; e.g., W, associated by us with r5, is associated by Curry with the scheme B5, and similarly for B', B, C_*, K, S, and C. (We owe the observation of this connection to Belnap.)

As in [3], we call a + *m.s. fitting* for a logic **L**+ got by adding axiom schemes from among B1-B12 provided that corresponding postulates from among q1-q12 hold in the *m.s.* Similarly, we call an Ackermann groupoid, or a positive Ackermann groupoid, *fitting* for **L**+ provided that corresponding algebraic postulates from among r1-r12 hold for the groupoid. We may now state the principal result of the paper.

Theorem 4. Let **L**+ be a positive relevant logic got by adding axioms schemes from among B1-B12 to **B**+. Then A is a theorem of **L**+ iff A is valid in all positive Ackermann groupoids fitting for **L**+.

Proof. The proof of theorem 1 will do in outline. Semantic consistency with respect to the algebraic interpretation may be determined by verifying axioms in an arbitrary fitting Ackermann groupoid, rules having already been dealt with in theorem 1. In each case it will be seen that the algebraic postulate in question suffices to verify the corresponding axiom. Semantic completeness, too, may be proved as in theorem 1. Indeed, given the semantic completeness results of [3], extended as necessary for the axioms B9-B12 newly considered here, it turns out in each case that the semantical constraints placed on positive model structures enforce the corresponding algebraic constraints on the positive Ackermann groupoid S(K) got on application of the technique of theorem 1. (13)

III

5. *Algebraizing entailment logics with negation.*

Like conjunction and disjunction, as we have suggested, negation does not add new kinds of problems. The essential point is merely to add negation to the underlying lattice structure, making it a DeMorgan lattice rather than simply a distributive one. We add at the same time excluded middle in view of the intended logical interpretation, though from a purely algebraic point of view this assumption is not required. (Even for weak relevant logics, we are sticking here to the truth-functional view of truth-functional connectives, following Anderson and Belnap.) Thus we arrive at the notion of a *DeMorgan groupoid* $\mathbf{G} = \langle G, o, \rightarrow, \wedge, \vee, -, 1 \rangle$, where $\mathbf{G}$ is a positive Ackermann groupoid (i.e., satisfying (1)-(4)) with respect to the positive operations, $\leqslant$ is defined in the usual lattice-theoretic way, and the following additional postulates hold:

(5) $a \vee b = -(-a \wedge -b)$, and
(6) $1 \leqslant a \vee -a$,

for all a,b in G. Correspondingly, we get the basic logic **B** by adding to **B**+ the following two axioms and rule:

C1. $A \vee \bar{A}$

C2. $\bar{\bar{A}} \rightarrow A$

R6. $A \rightarrow \bar{B} \Rightarrow B \rightarrow \bar{A}$

As possible additional axioms we list the following:

D1. $(A \rightarrow \bar{A}) \rightarrow \bar{A}$

D2. $(A \rightarrow \bar{B}) \rightarrow (B \rightarrow \bar{A})$

D1 and D2 may be thought of as strengthened forms of C1 and R6 respectively. The possible additional algebraic postulates corresponding to D1-D2 are

s1. $a \circ b \leqslant -b$ implies $a \leqslant -b$

s2. $a \circ b \leqslant c$ implies $a \circ -c \leqslant -b$

The DeMorgan monoids that stem from Dunn's [5] are simply DeMorgan groupoids that satisfy r1, r11, and q2; as is proved in [5], these structures algebraize the system **R** of relevant implication. Similarly, extending the definitions of preceding sections *mutatis mutandis*,

Theorem 5. A formula is derivable in the minimal relevant logic **B** if and only if it is valid in all DeMorgan groupoids. Moreover, any extension of **B** got by adding new axiom schemes from among B1-B12, D1-D2 to the axioms and rules of **B** contains as theorems just those formulas valid in all fitting DeMorgan groupoids.

Proof. All the axioms are valid and the rules preserve this property. Conversely, the Lindenbaum algebra of the extension in question is a fitting DeMorgan groupoid in which all non-theorems are invalid, ending the proof of theorem 5.

For its interest, theorem 5, like its predecessors, rests on conservative extension results. At this point, we abandon our lingering interest in systems formulated with axioms B8-B9, since, as is well-known, in the presence of others from among our possible axioms, systems formulated with these axioms break down into classical logic; similarly, though B12 is relevance-preserving when added to the system **R**+, it breaks the full logic **R** down into Dunn's system **RM**, which permits fallacies of relevance. Our chief interest, accordingly, lies in sub-systems of **R** — i.e., systems formulated with any of our other axioms — for which the conservative extension results of [9] are at hand to limit collapse. In particular, we note that the systems **E**, **T**, **R**, and **B** are conservative extensions with respect to negation of the corresponding positive systems listed above. The methods of [4] will show also that these systems are conservative extensions of the systems got by dropping the connective o and the accompanying rule R5. (For **R** this is known, since o is there definable as relevant consistency.) But the following results are new and interesting.

Theorem 6. Let the systems **T** of ticket entailment and **E** of entailment be formulated as in [15] with $\rightarrow$, &, $\vee$, —. The theorems of **T** are exactly the formulas valid in all DeMorgan groupoids satisfying r4-r5, s1-s2; the theorems of **E** are exactly the formulas valid in all such groupoids that satisfy r6 as well. *Proof.* As indicated, depending on the conservative extension results to be proved in [4].

Once Dunn monoids are stripped of commutativity, as we noted in I, $\rightarrow$ becomes merely a *left* residual. Question: is there in a natural way to be found also a *right* residual, call it $\rightarrow_R$, satisfying the law

(i) $a \circ b \leqslant c$ iff $b \leqslant a \rightarrow_R c$?

The answer, given s2, is "yes"; — (a o — c) will do.

IV

When he first noted these connections, Meyer was so excited that he dubbed them the Key to the Universe. They may fall somewhat short of *that*, but the connections are interesting and unifying. They would seem to lie behind the semantical analysis of [3], presenting in a more direct way the conditions that are put there on our ternary relation R. Cut-free Gentzen formulations of positive logics weaker than **R** are still wanted; hopefully the methods used by Dunn in Gentzenizing **R**+, which made essential use of both o and &, and passing use of 1 as well, are hereby rendered straightforwardly applicable to kindred relevant logics. Decision problems, too, are hopefully rendered more accessible, though we suspect that these will prove trivial only for systems that no one has thought or cared much about. What appears to be the most fun, though, is to take a course converse to the one adopted by Curry in directly introducing combinators into propositional algebra. (Dunn has made a related suggestion, though the scheme adopted here is our own.) Let a *combinatory Ackermann groupoid*, implicational or positive, be an Ackermann groupoid **G** in whose base set G occur some (possibly all) of the combinators of pure combinatory logic, governed by the usual rules, stated as conditions on **G**. As is well-known (cf. e.g. [6]) all combinators can be defined in terms of S and K, so let us take these as primitive and as elements of G, subject to laws

$$(K \circ a) \circ b \leqslant a, \text{ and}$$
$$((S \circ a) \circ b) \circ c \leqslant (a \circ b) \circ (a \circ c).$$

The combinator I may be identified with the identity of **G**, in view of its governing law

$$I \circ a = a.$$

We note then that, using the general order principles imposed under (1) on an Ackermann groupoid, we can state the postulates corresponding to particular logics very simply. Take, for example, **E**; we may formulate the class of algebras corresponding to **E**+ as the set of positive Ackermann groupoids satisfying

r3. (a o b) o c $\leqslant$ b o (a o c) B'
r5. (a o b) $\leqslant$ (a o b) o b W
r6. a $\leqslant$ a o 1 CI

The combinators in question are as noted those that, applied to the left-hand side of an inequality, yield the right-hand side. Identifying 1 and I, it would suffice to adopt instead

r3'. I $\leqslant$ B'
r5'. I $\leqslant$ W
r6'. I $\leqslant$ C o I

For, dropping o in favor of simple juxtaposition and associating to the left, we should have derivations as follows:

Derivation of r3 from r3':

(1) I $\leqslant$ B'	r3'
(2) Ia = a $\leqslant$ B'a	By (1), postulate P1
(3) abc $\leqslant$ B'abc $\leqslant$b(ac)	By (2), P1 [twice],

reduction rule for B'.

Derivation of r5 from r5' in like manner.

Derivation of r6 from r6'.

(1) I $\leqslant$ CI	r6'
(2) IIa = Ia = a $\leqslant$ CIIa	(1), P1 [twice], reduction

rule for I [twice].

(3) a $\leqslant$ IaI	(2), reduction rule for C
(4) a $\leqslant$ aI	(3), reduction rule for I

Thus, if this line of thought proves fruitful, the algebras of particular positive relevant logics are got simply by imposing an order on combinators; for the other cases are similar. The paradoxes of implication are blocked, not by banning the combinator K, but simply by holding I$\nleqslant$K and similarly for related evil combinators.

Putting the combinators in algebras and subjecting them to algebraic constraints would seem like dangerous business. In fact, in crucial cases we get the kind of conservative expansion central to completeness proofs. For, given propositional quantifiers, conservatively addable as in [1], we can interpret the combinators: e.g., B' as (p)(q)(r)(pqr → q(pr)), and similarly in other cases. [This shows, among other things, the undecidability of some of our thus extended logics — in particular, the system **BP+** got by adding propositional quantifiers to **B+**, since the pure combinatory logic, known to be undecidable, can be interpreted therein.]

Short of this move, a word of caution must be entered. For our propositional algebras, a, b, and c are supposed to be propositions, but what sort of proposition, really, is B' ? Nevertheless, what hangs together formally sooner or later hangs together intuitively, and the coincidence of several independent lines of investigation is surely more than sheer coincidence. Meanwhile, the purely algebraic results are themselves firm and solve the principal outstanding algebraic questions for logics weaker than **R** in the family of relevant logics; the proof that as such they constitute the Key to the Universe must wait for a sequel. ([14])

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NOTES

([1]) The theory of residuation, on which Dunn's and the present work rests, was developed by WARD and DILWORTH; cf. [7] for references.

([2]) P5, in fact, implies P1. For a general study of partially ordered algebraic structures, v. L. FUCHS, *Partially ordered algebraic systems*, Oxford, 1963, and G. BIRKHOFF, *Lattice theory*, Providence, 1966.

([3]) As noted below, P3 is a postulate for a left residual; we use a → b in place of some typographical variant of the algebraically natural b/a, in view of the intended logical interpretation of →, as implication.

([4]) When we speak of conjunction, we always mean the ordinary truth-functional conjunction & unless the contrary is explicitly indicated. (Thus the characterization of o a sentence ago is not circular.) And on algebraic interpretation it is & which is linked to the lattice-theoretic meet, ∧, while we use o both syntactically and for the corresponding algebraic operation. Note, too, that it is not necessary to be so agnostic as above about propositions, since the semantical analysis of [1]-[4]

enables one to take the UCLA view that a proposition is a function from reference points to truth-values.

(5) Thus Dunn's Gentzenization of **R**+ in [10] employs two kinds of sequences — extensional ones, to be interpreted as ordinary conjunctions, and intensional ones, to be interpreted as relevant conjunctions. We note too that ordinary deduction theorems using & are by no means inexpressible in relevant logics; they merely correspond to a different kind of consequence relation, called in [11] and [12] *deductive consequence* and tied to a relevantly definable, intuitionistically acceptable connective ⊃. Not disturbing underlying implicational insights in one's choice of axioms for o is, by the way, an important point, those given for **R**++ in [1] being too strong in general (though O. K. for **R**); the weakening of the axioms for o in [1] to rule form is what is wanted and is accomplished in R5 below.

(6) Even for **R**, proof that o as naturally defined is associative takes some work and proceeds most naturally through the commutativity of o; failure of the commutative law for o being as suggested, the hallmark of Lewis-style modal logics, including **E**, proof of associativity of o breaks down also.

(7) There are reasons of a sort to keep P4; it guarantees that *theories*, in the sense of [3], should be closed under *modus ponens*. The well-known relevant logics **E**, **R**, and **T** satisfy this condition, but we do not impose it in general, here or in [3].

(8) 1 turns up syntactically as t, going back in principle to Ackermann; cf. [13].

(9) But there is still room for a theory of the right residual, as we note below.

(10) The connection is not, however, Ackermann-specific, Ackermann groupoids (not otherwise restricted) being linked to our **B**+ rather than his Π' and Π'', identifiable with **E**.

(11) Notational conventions are as in [3].

(12) Except for A9, which is trivial.

(13) The utility of this theorem lies in the eliminability of o, and, to a lesser degree, of 1. Syntactically, we note that o only occurs in R5; striking o (and R5) from the syntax, the completess proofs of [3] show for all systems that the set of o-free theorems is not affected. 1, and its syntactical mate t, may be eliminated in interesting cases as in [13], [8], [12], and theorem 3 above. The arguments of [8] may also be applied to show pure implicational systems algebraized by fitting implicational Ackerman groupoids, eliminating &, V, and the distributive lattice structure introduced to accomodate them. Adequate postulates for specific logics are, in addition to (1) — (4), as follows: for **T**+, r3 — r5; for **E**+, r3,r5,r6; for **R**+, r1,r11; if all postulates hold, we have **J**+; dropping r7,r9,r11, we have S4+.

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