NEGATION

The tension between ontological positivity (negationless positivity) and anthropological negativity (positively described).

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A. Some remarks on negationless logic

1. *Introduction*

Is a complete description of the universe possible without using negation ? We think we must give a positive answer to this question, but we must add that a pragmatically usefull talk about the universe is not possible without negation. We are aware of the fact that the concept of complete description is only complete with reference to a given aim; our assertion can thus only mean that whatever the aim we pursue a relatively complete positive description is possible. The various types of negations we shall have to use will then be derived from their practical function, in the first place, and as abbreviations for positive propositions, in the second place.

These are the assertions we want to defend in the present paper.

2. *An adequate and complete description of the universe is possible without the use of negation.*

Ludwig Wittgenstein presents in thesis 2.04 of his 'Tractatus Logico-Philosophicus the following truth: "the totality of all existent facts is the universe." If a complete description of the universe is thus the description of the totality of all existent facts we do no need any negation; there are no negative facts.

The writer of the present article is not clear about the meaning of the word 'fact'; he would rather say that the totality of all existent things constitutes the universe. This paper is not the time or place to give a definition of the concept of

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(*) The reader whose interest lies foremost in our general theory of negation is invited to read part B, p. 272 and foll. first. This part will hopefully increase his motivation to read part A.

object or of thing. Let it be sufficient that we are thinking along the lines of Kotarbinski's pansomatism. Now it seems clear that there are no negative objects or things. For this very reason the use of negations cannot have a function other than practical.

It is a pity to be compelled to start an article about types of negations by a metaphysical discussion about the dispensability of negative facts or things, and yet this is exactly what we are forced to do.

First, negative facts are not observable. Everything that can be observed is of a positive nature. Apparent counterexamples are the best arguments in favour of this claim. When I say 'the dress I see is not black' I simply give incomplete information, or information that I think relevant to the practical needs of my hearer. In fact if I am at all in the possibility of making out by direct observation the colour of the dress I am in a position, if I have the right colour words at my disposal in my idiolect to describe positively the colour of the dress. Observing that "There is no noise" is again not a negative observation; it is the positive observation of silence. Observing that there is no money in my wallet is again a positive information, having regrettable practical consequences but reducible to the description of the positive state of the wallet (1).

In his book on Meinong, Findlay tries to defend the existence of negative facts by pointing out that even if the thesis were true that the universe consists of positive facts or things, we still have to assert that a description is complete and that this completeness can only be expressed by means of the negative sentence 'There are no other things in the universe'. We deny this claim and state that we can say exactly the same in a purely positive way 'These objects are all the objects in the universe' (2).

Let us then accept that there are no observable negative objects or facts. The only reason for the introduction of negative facts or objects would be that by inductive inference we derive their existence from our observation.

We think that whatever the system of inductive logic we use, we never are compelled to infer the existence of negative facts or of negative things. Why ? Because of the following two arguments:

a) Occam's rasor: if we infer from observation to negative objects or facts, we are compelled to infer inductively to the existence of a non denumerably infinite negative set. This is not an acceptable scientific procedure.

b) More deeply however we come to the following conclusion: a negative object or a negative fact cannot exist because its existence would be a contradiction. We accept indeed (and again we cannot defend this position here) that everything that exists must at least in principle have the ability to be a cause. The causation relation is such that a non existent thing or fact or event cannot be a cause. So we consider the expression 'existence of a negative object or fact' as a contradictory expression.

These two arguments seem sufficient to us to reject the existence of negative things or facts. *They are neither observable, nor inferable by induction.*

As there are no other sources of knowledge about the external world, we can dispense with those ghostlike entities (³).

If now we must come to the conclusion that nothing negative exists in the universe, this does not yet immediately compell us to assert that we can give a complete description of the universe without using negations. Indeed, often we need to introduce fictions (think about clasical mathematics) in order to describe reality. It could be the case that even in this completely positive universe we need negations to describe it. This could eventually follow from our own nature or from the nature of description. We do not wish to discuss this point here. We only accept the following rule: let us, as far as possible, avoid the use of fictional entities in the description of the real world, and if we use them, let us use them in such a way that they can be eliminated, in principle if not always in practice. This methodological rule seems to us defensible on the basis of the rule that we must try to introduce in our description as few as possible non real objects or facts. The desire to give a truthful description is the very simple reason for this attitude.

The very simple reasons we use here to reject the existence of negative facts and of negative objects are fundamentally different from the reasons that brought certain members of the intuitionist school (Griss) to the rejection of negation. They consider the statement 'not p', from the intuitionistic point of view: if not p is true, p cannot be constructed; being unconstructible, the phrase 'not p' where p is absurd is itself absurd and thus 'not p' is an absurd sequence of signs. If on the other hand p can be constructed then not p is necessarily false. This type of reasoning leads us to negative propositions that are only meaningfull if and in as far as they are false. They thus become useless for logical purposes. Freudenthal, and to a certain extent Griss, from the intuitionistic point of view reject the use of negation in logic for these reasons, connected with the constructibility of the propositions they want to handle. Our own argumentation is completely different, and independent from the intuitionistic point of view. This remark does not however imply that we reject the latter (4).

In his remarkable article "On what there isn't" (Review of Metaphysics, March 1972, pp. 459-488) Richard M. Gale comes to exactly the same conclusion "the world is fully describable in positive statements (thesis 2, p. 460) and "there are no negative events" (thesis 3 — ibidem). We want to take over some of his arguments but we can not share some of this other conclusions: a) the irreducibility of negative statements to positive statements b) reached as a result of his definition of positivity and negativity. In the paragraph that follows we want to attract the attention of the reader to some of the arguments used by him in favor of he possibility to describe reality in a purely positive way, trying moreover to strengthen the argument so as to reach conclusions about the necessity of this description. Moreover we think that our own definition of negativity will be prepared by some criticisms of Gale's interesting attempt.

Gale uses as an undefined relation "being of the same quality (id est: being both negative or both positive) and then formulates the following two conditions: CI: A property is positive if it is possible that there is a property of the same quality incompatible with it. C2: A property is negative if there could not be a property of another quality entailed by it. We are of the opinion that both conditions cannot be used: a) a counterexample against C2 would be the following circumstance: let us consider (as is indeed the case) that we have only a finite number of different colour words at our disposal. Let us then formulate the certainly negative property "having neither colour K1, nor colour K2, nor... asf." untill only one colour is not yet named. There we have before our mind a negative property that implies a positive property (the last colour of our list) and that yet, according to Gale's criteria should be positive. C1 is not satisfactory either because of facts that are in essence known to Gale himself and discussed by him on pp. 468: not all properties are counted as properties: neither properties of zero or universal extension, nor properties of heterogeneous type (conjunctions or disjunctions of negative and positive properties). But even these precautions do not eliminate the following counterexample: "being not red", implying that it is meaningfull to consider colour qualities for the objection in question, is incompatible with being even or uneven (no colours can be applied to numbers), and yet being uneven is of the same quality and thus non red should, applying the criterium C1, be positive while it is clearly negative.

It is important for us at the present moment of our discussion to be able to reject these two criteria because of the fact that by using these two criteria Gale comes to the conclusion that every translation of statements containing negations into statements using either the relation of "different from" or "incompatible with" is not really an elimination of the negative character of the statement (indeed he applies his two conditions to difference and incompatibility and comes to the conclusion that they are negative properties still). But if indeed we must accept that reality can be completely positively described and still that negative statements cannot be translated into positive statements, then we are *compelled* to use a set of sentencese (those referring to negative properties) that are *essentially superfluous*. To avoid this absurd situation, the consequence of Gale's point of view, we had to criticise his definition of positiveness and negativeness. Still we think that his paradoxical conclusion is rather close (but far from identical) to our own conclusions. We also think that for pragmatical reasons negative statements are needed but we moreover are convinced that to give an adequate description of reality it must be possible to translate all negative statements into positive ones.

We had to disagree with Gale in the former paragraph but we must strongly agree with him when he enumerates the following reasons to reject the existence of negative facts or events: 1) every positive fact implies an infinity of negative facts if they exist 2) (we did not use this argument yet): there is no valid procedure for counting the number of negative facts or events (if they exist) 3) purely negative facts or events are largely undeterminate, they have no clear positive and negative properties. 4) With Plato's Parmenides we deem it paradoxical to state that there *exist* facts or events that do *not exist.* 5) Moreover the failure of an event to arise, the privation of a property, the ommision of an act, the unfullfillment of a tendency that are "real" (in opposition to fictional) negatives facts can be purely positively described: when we say that a hard working student *fails* to meet the requirements of an examination, we simply conjoin the a priori large probability of his success to the unexpected event of his having the positive property of a low grade; when we say that we ommitt an action, (in Von Wright's terms: we "forbear" to do something) we simply say that we would have done something, perhaps should have done or wanted to do it and yet did something else, again positively described and incompatible with (one translation) or different from (other translation) the ommitted act. When an organism is deprived from an organ usually conferred upon beings of that description, then again we state that in general, and most probably, for members of a class a positive property holds while for the exceptional member another positive property holds.

The arguments, on the other hand, in favour of the existence of negative facts are easily refuted: if it is true that every true proposition must correspond to an existent fact, then still nothing compells us to state that all the properties of the proposition must be present in the fact (and in particular it is not needed that the negativeness of the proposition is mirrored in the negative character of the fact). If it is true that an individual must be different from other individuals and must have determinate frontiers, this does not imply that negative facts exist: the existence of frontiers can be expressed by means of the part-whole relation, some parts being inside an individual, some inside other individuals and some pairs of parts being in contact. If change does indeed imply, in everyday language (as Aristotle stressed already) that properties cease to exist and others that did not exist come into being, this description by means of negations of change is by no means the only or the necessary one; we can describe change by means of differential equations, including no negative whatever.

Let us then, as a consequence of the remarks made, subscribe to Gale's two assertions that we repeat here a) the world is fully describable in positive statements b) there are no negative events and moreover (in opposition to Gale) let us also subscribe to the first thesis: negative statements are reducible to positive ones. And for this reason let us discuss the motivations, the limitations and the techniques of the negationles logics developed untill now.

We thus have met both an epistemological (Freudenthal and Griss) and an ontological (Gale and ourselves, following Russell and Wittgenstein) reason towards a purely positive logic that, to our astonishment, has had so few practitioners while still ontological and epistemological reasons plead for its elaboration. The reader however will easily understand that the practically minded logician and mathematician is not willing to deprive himself from the powerfull tool that negation is. For this very reason we want to add (perhaps to the astonishment of the logician or the philosopher, but in the spirit of this collective research on negation) a third psychological argument.

We find it in the positivity bias that has been found according to G. Peeters (The positive-negative asymmetry: on cognitive consistency and positivity bias. European Journal of Social Psychology 1971). Heider and those influenced by him have examined graphs, the edges of which express either positive liking or negative disliking, positive resemblance or negative dissemblance. They have formulated a general concept of balance or equilibrium predicting (and verifying this prediction) that persons prefer balanced situations in which the cognitive and volitive factors are of the same sign and not of opposite signs (if a is positively evaluated and b resembles positively a, then there will be a tendency to evaluate also positively b). Within this theory of the interaction between attitude and belief, expressed in graph theorethic terms, there exists however a positivity bias. "Triads which included negative relations tended to be rated unpleasant, even if they were balanced" (Peeters pp. 456). Rosenberg and Abelson explain this positivity bias by the tendency to maximalise expected gain. However "a positivity bias has also been found in experimental settings where the subjects neither rated pleasantness, nor were asked to take the role of a stimulus person like P". "Several authors, for example Do Soto and Kuethe (1959) and recently McNeel and Messich (1970) have found that subjects were more inclined to assume a priori to any information a positive rather than a negative interpersonal relation" (p. 457). Peeters asks himself how this positivity bias can be explained. He comes up with the following fundamental explanation: (p. 471): behavior is classified in either avoidance or approach behavior. There is either a generalised preference for approach behavior (which implies the necessity to detect as swiftly as possible the negative elements in order to eliminate them: it gives them background properties opposed to a limited set of positive objects that receive then figural properties. This general or restricted approach preference implies — and this is the reason why we mention it — also a preference for positive thinking and logic, in opposition to negative one. *To negate is to avoid, to assert is to approach* (we come back to this point of view later on, on a biological basis). If this is the case then, on the basis of these psychological data, it is advisable that we should construct a language in which this positivity bias could be embodied. Negationless logic is such a language.

If our earlier assertions are true then we must reconstruct our logic. Various attempts have been made in this direction: a) the work of Griss; b) the work of Van Dantzig; c) the work of Vredenduin; d) the work of Gillmore; e) the work of Markov, Vorobev; f) the work of Nelson.

In this present article we wish to discuss Valpola, Gilmore, Nelson, Van Dantzig and Vredenduin. Nelson gives a comparative study of the earlier attempts; and is the most recent construction in this direction; Gilmore is generally accepted as an adequate formalisation of Griss' pioneer work; regrettably we could not consult the work of Vorobev and Markov.

The aim of our analysis will be the following one a) in these negationless logics, what types of negation-like constants can be introduced on a pure positive basis ? b) among these various attempts which attempts are most adequate to the positive ontology that we just announced in our introduction ?

It will appear that many different negations can be introduced on a purely positive basis; and that we shall have to combine them in many ways, in order to approximate to our positive ontology.

3. *Negationless logics*

In order to discuss and compare the various proposals towards a negationless logic we must ask ourselves what essential features they present:

1. In class logic we must certainly modify the definition of the complement (the class logic equivalent of negation) and deny the existence of a null class. Not all intersections exist.
2. The same is true for relational calculus (the zero relation disappears and the complement of a relation must receive a modified definition, while there are pairs of relations without intersections).
3. In propositional logic contradiction disappear as nega-

tions disappear. Griss himself, and also some of those who have tried to formalise his proposals modify equally the disjunction operator (looking at the negative $(\bar{p} \wedge \bar{q})$ or positive $[(p \wedge \bar{q}) \vee (\bar{p} \wedge q) \vee (p \wedge q)]$ normal form it will become clear that the meaning of the disjunction operator hides references to negation).

4. In functional logic, the status of quantifiers should come under scrutiny. Griss who, besides being in favour of negationless logic is also a radical finitist (more exacting than Brouwer: he denies even the potential infinity of the series of natural numbers) must agree with Van Dantzig's proposal to introduce only a restricted existential quantifier (meaning: existence up to the point p in a certain order). It would be more coherent however to introduce also restricted universal quantifiers. Only because of the mistaken impression that existential quantifiers have existential import and universals have not can one understand that unrestricted universal quantification is preserved.

Negative functions should also disappear (eg: expressions like (Ex) (— fx) are not possible).

If, as is usualy done, functional calculus is taken as the basic calculus, the modifications introduced there should entail the others. If however calculi of classes, of relations, of propositions and of functions are constructed in a way independent from each other, then one could have a positive propositional calculus combined with the existence of null classes, or the denial of null classes combined with the existence of negation in propositional calculus.

In the pages that follow, we are going to ask ourselves which attempts are most adequately adapted to the positive ontology sketched in what came before, subsidiarily also asking what type of negationless logic comes closer to the constructivistic and finitist position of Griss. He himself was not primarily interested in the reconstruction of logic on his foundation but concentrated on the reconstruction of mathematics. We shall quote his work primarily with reference to the "difference" or "distinctness" relation. However, we must make a preliminary remark: *negationless logic is something else than Hilbert and Bernays' positive logic.*

Hilbert and Bernays have examined a purely positive propositional calculus and functional logic (Grundlagen der Mathematik, I, p. 66)

I. Implication axioms

$p \to (q \to p)$
$(p \to (p \to q)) \to (p \to q)$
$(p \to q) \to ((p \to r) \to (p \to r))$

II. Conjunction

$(p \wedge q) \to p$
$(p \wedge q) \to q$
$(r \to p) \to ((r \to q) \to (r \to (p \wedge q)))$

III. Disjunction

$p \to (p \vee q)$
$q \to (p \vee q)$
$(p \to r) \to ((q \to r) \to ((p \vee q) \to r)))$

IV. Equivalence

$(p \leftrightarrows q) \to (p \to q)$
$(p \leftrightarrows q) \to (q \to p)$
$(p \to q) \to ((q \to p) \to (p \leftrightarrows q))$

V. Quantifiers

$(x)\ Ax \supset Aa$
$Aa \supset (Ex)\ Ax$

\+ The 2 deduction rules of Hilbert, modus ponens, and the substitution rules of propositional calculus. We have here a negationless logic at our disposal but, when we compare this positive logic with the other ones that claim this privilige, we see that the others try to find operators that will yield some of the results negation yielded. *In this calculus no attempt towards such an aim is present.* Moreover there is no awareness of the fact that the motivations that work for the rejection of negation modify some other logical constants too, and there is no fundamental avoidance of nullity or emptiness.

I. *Van Dantzig's negationless logic is a logic of acceptance*

Van Dantzig tries to construct a bridge between intuitionistic mathematics and classical mathematics. He does this in two different ways. First he replaces all formulae A of intuitionist mathematics by formulae — A that are stable, in the sens that they are equivalent to their double negations (triple negations are equivalent to simple one's). He then applies logical operations to stable propositions and constructs a structure closed under these operations. This part of his work is not the one that interests us here. In a second attempt he wants to develop an "affirmative mathematics", with this time not the weak negative interpretation but a strong affirmative interpretation, as a starting point.

It is typical for this point of view that no propositional or functional logic is the foundation of arithmetic but that indeed arithmetic itself is the first theory formalised. From our positive ontological point of view this also seems to be the right choice to make. One should first formalise a theory having a positive content and then only the logic used in the developing of the first. We would not in general consider however arithmetic as the first theory to be selected. But given Van Dantzig's aim, this is the only possible choice. His axioms are the following one's: I.o. is a number 2. If x is a number, the successor of x is a number 3. If x is a number, $x = x$ 4. If x and y are numbers, then if $x = y$, $x' = y'$ (where ' is the successor function) 5. If x, y and z are numbers, and if $x = y$, and $y' = z'$ then $z = x$. 6. If x is a number, and $x = y$, then y is a number.

It is important to see that equality is not defined explicitly, but only implicitly by means of a certain number of axioms.

It is equally important to note that nowhere either negation or disjunction was used in the axioms. We suppose that Van Dantzig equally introduces modus ponens and the rule for substitution of variables, the same way he introduces complete induction as a rule for deduction.

Peano's axiom stating that "Zero is not the successor of any number" could, in consequence of the affirmative character of the system, not be written(containing a negation).

Van Dantzig's motivation for his rejection of negation is to be found on page 1094 (Proceedings of the Section of Sciences, Academy of Sciences, Amsterdam) of his paper. He interprets logic as a system of assertions about formulae the mathematician is willing to admit. For this reason he can consider the conjunction as a normal propositional constant (it means that both p and q are admitted, or accepted). He can also consider the implication as a normal constant becouse it means to him that one is willing to admit q, the very moment p is accepted. But (and now we quote the text literally): "The symbol "not A", however has quite another nature. It does not describe the admittance of any formula, but the rejection of A, i.e. the mathematician's refusal to accept A. Of course he may refuse A, but why should he mention the fact at all ? We may make our list without telling anything about formulae we reject, or we eventually or conditionally would refuse to admit".

At the time of the composition of Van Dantzig's paper, no *logic of acceptance* was known. At the present time however assertion logic has been developped, and we could give the following paraphrasis of his crucial concept: "x is willing to accept p = (by definition)". There exists a proposition q such that when q is true, x asserts p" (willingness to accept would mean conditional assertion).

It is obvious that chapter XIV of Rescher's book "Topics in Philosophical Logic" will yield the formal basis of such a calculus of conditional assertions. When a logic of denials is explicitly rejected one has to introduce certain modifications in ch. XIV. 1. In system AI (p. 251) the two first axioms and the last one would be accepted but axiom A3, an axiom of refusal and not an axiom of assertion would be eliminated.

2. In system A2 only the second version of the assertion axiom could be accepted (and the second version would not be equivalent to the first version, this last one containing negations). In a similar way, in A3 the axiom of collective omniscience can only be accepted in its first version, the second containing negations 3. In system A4, where iterated assertions are studied negation is not involved. 4. System A5 introducing negations has to be rejected.

It is clear that next to the logic of acceptances (conditional or absolute ones) there should be a logic speaking about things and events. This realistic logic was the one we had in mind when we rejected negations in our ontological paragraph.

Van Dantzig asks himself if the acceptance of a disjunction is still an acceptance; he has doubts about the point: "our will to admit "A or B" is still not admitting anything at all nor (if we avoid negations: it is indeed the refusal to reject both A and B) committing ourselves to definite acceptances." We agree with him here and we are of the opinion that, as Vredenduin (see later) wants it, disjunction should be avoided if not as short hand for "1) if p then r, 2) if q then r, 3) for all z, if z then r implies $z = p$ or $z = q$, 4) and we accept r", a sentence where to avoid negation we had to utilise disjunction again. It thus seems rather unacceptable to utilise disjunction in a logical system inspired by the ideas of Van Dantzig. He retains it, however promising to get rid of it in the future. (This intention has however never been realised).

The restricted existential quantifier is introduced recursively as follows: $(Exo)\ (fx) = (def)\ fo$; $(Exn')\ (fx) = (def)\ ((Ex)\ (fx \vee f(n'))$ (where the —' is the successor). The Peano axiom about the zero, that could not be introduced, is replaced by the following one: "If x and y are numbers, and the sum of x and y is zero, then y is zero". Van Dantzig introduces a proof of non contradiction by giving the following model: If x is a number, x is equal to zero.

We now have to make two remarks with reference to this strong affirmative axiom system: 1) if our interpretation of Van Dantzig's intention is not false, then we should introduce conditional assertion in front of all the axioms. If we have axioms containing implications (and most of them, as the reader can verify, do present this feature) then Ax(p), — q: (the conditional assertion sign meaning: "x asserts p, under conditions") can be added in front of the formula, in front of the consequent, or in front of the formula and of consequent and antecedent. The different acceptance (i.e. assertion) logics will yield different arithmetics, all devoid of negation but all having different properties.

A second remark concerns the negation sign itself. The sentence that trivialises the system, called the sentence T, could be used to introduce a negation $-p =$ (def) $(p \rightarrow T)$. But as we can make our series of numbers start with any given number, varying the other sentences a accordingly, the sentence T, essentially equivalent to the sentence that the starting point is equivalent to its successor $(o = o')$ would become relative to this choice. There would be immediately an infinity of negations relative to the trivialising sentences chosen.

Negationless logic as we shall se with the following authors we shall analyse, contains in a very essential way the difference or distinction operator. This operator can be introduced in a axiomatic fashion or, to the contrary, by means of a series of definitions.

Van Dantzig introduces it in a very elegant way *by means of the restricted existence operator and by means of the order relation.* We want to present this way of introducing the distinction operator to compare it with the axioms presented by CFG Griss and with the ways our other systems handle it (we must do this because of the fact that in so many systems, negations are introduced by means of the distinctness relation).

1. "a is smaller than or equal to b" means by definition: (Exb) $(a = x)$

2. "a is smaller than b" means by definition "the successor of a is smaller than or equal to b"

3. "a is different from b" means by definition "a is smaller than b or b is smaller than a."

We must be aware of the fact that the feature of our system, implying that only one species of objects, ordered completely by one ordering relation, exists makes this definition of difference possible. We must also draw attention to the fact that a disjunction has to be used to give it (and the reader will remember our criticism of disjunction).

Griss in his book "Idealistise Filosofie" gives us a sequence of axioms about the identity and distinctness relation. It is interesting to see that Van Dantzig's definition makes certain of these axioms true, others false. I, II and III express the reflexivity, the symmetry and the transitivity of the identity relation, IV expresses the symmetry of the difference relation (an axiom certainly true for VD's definition by the commutativity of disjunction), Ax V states that if $a = b$ and $b \neq c$, then $a \neq c$ (the proof of this statement can be given by combining successively the two members of the disjunction with the identity relation, using the monotony of "smaller than" with respect to the identity relation). We have doubts about the excluded third formulated with reference to he identity or difference relation (it is certainly false that for every pair a and b, we can prove either that $a - b = 0$, or $a - b = n$ (where n may be a positive or negative number but must be one of the successors of 0). We do not see either how to prove from VD's definition the seventh and last Griss property for distinctness: when for every c, b distinct from c, implies a is distinct from c, then $a = b$. Yet intuitively this property seems acceptable.

The fact that we can not prove this last axiom from the VD definition throws some doubt on it; these doubts are confirmed when we go from natural numbers to real numbers. They are defined by a double series of approximations to them, the velocity of which must be given (the typically intuitionistic demand) "we can not define affirmatively the relations "smaller than" or "different from". This is to be expected, as e.g. the relation $x \neq y$ means that a natural n exists, such that $x - y$ larger than or equal to 2^{-n}. This however implies an unrestricted existence statement" (p. 1098). From the more general point of view that is ours here, we are not specially interested in the foundation of the arithmetic of real numbers and we can generalise: *for every set of objects ordered along certain relations where the types of possible differences between these objects cannot be expressed by means of a restricted existential quantifier, the type of definition of the distinctness relation that VD uses cannot be applied.* For most sets of objects, this will be the case. This implies an important consequence: exactly as an infinity of negations relativised to trivialising statements independent from each other, arises, exactly in the same way an infinity of distinctness relations and of ordering relations arises: we have to select a given n for the restricted quantifier and instead of merely expressing the fact that there exists a difference we have to state the minimal difference in question. VD calls this the "dispersion" of the ordering and the distinctness relations.

In conclusion we can now state: *the motivation of VD's negationless logic is to build a logic of conditional assertions about a completely ordered set*; we brought this out more clearly and showed the possible formalisation of this intention by means of assertion logic and the multiplicity it entails. We must also stress that in our ontological introduction our motivation in the defense of negationless logic was fundamentally different from Van Dantzig's one. The convergence towards the same point of two so different intentions constitutes another reason for the study of negationless logic. Griss' motivation is again different both from ours and from Van Dantzig's. In his book "Idealistise Filosofie" ("Idealistic philosophy") it becomes clear that a) G. F. C. Griss comes to the conclusion to reject negation in his logical system on the basis of the following beliefs a) "To show that something is not true, id est: to show the inexactness of an hypothesis, is not an intuitively clear way of acting. It is in fact impossible to have an intuitively clear representation of a supposition that later appears to be false". b) Bergson in his "Evolution Créatrice" (pp. 298-322) translates "the object a is absent" as "I expected to find a and I find something else". Bergson tells us "an intelligence that were only intelligence having no regret or desire could not even conceive absence or emptiness" (Bergson, pp. 306) "Pegasus does not exist" means "I have a representation of a winged horse and this representation is different from any real representation".

Bergson's arguments go in the same direction as our own in our earlier section that came before. It would however be possibte to disagree with Griss' and Freudenthal's argument and still, on the basis of our own, hold the position that a negationless logic should be construed.

It is namely by no means evident that one could not clearly conceive of false hypotheses. I can clearly conceive my being at the present moment in Athens and still I am not there. Realism is possible and for Griss his negationless logic derives from an idealistic metaphysical point of view (where "to be" is "to be constructed") that we certainly not have to share. We only want to stress that the discussion about the interest of a negationless logic clearly depends upon our general world view.

We criticised his analysis of disjunction and came to the conclusion that a serious problem exists as to the properties of the "distinction relation": it should be *definable*, any axiomatic characterisation of it appearing to be either insufficient or taking a very complex relation as a primitive; but the type of definition we met can only be applied in very special ordered sets on the one hand, and on the other hand we cannot prove for it all desirable properties it should have. Still when we compare it later on to other distinction relations it will become clear that nothwithstanding these undesirable properties, it still is one of the best proposals made up to date.

2. *Gillmore's Negationless' logic has to be completed by a logic of existence*

(We quote from Gillmore's dissertation "The effect of Griss' criticism of the Intuitionistic logic..." (Amsterdam 1953)).

The author is concentrating on propositional logic and functional logic and his rejection, with Griss' of emptiness and negation is to be placed there. It is only with Storrs Mac Coll and in certain versions of Aristotelian syllogistics that we encounter the beginning of the study of a rejection of negation in class logic and we did not meet during our studies a negationless relational calculus.

Glilmore's technique consists in the construction of various approximations to negationless logic. He starts with intuitionistic logic to construct Griss' attempt as a subsystem of it.

We have to remind the reader of the well known axioms and rules of inference because we shall have to remodel them afterwards and we shall do this in a way different from the one used by Gillmore himself.

Axioms of H.

I1 : $P \rightarrow (P \wedge P)$
I2 : $(P \wedge Q) \rightarrow (Q \wedge P)$
I3 : $(P \rightarrow Q) \rightarrow [(P \wedge R) \rightarrow (Q \wedge R)]$
I4 : $[(P \rightarrow Q) \wedge (Q \rightarrow R)] \rightarrow [P \rightarrow R]$
I5 : $P \rightarrow (Q \rightarrow P)$
I6 : $[P \wedge (P \rightarrow Q)] \rightarrow Q$
I7 : $P \rightarrow P \vee Q$
I8 : $(P \vee Q) \rightarrow (Q \vee P)$
I9 : $[(P \rightarrow R) \wedge (Q \rightarrow R)] \rightarrow [(P \vee Q) \rightarrow R]$

$$\left.\begin{array}{l}\text{I10: } (Ax_i)\, P(x_i) \rightarrow P(x_j) \\ \text{I11: } P(x_i) \rightarrow (Ex_j)\, P(x_j)\end{array}\right\} \begin{array}{l}(x_i \text{ not being in the range of any} \\ \text{quantifier of } P(x_i))\end{array}$$

I12: $P \rightarrow (\neg P \rightarrow Q)$
I13: $[(P \rightarrow Q) \wedge (P \rightarrow \neg Q)] \rightarrow \neg P$

Rules of inference:

$$IR_1: \frac{P \rightarrow Q}{P \rightarrow (Ax_i)Q} \qquad (x_i \text{ is not a free variable of } P)$$

$$IR_2: \frac{P \rightarrow Q}{(Ex_i)P \rightarrow Q} \qquad (x_i \text{ is not a free variable of } Q)$$

$$IR_3: \frac{P,\ P \rightarrow Q}{Q}$$

A system H is defined by introducing into the formal system described constant propositions, and atomic predicates both of a unary and of a relational nature, among which comes again to the front the relation of difference $\neq$. The axioms of this applied propositional calculus that becomes indeed a fragment of functional calculus are not specified except those for the difference relation. It is interesting to note that these are again different from the one's Van Dantzig and Griss use, and are indeed very weak. a) the difference relation is symmetrical b) it is anti reflexive (to express this fact a negation is used, because we are still only preparing a general framework for negationless logic: for all x, not "x different from x" is a theorem) c) finally there are at least two individuals so that the difference predicate is not an empty predicate. The reader might wish at this moment to compare these axioms to the earlier ones.

We are now presenting Gillmore's version of negationless logic, and simultaneously our own criticism of it.

Our fundamental criticism is the following one: the existential quantifier is used in Gillmore's proposal as indicating the non emptiness of the predicate that follows it. This is the familiar error also committed by Quine in his theory of ontological committment. The work in free logic however, begun by Henry S. Leonard "The Logic of Existence" (Philosophical Studies 1956) and continued by Karel Lambert "Existential Import revisited" (Notre Dame Journal of Formal Logic, 1963), summarised and extended in Rescher's "Topics in Philosophical Logic", chapter on the logic of existence, introduces a specific method different from existential quantification to express existence: an entity exists (in one definition) when either it has properties that it could eventually not have or it is a property of an individual it could eventually not be a property of and when it possesses properties not possessed by all individuals. In another definition not yet formalised an entity exists(we used this definition in our metaphysical introduction) when it is either effect or cause of at least one event. Following Leonard's usage this strong existence that has ontological meaning is expressed as follows (E!x) (fx). The proposal defining (E!x) by means of the fact that x possesses non-necessary properties uses negation. To avoid this we shall have to use the usual way to get rid of it: $\Diamond\sim p$ will mean that "$\Box p \rightarrow (\Diamond p \leftrightarrows \Box p)$ (the necessity of p implying a modal paradox).

We are of the opinion that by means of such a strong existence predicate Gillmore's requirements (pp. 5, Gillmore 1953) have to be rewritten as follows:

a) negation may not appear as a primitive symbol (it is either absent or introduced by means of definitions not including negative concepts)

b) the atomic predicates and proposition must all be such that theorems of the following form are provable for them: (E!x)p or (E!x) (P(x)) (Gillmore does not use the strong existence expression but simply the existential quantifier: this we consider not an adequate method to construct a pure positive logic because of the fact that for all x, fx implies that there is at least one x, such that fx, and for all x, fx may be a sentence speaking about a completely fictitious entity like "all angels are immortal")

c) the molecular predicates built by means of conjunctions, disjunctions, implications and quantifiers, from atomic predicates must obey a similar restriction in terms of the strong existence predicate: for disjunction and implication the two terrms of the molecular sentence must be applicable to at least one strongly existing individual; for conjunction at least one strongly existing individual must exemplify simultaneously both propositions or predicates of the conjunction. Predicates so constructed are called "C-admissible."

Even if those modifications by means of Leonard's ideas (or modified versions of them) are introduced into Gillmore's calculus, we must make the following two remarks: a) the negationless logic is introduced by means of restrictions introduced in another logic. We are of the opinion that to the contrary it must be developed as an independent system b) the proposal of Gillmore is equivalent to the following one: not all predicates that are well formed are necessarily not empty, but all well formed parts of theorems are not empty. This allows us to have empty well formed predicates but eliminates from wellformed parts of theorems such empty concepts. This intention is too weak, if we take into account the motivation for the construction of a negationless logic: if indeed reality can be described in a purely positive way and if the language we seek to construct must be the exemplification of this purely positive approach, not only proofs and parts of proofs but also every admissible predicate must be ontologically non empty.

If we apply however Leonard's concepts to atomic or molecular predicates, we conjecture that this requirement, stronger than the one Gillmore formulates, will be met.

This will only be the case however if the predicates called by Gillmore's "G-admissible" are eliminated (these predicates are contradictory ones and in their definition a negation is used because the following theorem is asserted: for them: "Ax (not P)". If we can describe purely positively everything we want to say we must not allow these predicates: every predicate must be positive and non empty. Gillmore is well aware of this fact and one could characterise his work as an attempt to justify the introduction of G-admissible predicates into negationless logic. This is done by means of the introduction into the language of two specific variables u and v, that always remain free, and that can only be used in the following expression: "$u \neq v$.

On p. 10, this becomes very clear: 1) every occurrence of negation in a formula is to be replaced by $p \rightarrow N \wedge \sim N$. 2) Every occurrence of $N \wedge \sim N$ is to be replaced by $u \neq v$. $u \neq v$ has the properties of the empty predicate in virtue of T5. 1: $(u \neq v) \rightarrow$ all well formed formulae. On page 8 Gillmore points out that no special assumptions are made as to u and v. This implies that their range is the set of all individuals in a given model, and so $u \neq v$ includes all instances of $x_1 \neq x_1$, $x_2 \neq x_2 \ldots$

In a negationless logic the equivalent of an empty prodicate can thus be constructed. Why do we need it so much, according to Gillmore ? Why can we not only use C-admissible predicates, but have to use G-admissible predicates ? The reason is clearly that in Axiom I10 of H, one can have the case that one uses a C-admissible predicate in the antecedent, and gets a G-admissible one in the consequent (take for P, $x_i \neq x_j$ and one gets $(Ax)[(x_i \neq x_j) \rightarrow (x_j \neq x_j)]$. More generally to say that something is a theorem, it is convenient to be able to say that "$p \vee (u \neq v)$". Gillmore wants to have replacements for all negation statements in H. From the point of view of our ontological epistemological and psychological basis, we must reject this empty predicate. Gillmore's logic is *not* a purely *positive* logic; it uses empty predicates. Gillmore tries to defend his introduction of a null predicate by saying (p. 23) that the concept of null predicate is relative: "nullity" is a relative concept; there is no absolute nullity within mathematics. There is however a non relative nullity for any realistic ontology or for any epistemology. And the arguments used by Gillmore are not convincing ! They consist in pointing out that the nullity predicate can be expressed as $u \neq v$, or as $N \wedge \sim N$ or as "being an element in all classes", in various formal systems. This is no proof of the relativity of nullity (nor of existence) as it is easy to show that we have here only 3 names for the same entity. The purely positive logic to be constructed in the future may however keep a certain number of the axioms of his final K-system. We are going to make some comments upon the deduction rules first, because they contain (pp. 14) a characterisation of the (Ex) operator (x is the list of all variables in the formula that follows) and because (E!x) has, we think, not all of these properties in any of our interpretations for it.

The deduction rules are the following ones:

RP1: if Q is awff of K constructed from the wff R_1, R_2 ... R_n without using quantifiers then

$$\frac{(Ex)\ ((\ldots(R_1 \wedge R_2) \wedge R_3 \ldots) \wedge R_n)}{(Ex)Q}$$

RP2: $\dfrac{P,Q}{P \wedge Q}$

RP3: $\dfrac{(Ex)P\ P \rightarrow Q}{(Ex)\ (P \wedge Q)}$

RP4: $\dfrac{(Ex)P \wedge R,\ \ P \rightarrow Q}{(Ex)\ (Q \wedge R)}$

RP5: $\dfrac{P}{(Ex)P}$

RP6: $\dfrac{P,(Ex)Q}{(Ex)\ (P \wedge Q)}$

RP7: if P and Q have no free variables in common, then

$$\frac{(Ex)P,\ (Ex)Q}{(Ex)\ (P \wedge Q)}$$

In order to verify that the replacement of (Ex) by (E!x) has some consequences let us chose one interpretation for the ontological quantifier: $[(E!x)P] \leftrightarrows [(Ex)\ (Px) \wedge \Diamond(\sim Px)]$
Consider *RP3*: $(Ex)Px \wedge \Diamond(\sim Px)$ is given.
But nothing is said about $\Diamond(\sim Qx)$. Thus: non sequitur.
Consider RP5 and RP6. It could be the case that in both P, present there, only individuals with necessary properties are mentioned. In that case, RP5 and RP6, cannot be kept. The objections wil however suggest to the reader all necessary corrections (strengthening of the presuppositions by means of assertions of contingency) Even RP7 might be questioned: for both predicates P and Q, there are individuals possessing them (1) and possessing them contingently (2) (if we use E! instead of E) but it could be the case that when both are present, the only individuals possessing them do so necessarily. So again stronger conditions have to be added.

Having looked at the rules of deduction let us now characterise the axiom's themselves (because they can be preserved for any purely positive logic).

They are of the same type: allowing the usual deductive operations but only if the necessary existential conditions are given:

Ko: (Ex)p for any atomic predicate or proposition

K1: $\dfrac{(Ex)P}{P \rightarrow (P \wedge P)}$

K2: $\dfrac{(Ex)\ (P \wedge Q)}{(P \wedge Q) \rightarrow (Q \wedge P)}$

K3: $(Ex)\ (P \wedge R),\ (Ex)\ (Q \wedge R),\ (Ex)\ P \rightarrow Q$
$(P \rightarrow Q) \rightarrow [(P \wedge R) \rightarrow Q \wedge R)]$

We are not going to transcribe the whole series (to be found on p. 13 of Gillmore's thesis) but we wish to point out that 1) all parts of the deduction, atomic or molecular, must be shown to have ontological import in the premise 2) if we replace Ex by E!x we are in some cases going to meet the same difficulties as in the case of the deduction rules: if existence is contingency then the individual existential quantifier might pick out cases in which a property exists contingently, but when all the conditions together are taken to be true it could be the case that be individuals satisfying the collective condition do so with necessity. In that case, another premise excluding this has to be added.

As conclusion of our remarks on Gillmore we come back again to our main point: on p. 18, Gillmore himself states that $(Ex)p \vee (Ax) \neg p$ might be added in the system K as an axiom, while we are of the opinion that *(E!x)p* should be added as an axiom to each negationless logic. Moreover we think that Van Dantzig's *acceptance* logic must be combined with a version of Gillmore's modified *existential* logic (this cannot be done without substantial changes as the first system rejects disjunction while Gillmore does not do this).

It is a fact that the very moment we think to have at our disposal a positive logic, we are going to search for a positive definition of negation. This is done by Gillmore in the following fashion:

Let

$$\mathrm{D}(U \neq \mathrm{X}) \text{ mean } [(\mathrm{U}_{i1} \neq \mathrm{X}_{i1}) \vee (\mathrm{U}_{i2} \neq \mathrm{X}_{i2}) \ldots \vee \mathrm{U}_{in} \neq \mathrm{X}_{in})]$$

and let

$$\mathrm{C}(\mathrm{u} = \mathrm{x}) \text{ means } [(\mathrm{U}_{i2} = \mathrm{X}_{i2}) \vee (\mathrm{U}_{i2} = \mathrm{X}_{i2}) \ldots \wedge (\mathrm{U}_{in} = \mathrm{X}_{in})]$$

Griss-negation is defined as follows:

$$\mathrm{nP(x)} = (\mathrm{A}u)\ (\mathrm{P}(U) \rightarrow \mathrm{D}(U \neq \mathrm{X})$$

while intuitionistic negation means:

$$\mathrm{NP(x)} = (\mathrm{A}u)\ [(\mathrm{P}(u) \wedge \mathrm{C}(U \neq \mathrm{X})] \rightarrow \mathrm{D}(U \neq \mathrm{X})$$

obviously $n \rightarrow N(by(p \rightarrow q) \rightarrow [(p \wedge r) \rightarrow q])$

By propositional calculus we can rewrite N as follows:

$(Au)\ [(P(U) \rightarrow [C(U \neq X) \rightarrow D(U \neq X)]]$

(here negation means obviously absurdity of the negated sentence).

We shall meet again and again this defining negation by difference, replacing an operation on propositions by an operation on individuals but we can easily verify, when comparing the definition of $\neq$ in Van Dantzig or the stronger axioms for it put forth by Valpola, that $\neq$ is here too weak.

III *Valpola's negationless logic presupposes a calculus of truth and falsity as modalities.*

Valpola's motivations, leading him towards the elimination of negation, are similar to ours (see op. cit., p. 113). He begins his construction by asking what would happen if negation were eliminated:

1) it would be impossible to express incompatibilities: (f.i:$(Ex)\ (Ey) \sim (x = y)$ or $\sim [(Ax)\ (Ay)\ (x = y)]$).

2) all *indirect* proofs (by means of reduction ad absurdum, or by means of the transposition principle) have to be replaced by *direct* proofs

3) many antinomies disapear (*not* all: see Curry's paradox)

4) we may add that by the disappearance of the De Morgan theorems the conjunction, disjunction and implication become independent symbols, not interdefinable.

Valpola's work incorporates the idea to use the concept "true" in the object language as independent predicate and a series of restrictions on the substitution rules, restrictions having the purpose to avoid going from non empty to empty predicates. Some examples of these restrictions are the following ones:

1. From $(x)\ (y)[P(xy) \supset Q(xy)]$ one can derive $(x)[P(xa) \supset Q(xa)]$ *only* when $\vdash (Ex)\ P(xa)$

2. From the same sentence $(x)\,(P(xx) \supset Q(xx))$ is only derivable when $\vdash (Ex)\,P(xx)$

3. From the same sentence, when proved as a theorem, one may come to the conclusion $\vdash (x)\,((y)\,(Ry \supset Pxy) \supset (y)\,(Ry \supset Q(xy))$, when $\vdash (Ey)\,(Ry)$

4. But to conclude that $\vdash (x)\,((y)\,(Qxy \supset Ry) \supset (y)\,(Pxy \supset Ry)$ is not permitted in all cases, because a value of x verifying Q may falsify P. We eliminate this danger by adding $(Ez)\,(Pxz)$ to the antecedent of the implication.

5. In general, implicational sentences the consequent of which is again our implication, hold only when the antecedent contains a condition eliminating any values of the common variables making the consequent empty.

We shall meet again these restrictions on the substitution rules when we define the "regularity conditions" characteristic for Valpola.

The aim of this introduction of truth as a predicate and of the restrictions on the substitution rules is to avoid any speaking about classes or relations that are logically or empirically empty, and all talk about "logically contradictory" propositions. In order to reach this aim we *have* to eliminate (p. 122) the negation operator because for any proper predicate, by applying negation and abstraction we obtain the non-proper class $(\hat{x})\,(\sim P(x))$. The reduplication of the universe achieved by means of negation must be eliminated.

The proper task however of all those who construct negationless logics, is the replacement of negative tools by positive ones. (This is precisely the positive analysis of negation that we are looking for). The question arises: what part of the classical propositional calculus is to be kept ? What replacements are to be introduced and what interpretations are to be given to the propositional constants we preserve ? Truth tables will not do anymore (they imply tacitly the use of negation in the T — F duality).

The fundamental constants that are taken as building stones are *conjunction* and *general implication*. Singular implication is eliminated because it is proved either as a special cause of general implication, or because one knows the truth of the consequent, or the falsity of the antecedent. According to Valpola this is an argument in favor of the elimination of singular implication because in order to prove it we must have stronger information at our disposal. *Disjunction* is equally eliminated because to Valpola it is either a question, or a provisional hypothesis (i.e.: a proposal to accept temporarily one of the members of it. We have already a logic of acceptances; we are still waiting for a logic of proposals: some postulates for it could be:

$$\mathrm{Prxp} \rightarrow \sim [\mathrm{Prx} \sim \mathrm{p}];\ \mathrm{Prxp} \rightarrow [\mathrm{Bx}\Diamond \mathrm{p} \wedge \mathrm{Bx}\Diamond \mathrm{Acq} \wedge \mathrm{Bx} \sim (\mathrm{Ay})\mathrm{Byp}];$$

(where Prxp = "x proposes p" and B = belief); Both these types of utterances are not usable in the descriptive positive ontology we are building a logic for. We agree with Valpola's analysis of disjunction (remembering Van Dantzig's misgivings about it, and Gillmore's two definitions for it) but we would not analyse implication the same way he did it, given the absence of a truth functional interpretation.

As does Griss himself, Valpola tries to keep some of the advantages of the disjunction by introducing the union of two classes (but not the disjunction of two sentences).

We would not speak to such length about connectives different from negation, if we could not show that the same desire to develop a logic suitable for a language describing a real world has led to the *rejection of disjunction and negation, the privileges for conjunction and for law — like universal implication.* We could state that all molecular sentences should build true sentences from true sentences. This $\sim p$ and $p \vee q$ do not do, $p \wedge q$ and $(x)\ (fx \supset qx)$ do. For the same reasons the system is developed immediately in functional calculus: we must give ourselves the means to speak about individuals and about laws if we want a logic, adapted to an ontology. The union of two sets is defined as the set that is included in all sets that include the two original sets. Formally C is in the union of a and b if $\mathrm{Vabc} \equiv (x)\ (x \in C \equiv (S)\ ((a \subset S) \wedge (b \subset S) \supset (x \in S))$. The three usual axioms for *disjunction* (its introduction and its elimination) have analogs in theorems about this union set. Finally, on the level of a syntactic metalanguage some remarks are made upon the functions of negation in axiomatic systems. In most axiomatic systems pairs of predicates occurr that are related to each other as each other's negations. Such predicates either must be positively defined in a negationless logic, or else suitable axioms have to be proved from these axioms. Such a situation occurs when we examine the "identity-difference" pairs. As we saw with Van Dantzig, in some systems (depending upon the ordering relations present in them) diference can be easily defined. In systems where the ordering relations are either absent (theory of groups, of fields, of rings) or complex (geometries) it is not te be expected that definitions can be found. Here the two relations must be independently introduced; if we do this, in general they become independent and we have to introduce special axioms to reestablish the usual relationship. The axiom $((x)\ (y)\ (x = y)$ or $(x \neq y))$ yields an excluded third; How could one however reintroduce an axiom of noncontradiction ? We do not have negation as a promitive connective; but one can introduce

$$\sim Pa \equiv (\text{def})\ (x)\ (Px \rightarrow (x \neq a))$$

Even then however $\sim (p \wedge \sim p)$ would become

$(x)\ (y)\ [Px \wedge (z)\ (Pz \rightarrow z \neq y)] \rightarrow (y \neq x)$, an implication with an empty antecedent (excluded in a positive calculus)

The other axioms for difference:

1. $\vdash (Pa \supset Qb) \supset ((x)\ (Qx \supset (x \neq b)) \supset (x)\ (Px \supset (x \neq a))$
   (the transposition principle for difference)
2. $\vdash Pa \supset (x)\ ((y)\ (Py \supset (y \neq x)) \supset (x \neq a)$
3. $\vdash (x)\ ((y)\ (Py \supset y \neq x) \supset (x \neq a) \supset Pa)$
4. $\vdash (a = b) \vee (a \neq b)$
5. $\vdash (a \neq a) \supset Pb$
6. $\vdash (x)\ [(a \neq x) \supset (b \neq x)] \supset (a = b)$

(allowing to prove $(a \neq b) \equiv [(x)\ (b = x) \supset (a \neq x)]$
or $a = b \supset (x)\ (b \neq x \supset a \neq x)$ and also $(x) \sim (x \neq x)$

can not all be accepted because again in (1) the antecedent may become empty (f.i. in case $\vdash Pa \wedge Qb$). One can give weakened versions f.i.: $(z)\ (Pza \supset Qzb) \supset (z)\ ((x)\ (Qzx \supset x \neq b) \wedge (Ex)Pzx \supset (x)\ (Pzx \supset (x \neq a))$, a consequence of (1). These weakened versions do not yield however the desirable consequences.

It becomes clear, when we meet so many different axiom systems for $\neq$ that we should look for a semantic model of $\neq$. We could put forward, in a part — whole calculus (Goodman), "$(x)\ (y)\ (x \neq y)$ means $(Ez)\ (zPx)$ (z is part of x) and (zy) (z is separated from y). The concept of separation is however exactly as the concept of difference again to be defind axiomatically. We conjecture however that the model of difference presented here is the only one that is general enough, presented untill now (the models of difference-using order presuppose much more about the model than our present version).

We must finally come to the conclusion that a principle of non-contradiction cannot be formulated.

Still, the contradictions of a system can be formulated in many different ways: $(x)\ (y)\ (x = y)$ may be the expression of a contradiction or, in an ordered system: $m = Sm$ (where S means successor). The contradictoriness is defined, with the help of a negation, in metalogic as the non-derivability of such sentences (and we have always only relative non-contradiction, with reference to a type of positive contradiction sentence).

We have however sufficiently commented upon the type of system, Valpola is presenting. We must now proceed to the examination of our central point: how does he realise the intention of building a positive system ? We can answer by a comparison: *Van Dantzig develops a logic of acceptance, Gillmore a logic of existence, while Valpola develops a logic of truth.* Indeed: his general plan is the following one: 1) one defines the concept of regularity (by means of the concept of truth, predicate *in* the object language), regularity meaning non emptiness, positivity for different types of expressions 2) axioms for a strong, typeless theory embedded in functional logic are introduced, and also rules of derivation: it is shown that the first are regular, and that the second preserve regular-examine the relation between Gillmore's system and Valpola's. ity 3) E-statements are defined by means of which we can

Our own contribution in this discussion will be the following one:

1. We comment upon the truth concept in the object language; suitable axioms for it should be presented. Valpola does not give them.

2. We comment upon the regularity concept asking ourselves if, for the adequate expression of a positive ontology it is either to strong or too weak.

3. Is there an interaction between the truth-axioms and the application of regularity ?

4. We remark here the obvious, but not yet examined fact, that Valpola's E statements build a bridge towards Gillmore (and also Vredenduin's) positive logic.

I. Truth.

Truth is here essentially treated as a modality. It is not the semantic metalinguistic concept of Tarski, but a specific predicate. Let us write down some specific axioms for it:

$$Tp \rightarrow p;\ TTp \rightarrow Tp;\ T(p \wedge q) \rightarrow Tp \wedge Tq;$$
$$T(p \rightarrow q) \rightarrow (Tp \rightarrow Tq);\ T(p \vee q) \rightarrow Tp \vee Tq;$$
$$(Ep)Tp;\ (Ap)\ (Eq)\ (q \neq p) \wedge (Tp \rightarrow Tq).$$

Except the two last axioms the earlier ones are typical for a modality. Except the disjunction axiom, they all hold if truth is replaced by necessity. The two last axioms are specific. We cannot prove that this set of axioms is complete, except by saying that we have put T into a relation with every logical operator.

We have to add two remarks: if we define T as a modality we can do the same for falsity, and if we define both as modalities we have to examine if T or F or both can be replaced by other predicates in the predicate calculus:

$TFp \rightarrow Fp$; $F(p \wedge q) \rightarrow Fp \vee Fq$; $F(p \vee q) \rightarrow (Fp \wedge Fq)$; $(p \rightarrow q) \rightarrow (Fq \rightarrow Fp)$.

Here iterations do not reduce: $FFp \rightarrow p$ does not hold. $FTp \rightarrow Fp$ is a very doubtfull and questionable axiom.

All our axioms have the form of implications, none of them have the form of equivalences. $p \rightarrow Tp$ does not hold; $\vdash p \rightarrow Tp$ does however (it is important not to identify these two assertions).

As we have no negations we cannot define T by F or F by T. Still, some relations hold: $(Tp \wedge Fp) \rightarrow Fp$; $Tp \rightarrow FFp$;

$$(Tp \vee Fp) \rightarrow (Eq)\ (Tq \wedge [(q \rightarrow Tp) \vee (q \rightarrow Fp)]$$

Both modalities can be defined by means of relations:

$$Tp = \text{Def } p = R(p_1 \ldots p_n) \wedge (Ex)\ (x = S(x_1 \ldots x_r) \wedge Q(R,S) \wedge Q'(p_1 \ldots p_n, x_1 \ldots x_r).$$

Wittgenstein's definition would make 1) Q an isomorphism and 2) in consequence, Q' a one-one relation. We want to have more generality and take Q and Q' unspecified. We know that the real solution must be in between. However we only wanted to point out (knowing that we would have to procede to higher order functional logic) the possibility to define T by a relation between relations. F would be definable in the same way: in Wittgenstein's terms Q' would here be many-many, or one-many or many-one, and Q would be any morphism (not an isomorphism, and not even a homomorphism). We stress that the definition of F would be purely positive. From our more general point of view, we only have to define a set of Q's and Q''s and state that Fp holds if any member of the set can hold (notice the modality) while Tp holds only if a member of a restricted sub-set holds. This is much too general naturally but our aim at the present moment is only to show the possibility of introducing positively T and F in the object language. Later it will be necessary to verify if the axioms introduced for T and F as independent positive predicates correspond to the axioms imposed upon them. Our fundamental remark upon Valpola's treatise is (cfr. our main remark about Gillmore) that his central concept: truth, cannot be an arbitrary predicate and must be introduced in a way similar to the one we just used. Let us now see the truth concept plays its role in the definition of "regularity".

Let us first quote Valpola's intention (p. 196, op. cit.): "Regulär werden hier solche wohlgebildeten zeichenreihen des Kalküls genamt, die in dem früher beschriebenen Sinne inhaltsvoll sind und *keinen solchen Prädikator enthalten, dessen Bedeutung logisch absurd oder empirisch unrealisierbar ist.*"

The definition is a recursive definition:

1. a name is regular if

a. the name is atomic, and occurs in a *true* and regular sentence, or in a *true* atomic or both characteristic and regular (a name is characteristic if it specifies one unique individual). Comment: it is not easy to interpret this sentence in a non circular fashion when one looks at the lastpart of it (a name is regular if atomic... and occurring in a true atomic sentence all names of which are regular and characteristic). One part of the definition is certainly circular.

b. the name is characteristic and there exists a regular and true sentence constructed from the predicate Pc, having the form $(Ed)\ (Pc \equiv_c (hc \equiv_h hd))$. Universal quantifiers here as in Vredenduin's work, are attached to equivalences and implication.

Comment: To define the regularity of names we need to know the regularity both of atomic and molecular sentences. The fact that Valpola uses two conditions for name-regularity mirrors the fact that either the true atomic sentence in which this name is characteristic can be found, or that, one knows that there is an individual having the properties of one characterised by such a name. From our point of view the quantifier should be a strong existential quantifier. There are no non atomic non characteristic names. This might demand another condition.

2. A *predicate* is regular iff:

a. the predicate is atomic and one of its complete specialisations is such a true atomic sentence, in which every name is atomic or both characteristic and regular.
b. the predicate is atomic, the existential sentence formed by means of it is true, and all names that might be present in it are regular.
c. the predicate is molecular ($\rightarrow$, $\leftrightarrows$, $\wedge$, Ex) and the existential sentence built upon it is regular.
Comment: Here as with names we see that in the recursive definition of regularity there ought to be built in precautions against circularity (we use the regularity of names here). A more important comment however is that the existential quantifier in front is considered as sufficient to confer existential import. In an earlier remark concerning Gillmore we have already made clear that this cannot be the case. We are to the contrary of the opinion that in Valpola's condition Ia, and in his conditions 2b and 2c strong existential quantifiers must be introduced. This would mean a modification of his regularity conditions. On the other hand however, we may ask if in Gillmore's proposal the concept of truth ought not to be introduced in the object language ? We think it should, striving thus towards *a synthesis of logic of acceptance, of truth, and of ontological existence pragmatic positivity, syntactic positivity and semantic positivity.*

3. A *sentence* is regular in one of the following conditions:

1. The sentence is atomic and true, containing only names that are either atomic, or characteristic and regular.
2. The sentence is an atomic sentence constituting a specification of a regular predicate and containing only regular names.
3. If the sentence is molecular, both its constituents must be

regular if it is a conjunction; its predicate must be regular if it is an existential sentence, and if it is an implification both antecedents and consequents must contain only regular predicates but moreover for every true regular and complete specification of the antecedent the analogous specification of the consequent is a regular sentence.

Comment: we must make the same comments here as for the regularity rules, we met before. The existential quantifier must be strengthened, and moreover we ask the following question: if we want a purely constructive definition of regularity, should not case 2 (defining the regularitive of an atomic sentence by means of the regularity of more general and complex one, be eliminated ? If this is done will however the system remain strong enough ? We can not answer the question but only ask it here.

There is a strong relationship between *regularity* and *truth*. All regular existential sentences are true; but it can occurr that regular implicational sentences are false, because next to the true specifications of antecedent and consequent, there exist false specifications. The system contains six rules of deduction that all preserve regularity. We are not going to copy them because we have made in what went before the most important comments we have to make with reference to this calculus. We shall simply mention them by name and point out the most characteristic one for the calculus in question.

1) simple substitution of one variable for another (under condition that no collisions occurr; if one of the variables occurs free then it must occurr so in other predicates than the first variable).

2) substitution of one variable by "a sequence of variables, the sequence being uniform everywhere when the substitution occurrs, (rule characteristic for this calculus).

3) change in the sequence of terms (we must make the following comment: the properties of the truth modality that we described before are relevant for all the rules in question because all are formulated in such a way that the deduction can only occurr from a true sentence to a true sentence. This holds also for all the following deduction rules. The reader might

be sceptical as to the real interaction between the axiomatic definition of T and the deduction rules. But we think his scepticism will subside when he looks at rules 4, 5 and 6 that are rules for conjunction elimination, and modus ponens for names and for predicates. Equivalents of these rules occur in the list of properties of T we tried to construct before. The axioms of the calculus can be found on pp. 167 op. cit. We again make the remark that they should be completed by adding assertions about the *truth modality* and the ontological *existence quantifier E!*. The axiom system is remarkable because it makes no difference between signs of different types: individuals, predicates or relations that have to be distinguished by their function in the sentence, and by the fact that universal quantifiers are only used as indexes of implications and equivalences. These two decisions, though Valpola does not mention this fact, are probably again related to the intention of constructing a pure expression of necessary connexions in the real universe and all types of signs are only designating features of the real inverse.

As a final remark we want to say something about the relation between Gillmore-Vredenduin and Valpola. Valpola defines the concept of E-sentence, for a sentence, in all places as which in the original sentence an implicational predicator is present. The E-sentence adds an existential quantifier to the antecedent of implication or equivalence and conjoins this existential proposition to the original implication or equivalence. On p. 209 we find the theorem that can be proved by looking at the definition of regularity: a sentence is regular and true, iff its corresponding E sentence is true. If we can consider the deduction rules of Gillmore as the constitution of so many E sentences we can at least say that with reference to implication this shows that Gillmore's demands and Valpola's are in fact the same. But we cannot assert an equivalence between the two systems because disjunction is present in Gillmore, absent here and because we have here in fact, as a consequence of the systematic assimilation of types to other types, a higher order functional logic (even though it is senseless here to speak about the order of a symbol).

As an example of what we mean we can look at axiom AIO $\vdash (\mathrm{aef} \supset_{\mathrm{ef}} \mathrm{bef}) \wedge \mathrm{Eg}((\mathrm{agg}) \supset_{\mathrm{ab}} (\mathrm{ahh} \supset_{\mathrm{h}} \mathrm{bhh})$ where the natural interpretation of aef can be both a(ef) a relation) or a(e(f)) (a predicate of 3rd. order).

Let us thus, aware of the fact that Valpola's is a very strong system (with non contradiction not yet proved) only assert a partial equivalence as to E sentences between Gillmore's attempt and Valpola's. We cannot analyse the whole system here as we are only interested in the consequences of the "positive bias" of our examples, consequences that are manifestly deep and difficult, and affecting also other operators except negation.

We leave then our analysis of Valpola's attempt by reminding the reader that also in the axiom system whenever the existential quantifier occurs, the author should use the ontological quantifier (here we see that axioms A9 till A18 would be affected).

4. *The negationless logic of Vredenduin is a logic of definitive acceptance and can be analysed by means of automata.*

Vredenduin (op. cit., pp. 206) makes a strong difference between the dynamic growing of a logico-mathematical system, and the finished end-product. In the dynamic process negation will be used (e.g. for proving that some sentence cannot be preserved in the system) but among the parts of the final system no theorems about negations or impossibilities have to be introduced.

Vredenduin himself describes the situation as follows "in the mathematical systems only those propositions will occur that are true. And as they are all affirmative, ... a sign for negation is useless in the system.

If we remember the assertion calculus that we used to express Van Dantzig's intentions, we can, taking into account this opposition between the dynamic becoming and the static finally state of the system say that Vredenduin wants to construct a calculus for "definitively asserted propositions". His reference to truth comes only in the second place, when we look at his description of Griss' intentions. The concept of being *definitively asserted* can be defined as follows "p is asserted at the moment t and p will be asserted at all later moments". In this interpretation we need a) chronological logic (see Arthur Prior) introducing time and b) assertion logic (as in our interpretation of Van Dantzig). The question: "will there be negative statements among those accepted definitively ?" will then become a topic for discussion. We think (reconstructing) that the following idea prevails: either for all negative statements there exists stronger positive statements implying the first so that the negative statements will all eventually be discarded earlier or later or negative statements are not really statements but rejections (and as such no part of the finished system ,though essential episodes of the construction).

We want to point out that our own motivations to search for negationless logics, deriving from a positive ontology, again (cfr. Van Dantzig) are fundamentally different from these ones, presupposing the opposition between dynamic constructing and finished system.

A second remark seems obvious to us: when we look, p. 205 and 206 at the interpretation of the signs introduced it is obvious that a positive class logic is the model that is used by Vredenduin (following here Griss) to interpret a functional calculus (propositional calculus losing its importance for reasons we shall come back to later on). The meaning for instance of $p(x) \supset (y)$ is that the class of objects satisfying $p(x)$ is included in the class of those satisfying $p(y)$. The same interpretation holds for the implication between n-ads. While $(Ex)(px)$ is called proposition and not further interpreted, $(Ex)p(xy)$ is the class of those y for which an x exists satisfying $p(xy)$. And a wff without free variables (206) is interpreted as a universal class either of elements, or of pairs or of n ads according to the form of the formula. No empty class is introduced and so all contradictions and false formulas must be eliminated in so far as for every formula an interpretation is to be produced by non empty classes.

Disjunction and implication are superfluous again from the point of view of definitive acceptance. It may be true that from p follows r and equally from q follows r (expressed by p or q implies r) but in the definitive system we shall never derive anything from unproved propositions and wait untill either A or B is proved to derive anything from them. The same remark holds for implication. Not sharing Valpola's *nomological positive world view*, Vredenduin remarks that it might be usefull outside the system to know that when p is proved, q can be proved. But we shall never include this *in* the system. There we shall find a proof of p followed by a proof of q, and nothing else. The reader should observe here how we met in Valpola and Van Dantzig an opposite remark concerning implication.

We see once more that the theory of negation is closely linked with the theory of other propositional connectives. This fourth version of negationless logic with its strong separation of the static definitive from the dynamic provisional rejects both disjunction and implication, while Valpola preserves at least general implication and Gillmore both disjunction and implication. For us, who try to bring these attempts together, we see our only hope in doing so, in taking seriously the acceptance idea of Van Dantzig, the ontological existence idea of Gillmore, the truth idea of Valpola and the definitive assertion idea of Vredenduin. The different decisions made as *to the destiny of our propositional connectives might all be true within the scope of a given quasi modal operator (of acceptance, truth, existence and definitive acceptance).*

We cannot construct a definitive synthetic system here. We only investigate the various proposals to discover the general lay-out of such a system.

When we procede to the detail of Vredenduin's investigations we shall only mention those parts that are relevant for our purposes. A2.0 and A2.1 give his rules for conjunction.

The A.3 axioms are 1) A3.0: usual modus ponens 2) A3.2 the distinctive: p/(Ex)p 3) A3.3: the equally distinctive $p \rightarrow_{vs} q/(E_{vs}p)$, $(E_{vs})q$ 4) A3.4: if p, r/q then $E_{vs}p, r/p \rightarrow_{vs} q$. No variable of vs must be free in r. The premiss r may be dropped (the vertical stroke) means: "— derivable from —".

We see that this work that precedes Gillmore's, uses in fact the existential quantifier for very much the same reasons as he. Except that we may here, in view of the classic interpretation, envisage translation like A3.2 into: whenever a sentence occurs as a conclusion, id est: a class is described as containing another class, then the first class is not empty. A3.3 will mean: whenever a class is included in another class both are not empty.

There is a problem with reference to A4.3: by A3.2, from p follows (Ex)p, so in fact the $E_{vs}p$ condition in the second member seems always satisfied. Yet Vredenduin wants to keep this vacuously satisfied premise, to avoid the following conclusion: p,r/p. Without the questioned premise, r would imply that p implies p (and so the existence of p, by A3.3).

We cannot see how Vredenduin can avoid with his notation the fact that from an arbitrary premise r, the existence of an arbitrary premise p follows. He could have written: If p is a theorem, then p exists (the class satisfying p is not empty). But if he writes A3.2 then any well formed sentence must correspond to a non empty class and if he writes p,r/p, then he declares p and r to be well formed sentences.

It is astonishing to see how far the realisation of the intentions of Vredenduin (with implication and, as we shall now see disjunction present), lies from the declared intentions. The 4 axioms are axioms on disjunctions: A4.0 p,Eq/p or q A4.1 is the commutativity and A4.3 the introduction of the disjunction when an identical consequence follows from its two components. No existential conditions are added here. Yet by virtue of the all powerfull (perhaps too powerfull) p/(Ex)p they are tactitly present. It is the more astonishing that in some other axioms the existential premise is explicitly present but by no means in an evident way: A4.0 yields p, Eq/p or q (why should not equally Ep be demanded and why should not A3.2 guarantee without any supplementary trouble both ?) A4.4 is the distributivity law, with the existential conditions for the introduced conjunctions added.

A4.4. $[(p \vee q) \wedge r]$, $Ep \wedge r$, $Eq \wedge r/(p \wedge r) \vee (q \wedge r)$. Here in A4.4, the A3.2 axiom could not have supplanted the existential axioms. We are however of the opinion that adding specific existential premises is a course in principle different from A3.2. A3.2. is more radical, the specific axioms are more prudent. We have kept untill now the axiom A4.2: $E(p \vee q)/Ep$. A4. If commutativity holds without restrictions then we have also Eq deriving from the same rule. Adding those two results together we came necessarily to the fact that $E(p \vee q)$ implies $Ep \wedge Eq$. As the converse is certainly true, *conjunction and disjunction both between exemplified propositions, become equivalent.* This is a highly undesirable result for somebody who wants to introduce disjunction: it seems to us to eliminate its usefullness. For this reason we would propose the weaker $E(p \vee q)/Ep \vee Eq$ but this is perhaps too liberal for one who never wants to use a proposition, that is not a propositional function and no propositional function that is not satisfied. In the class interpretation Vredenduin would certainly demand that if the union of two classes is not empty, neither of these two classes is empty. The difficulty derives from the fact that we should be able to distinguish between: being universal, and having at least one element (for a given class). Here we never did. We would like to formulate the proposal radically to rewrite Vredenduin's system with all propositions replaced by propositional functions and with the existential quantifier (both the fictive one and the strong ontological one) clearly distinct from the universal quantifier (in a calculus similar to protothetic). It is certainly not true in a positive system that the union of two classes is identical to their intersection (indeed the union might exist while the intersection, being empty, does not even exist). Vredenduin's system shows clearly a discrimination between the existential operator used untill the present moment and usual existential quantifiers: he writes the first E and the second (Ex). For us who make equally this differentiation, our discussion about the E! operator of Leonard is a necessary prerequisite for the distinction that shows in Vredenduin's notation but is nowhere noticed in his axiom or rule system. A5.0 is the familiar instantiation axiom $(x)p/p$. A5.1 is an insertion axiom. If $p, q/r$, then $p, (x)q/(x)r$, in as far as x is not free in — p (p may be dropped). We must make the remark that Gillmore realises better than does Vredenduin that even when universal quantifiers are used it is needed to let them be preceded by existential (for us: strong existential) operators. The substitution rules and the rules for the use of implication have no relationship with the positivity of the calculus. But we return to the essence of our topic with the discussion of the properties of identity and difference (par 9, pp. 223-225). We shall use the occasion to relate the properties mentioned to some more general remarks about identification and differentiation (considered as actions). When no empty predicates are tolerated, and when the distinctness relation is introduced we need at least two individuals. Vredenduin expresses himself as if untill par 9 no field of individuals was introduced. This can obviously not be the case because we need a domain over which the variables range. So at least one material axiom was presupposed (Ex) $(x = x)$. In order to make many of the earlier axioms trivial (by making the existential conditions collapse) we have at least what Vredenduin does not wish to introduce, namely the material axiom (Ex) (Ey) $(x \neq y)$, valid from the beginning.

However following Griss, Vredenduin proposes:

Ag.0: $x = y, p(x)/py$ (p has not y as variable)
Ag.1: $(x \neq z) \rightarrow_z (y \neq z/y = x)$
Ag.2: $(x \# y)/(z \# x)$
Ag.3: $(x \neq y)/(y \# x)$

We are going to discuss the content of these axioms and of some of their consequences taken together. For this reason, we now mention some of these consequences.

1. If $\#$ is not an empty predicate (Ax) (Ey) $(x \neq y)$
2. By means of Ag.1, $E\#/(x)\ (x = x)$ (indeed if (Ey) $(x \# y)$ holds, then $(x \# y) \rightarrow_y (x \# y)$ follows and by Ag.1: $x = x$. So; if $\#$ is not empty, $=$ is not empty neither.
3. Transitivity, symmetry and reflexivity of $=$
4. Combinations of properties of $=$ and of $\neq$: a) $x \# z$ $(x = y)/(x \# z)$ b) $(x = y)/\ (x \neq z) \rightarrow_z y \# z$.

Comparing these axioms to those of Valpola that are stronger (except in the case of A9.1), to those of Gillmore that are weaker and to those of Van Dantzig we feel again the need, expressed already before, to evaluate these syntactic characterisations of difference by means of a semantical model. The first model we used was a mereological one (and we could not pursue its examination very far because of the separateness axioms that would lead us into yet another unexplored domain of logic). Now we can think about an automaton model of identity and difference: two objects a and b are identified by the automaton if it produces always the same outputs when the objects are presented as inputs, and they are differentiated if there exists at least one output provoked by one of the two objects and different from any output provoked by the other. It is obvious that we define here the difference relation among all objects by means of the difference relation on a particular domain of entities, namely automata reactions towards presented objects. This is thus no elimination and no complete model of the concept we study, but simply a reduction of the field on which we have to presuppose it.

Let us now in this light look at the axioms: A9.0: if x and y are identified by the automaton all reactions towards the one are identical to reactions towards the other. If a predicate is characterised by a reaction, then axiom A9.0 immediately follows. A9.1 on the other hand states that for all objects, if x and y are differentiated from them in the same way, by the same reactions, then x must be identical to y. This however is not true in the automaton model: indeed x and z may be differentiated from the same objects each time by other reactions for x and for y and the solidarity in differentiation might thus be the cause of a strong no-identity. A9.3 on the other hand is certainly true: the relation is symmetrical in this model. There is no doubt in the model about A9.2 either. And it is very natural that at least one theorem depending upon the axiom that becomes false in this intuitive model, is also unacceptable in intuitionism; namely: if differentiation occurs, identification must also occur. In the model we can easily conceive of an automaton reacting otherwise upon each input and identifying no input to another one.

It thus seems that this model can yield some fruitfull results. However it speaks about processes of identification or differentiation rather than about relations of identity and difference, and paradoxically either the automaton itself or at least the observer must be able to differentiate the objects that are identified if the process of identification does not lose all meaning. The main problem becomes here: if the automaton is also able to receive information about its own functioning how would it differentiate its own identification from its own differentiation (this question was suggested to us by E. Vermeersch). A differentiating automaton for which we introduce a time order and that had been used to identify two stimuli or inputs that it will discriminate after a certain moment will increase necessarily the number of his different outputs, and an identifying automaton will decrease the number of its different outputs. If thus the degree of complexity is growing (in the supposition that we can measure degree of complexity) and if the automaton is able to be informed about this fact, it can come to the conclusion that it is differentiating; if the degree of complexity is decreasing it can come to the conclusion that it is identifying. In this dynamic process, paradoxically enough we see that completely identifying or a completely differentiating automaton would necessary destroy themselves either by collapse or by explosion, so that a pragmatic corespondent of the unacceptable theorem we have commented upon, must be introduced for not self destructive automata: if there is identification, there must be differentiation (and conversely). This theorem cannot however be expressed in Vredenduin's language as it introduces time.

Finally the model is really defining difference in general by means of differences between degrees of complexity. Is then the concept of difference so deeply primitive that we always have to presuppose it in some hidden way when we try build a model for it and if we build such a model is it always only pointing out relationships only and never giving a real reduction of the concept under study ? We think this to be true for the two fundamental relations of identity and difference.

Finally, after introducing identity and difference, the so questionable disjunction sign appears (much to our surprise if we look at the introductory remarks of Vredenduin).

A10 tells us: $p(x) \vee q(x)\ (p(y) \rightarrow y \neq x)/q(y)$ and the class logical interpretation of this becomes: if x belongs to the sum-class of p and q, but is different from all members of p, then by means of an expression that yields the equivalent (by now well known to the reader of this article) of negation: If x is the only free variable of p(x) then $\sim p(x) =_{\mathrm{Def}} p(y) \rightarrow_{y} y \# x$. If x and y are the only free variables of p(x,y), then $\sim p(x,y) =_{\mathrm{Def}} p(u,v) \rightarrow_{uv} u \# x \vee u \# y$ asf. If is interesting to note that Vredenduin denies the negativity of this pseudo negation as follows: it is not a negation in the proper sense, as it has nothing to do with refutation or contradiction (p. 226).

Negations for functions of an arbitrary number of places can be defined, and if two places become identified, the function remains still of the same order and the negation definition remains identical. It is always a delicate question when identity and difference are both positively defined, to see how they relate to each other. It can be proved that "x identical y" is the negation of "x different y" and conversely "x different y" is the negation of "x identical y" but only if we accept the intuitionistically unacceptable axiom that is equally false in our automaton model. $\sim(x \# x)$ is rejected because $x \neq x$ is empty, and thus unacceptable. This is only a special case of the two theorems $Ep/E{\sim}p$ and $E{\sim}p/Ep$ (in the class logical model: all classes *and* their complements must be non empty). This excludes *both* the null class and the universal class. These extremely strong theorems make often the existential presuppositions that are added in expressing the usual relations between negation (as here defined), and conjunction implication asf superfluctions. We can only state that more or less E-premises should be present more, systematically covering every part and combination of parts, making these theorems independent from the Ep/Ep condition (or from the p/Ep, we earlier encountered), or less, putting all the existential strength in these strong propositions. In part III of Vredenduin's work, quite different from who remains within the framework of first order functional logic, or from Valpola for whom no types exist, Vredenduin applies his ideas to higher order functional logic. For each type level an existential operator of that level will be introduced and also a negation of that level (this is a consequence of the feninition of by means of quantifiers, having ranges of given levels as their domain).

The formal properties of the system remain the same on different levels, with suitable adaptations. For instance Ag.1 becomes $p \#_1 f \rightarrow_f q \#_1 f/q =_1 p$.

For non mentioned reasons on this level the equivalent of Ag.1 seems completely unacceptable to Vredenduin and he proposes its replacement by two of its consequences:

q.1 $E_1 \#_1/(f)f =_1 f$
q.3 $p =_1 q/q =_1 p$
A definition of identity and difference can now be given:
D13: 0: (p. 243) $p_1 = {}_1q_1) = {}_{Def}p_1 \rightarrow {}_{vs}q_1 \wedge q_1 \rightarrow {}_{vs}p_1$
$$(p_1 \#_1 q_1) = {}_{Def}E(p_1 \wedge q_1) \vee_1 E(\sim p_1 \wedge q_1)$$

This definition will make symmetry of $\neq_1$ derivable and also axiom Ag.2. But this reduction of higher order difference to higher order negation should not make us forget the original aim (eliminating negation).

Some interesting properties are
1) $E\,/E_{1\ 1}$ 2) $E\,/(Ef)\ (x)f(x)$ but also 3) $E\,/(Ef)\ {}_1(x)f(x)$
4) $E/(Ef)\ (Eq)E_{vs}f\ g$, 5) $E/(Ef)f(x)$

It is not our purpose here to put down all the properties of higher order difference or negation. Our task is a) to ask in how far the way it is introduced is compatible with Vredenduins own idea of constructing a system of definitively accepted propositions and in how far it is compatible with the expression of a purely positive ontology; b) if we can throw some light on it from the point of view of one of our models of difference or identity. From the point of view of ontology, we have to decide between an existence that is one type of existence for all entities (Plato's solution) or an existence that has different types for different types of entities. Vredenduin has made his choice a) he chooses the Aristotelian multiplicity of existences (this would be even more clear if the reader would think again for a moment about E as about a strong existential operator) b) but he tries to make their properties, though not identical, yet as similar as possible. We should not forget that we should bear in mind the possibility of the other solution: a type free existential operator. Perhaps (this is only a weak hypothesis) it is more in agreement with a positive ontology to have a unique existence operator. From the point of view of the theory of definitive acceptance, we ask ourselves if, exactly for the same reasons that made us reject in this theory negation (because stronger affirmative assertions are available), we should not dispense with properties of properties because all that can be said about such higher properties should be derivable from first order ones and their internal relations. In our automaton model for identity higher order identifications or differentiations can only be expressed by means of feedbacks: an object having a property is here represented by an input exciting an output. If this output is fed back into the automaton, and if it has again a property, this means that it excites also a specific output. The identification of second order will be the production of the same output for two different inputs that are themselves outputs of the same automaton. In this semantical model of identity (and difference: the reader can complete for himself) higher order entities are simply characterised by their temporal history but are all in fact first order entities. This is a strengthening of our former remarks.

In function of all this we are of the opinion that the introduction of higher orders into negationless logic, should be followed by looking for procedures of reduction to a typeless system. Intuitively anyway, in the classlogical model that Vredenduin borrows from Griss properties of properties are simply classes of classes.

We end our analysis of this work by a comparison between Griss own class calculus, adjoined to a propositional calculus without negations and with existential restrictions, and Vredenduin's system as we have worked it through.

Griss' system itself contains in stead of the existential presuppositions, assertions about the non emptiness of classes (written as: $\alpha \chi \beta$: a and b have a non-empty intersection). We give the complete system even though only part 3 is characteristic

1. Axioms concerning propositions

$p \rightarrow (p \wedge p) \quad (p \rightarrow q) \rightarrow ((p \wedge r) \rightarrow q \wedge r))$
$(p \wedge q) \rightarrow (q \wedge p) \quad (p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$
$(p \wedge q) \rightarrow p$

2. Rules of Substitution

From the proved formulas P and Q follows $P \wedge Q$
From P and $P \rightarrow Q$ follows Q
From R and $P \rightarrow Q$ follows $P \rightarrow (Q \wedge R)$

3. Axioms concerning classes

a. intersection
$\alpha \subset (\alpha \cap \alpha) \quad a \chi b \rightarrow a \wedge b \subset b \wedge a$
$(a \chi c) \wedge (b \chi c) \wedge (a \subset b) \rightarrow [(a \cap c) \subset (b \cap c)]$
$((a \subset b) \wedge (b \subset c)) \rightarrow (a \subset c)$
$a \chi \beta \rightarrow (a \cap b) \subset a \quad a \subset a$
b. union
$a \subset a \cup b \quad a \cup b \subset b \cup a \quad [(a \subset c) \wedge (b \subset c)] \rightarrow [(a \cup b) \subset c]$
$[(a \chi c) \wedge (b \chi c)] \rightarrow [(a \cup b) \cap c] \subset [(a \cap c) \cup (b \cap c)]$
c. axioms for χ and $\neq$
$a \chi a \quad (a \chi b) \rightarrow (b \chi a) \quad [(a \chi b) \wedge (a \subset c)] \rightarrow (b \chi c)$
$[(a \neq u) \wedge (b \subset a)] \rightarrow b \neq a$
d. axioms for the complement
$(a \chi b) \wedge (a \chi c) \wedge (c \neq u) \wedge [(a \cap b) \subset \neg c) \rightarrow (a \cap c \subset \neg b)$
$(a \neq u \rightarrow [[(a \cup b) \cap (\neg a \cup b)] \subset b]$

e. axioms for $\neq$

$x = x$; $(x = y) \rightarrow (y = x)$; $(x = y \wedge y = z) \rightarrow (x = z)$; $(x \neq y) \rightarrow (y \neq x)$;

$[(x = y) \wedge (y \neq z)] \rightarrow x \neq z$

f. axioms for $\#$

1) all axioms in e+

2) $x \# y \rightarrow (z)\ (z \# x) \vee$
$(z \# y)$

3) $(z)\ [(z \# y) \rightarrow (x \# z)] \rightarrow (x = y)$

(intuitionistic negation uses e and fl-2; classical the whole of e and f.

We must stress how Gillmore, Vredenduin, Valpola constitute progress: here in Griss' attempt there are no existential or thruth conditions or regularity conditions on rules of derivation or on propositions. But on the other hand Griss brings to the mind clearly and directly a class-logical model that is derived directly from the idea of a purely positive ontology.

Vredenduin has shown that most of Griss' axioms are true in his system. But there is no complete equivalence of the 2 theories (for the reason we just mentioned).

Our conclusion at the end of this analysis is again that Vredenduin's claim is complementary to and neither identical to nor contradictory with the claims of the other studied systems. We met here in a very instructive way the difficulty of selecting the exact existential premises, and we were aware all the time of the dominance of a class logical model, only explicit in Griss but guiding Vredenduin's more general construction. We had also the occasion to examine briefly another semantical model for identity and difference (the automaton model).

5. *David Nelson's logic of strong positive negation is a logic of epistemic negation.*

It is well known that S. C. Kleene has given an interpretation of intuitionistic logic by means of recursive functions (functions the values of which are computable in a finite time, wherever they are defined). An automaton is a recursive function. The model we met before for difference could be generalised for the whole of logic (and not only limited to identity or difference). This interpretation of propositions by means of recursive functions, can lead us to an independent decision making process for an assertion and a denial, for truth and falsity. By these means a positive definition of negation will be introduced in a new way: it will be defined by an algorithm. In the field of positive logics, based upon a positive ontology, we meet here a not yet studied feature: the way by which a positively defined knowing subject decides about the existence of positively defined features of reality.

We briefly compare three studies by Nelson: "Constructible Falsity", "Negations and separation of concepts in constructive systems" and "Non null implication".

In a certain sense the work we are now going to study cannot be called a negationless logic; negation is present (but then it is also present, as a defined symbol in all other so called negationless systems we met, except in the unproblematic positive logic of Hilbert-Bernays) but instead of defining it by means of difference, one defines it by means of falsity, taking guard however to give a positive interpretation of falsity by means of a recursive function. *If a positive ontology implies a positive theory of knowledge and action, then this attempt must be added to the earlier ones.* Our own modest contribution will be to indicate where it has to be inserted.

That, exactly as we indicated, Nelson's inspiration derives from epistemological presuppositions becomes very clear when we look at the introduction of his "Negation" article in the work "Constructivity in Mathematics". A statement ascribes a property to an object. This is empirically verified when we observe that property. It can however also happen that we do not observe that property, and then the situation is ambiguous as to its significance: either the property did not exist, or the observation was badly executed, or the observation was not even executed at all. It appears to Nelson that every observation of the absence of a property is in fact the observation of the presence of another property, taken as a token of this absence. If this epistemological view on negation is taken, certain formal consequences follow:

1) "p or not p", the law of the excluded third, cannot be valid: it is not true in general, that observations can be found either verifying p or verifying not p. Often neither for or against p evidence is forthcoming.

2) but (and here Nelson deviates strongly from classical intuitionism, where the link between assertion and internal constructive experience has indeed provoked only the rejection of the excluded third) it can also be the case that every possible relevant observation both yields evidence in the favor of p and in favor of not p (on p. 220, the concept of evidence is related to the concept of interpretation). In that case the contradiction "p and not p" is valid and so in general (not (p and not p) the law of non contradiction is not valid). This paradoxical consequence derives immediately from the fact that the verification of not p is the observation of a positive property.

The fact that a contradiction might be provable does not make the system trivial. As Jaskowski has shown, it is possible to construct a system that has not all well formed formula's as theorems, and in which however contradictions can be demonstrated. The remark that precedes shows that both the verification and the falsification of a proposition are defined by means of independent definitions (only then can both p and not p be confirmed to an equal extent). This yield us another attempt towards a positive definition of negation.

Nelson expresses this intuition by introducing an arithmetisation of the evidence in favor of p and also of the evidence in favor of not p. To express the independence of these two concepts he defines P-realisation (positive realisation) and N-realisation (negative realisation) by means of positive and recursive definitions.

Because he is primarily interested in the formal sciences, the fact that a given experiment or observation is performed, is expressed by the assertion that a given interpretation is adopted for the propositions to be realised. If i is an interpretation, then if a formula A is realised it is either P_i-realised, or N_i realised. Either P or N realisation is designed by R and anything that realises a formula is a natural number that is the arithmetical representation of a set of sentences. If for all interpretations, (i.e.: observations) a sentence is positively realised we omit the index and we write aPA, and if for all of them it is negatively realised, we say aNA. If we want to refer to a given interpretation, we write aN_iA or aP_iA. The negation defined by the negative realisation rules is called strong negation and Nelson rightly claims that we do not need any reference to absurdity or other concepts of that kind so that Griss' and Freudenthal's objections against negation must disappear. Our ontological objections do not disappear however and the reader is thus entitled to ask why we mention this system in the present context. We are going to give the answer immediately adding a remark, similar to some of the remarks we had to add to the earlier proposals (a theory not mentioned in the definition of strong negation is presupposed by this very definition)

a) falsity and truth, P realisation and N realisation are independently defined and both are positively defined. This system thus shows that "x knows that p" and "x knows that not p" can be independently and without use of negation be circumscribed.

b) inductive logic, namely the theory of confirmation is really presupposed by Nelson's motivation. It is indeed certainly not possible that p can both be completely verified and completely falsified by experiment and observation. It is however possible that both are equally confirmed (confirming a proposition is less than verifying it: it implies only that a certain probability increase is effected). Either belief logic or epistemic logic have to be presupposed by Nelson's ideas (x may be entitled or even obliged by the given evidence to believe both p and not p, as he may be entitled or even obliged by the same evidence to believe neither p nor not p. Or, more accurately the inductive logic of confirmation must be presupposed and even more, a theory of confirmation having non-usual properties: p and not p may be both arbitrarily highly confirmed (the rule c(non h,e) = 1 = c(h,e) may not exist).

We are now going to look into the definition of P-realisation and N-realisation, and the consequences of these definitions for a) the presupposed confirmation function or b) the presupposed epistemic logic. As Nelson defines his concept of positive and negative realisation in his first article "Constructible Falsity" we are going to use the information found there and come back to the epistemological foundation after seeing how the concept is exactly defined: We quote Nelson, 1, p. 17. If A is a number-theorethic formula and a is a natural number then a is said to P-realise or to N-realise A according to the following clauses.

1. If A contains the free variables $x_1 \ldots x_k$, then aP-realises A or aN-realises A when this is the case for the closure $A_{x1} \ldots A_{x2}$ A.

2. For elementary formulae,
1. a PA when a is o and E is true
2. aNA when a is any natural number and E is false

3. For conjunctions $B \wedge C$
1. $aP(B \wedge C)$ when $a = 2^b 3^c$ and b P B, while c P C (here a is a function of 2 numbers, one of which realises one term each of the conjunction).
2. $aN(B \wedge C)$ when $a = 2^0\ 3^b$ and bNB or $a = 2^1\ 3^c$ and cNC (here we have 2 cases, and in each case a is only function of one of the 2 numbers, the other being a constant).

4. For disjunctions $B \vee C$, we may expect the dual situation:
1. $aP(B \vee C)$, when $a = 2^0\ 3^b$ and bPB or $a = \alpha^1\ 3^c$ and cPC
2. $aN(B \vee C)$ if $a = 2^b 3^c$ and b N B while c N C

5. For implication $B \supset C$,
1. a is the Gödel -no of a partial recursive function p such that for every bPB, p(b) P C
2. $aN(B \supset C)$ if $a = 2^b 3^c$ and bPB while cNC

6. For negation $\neg A$, 1. $aP\neg a$ when aNa and 2. $aN\neg A$, when aPA.

7. For the existential quantifier (Ex)(Ax),
1. aP(Ex) (Ax), $a = 2^n 3^b$ and b P A(n)

2. aN(Ex) (Ax) if a is the Gödel-no of a general recursive function p such that, for every n, p(n)NA(n)

   8. For the universal quantifier (Ax) (Ax);

1. aP(Ax) (Ax) when a is the Gödel no of a general recursive function p such that for every n, p(n)P(An)
2. aN(Ax) (Ax) when a $= 2^n 3^b$ and bNA(n).

Let us comment upon this very important definition:

1. It uses in an essential way the concept of truth; we might remember that we proposed, analysing Valpola to analyse truth and falsity as independent modalities. The axioms for T and F we proposed there are in part the same as those present here (see for example conjunction and disjunction).

2. The fact that for an atomic formula positive realisation and negative realisation might coincide (for negative realisation a has to be an arbitrary number, for positive realisation it must be a specific number: it is obvious that both might be the case, o being a member of the set of numbers) shows the falsity of the law of non contradiction. The reader is asked to consult our earlier definition of truth and falsity (in the paragraph about Valpola) and he shall find that the inspiration of the truth definition is again similar to the one used for atomic sentences here: the fact that a = o is unimportant, even the fact that a is a number. The only important feature is that true atomic sentences are mapped upon definite constants while false atomic sentences are mapped upon a universal set (or rather upon any element of such a set).

3. For implication, and the two quantifiers recursive functions are introduced: it thus must be possible from a realisation of the antecedent to find in a finite number of steps the realisation of the consequent and to prove for positive realisation of the universal quantifier in a finite number of steps the truth of the predicate and for negative realisation a definite counterexample, while for positive realisation of the existential quantifier we have to find an example of the predicate and for negative realisation an algorithm yielding its refutation

for any constant whatever. The constructive feature of this definition is what was absent in our earlier definitions: in other respects the inspiration is the same.

Going back now to the epistemological motivation of this system we have to replace "positive realisation" by "increase of the degree of confirmation" and negative realisation by "decrease in the degree of confirmation". The places where these terms are not present may remain what they are. We claim that Nelson's explanations in his "Constructivity" article prove that in fact he wants to give a definition of truth and falsity as limits of non classical degrees of confirmation. And we can preserve the definitions, making the replacements proposed here and now.

By means of the concept of negative realisation, the concept of strong negation can be defined. As H. Rasiowa remarks (Constructivity in Mathematics, "Algebraische charakterisierung der Intuitionistischen Logik mit Starker Negation" pp. 234-240) the "strong negation" in Markov and Vorobev are identical to the one defined by Nelson. Vorobev has axiomatised this strong negation and it appears that it has all the properties of classical negation, but can be added, on the basis of the Nelson interpretation, to a calculus with intuitionistic negation without identifying it with classical propositional logic.

This calculus admits also empty predicates because they are defined by means, not of the intuitionistic absurdity, but by means of strong negation. We have ontological and not only constructivistic reasons to avoid empty predicates. For this motive, we deem it not less necessary to add the calculus of positive and negative realisation to a positive calculus (any of the ones we earlier studied) than we think it necessary for epistemological reasons to add the N-P calculus to one of the positive logics (Griss, Gillmore, Vredenduin, Van Dantzig or Valpola). As the topic about which the calculus of strong negation speaks is again essentially different from the earlier ones there cannot be any incompatibility.

P 236 and 237, op. cit., H. Rasiowa gives a good characterisation of the differences between the intuitionistic negation and strong negation by means of a topological model: a) intuitionistic negation mapped upon a topological space corresponds to the interior of (X — x) where X is the name of the space, x is the set we take the quasi complement of, — is the intersection of X with the classical complement of x, and the interior is the set of points of that set that are in the set and not on the frontier of the set. b) strong negation however is the difference between X and g(x) where g is any involution function (a one-one mapping function, with the property g(g(x)) = x). The reader can consult the same article pp. 234-235 for suitable axioms for strong negation. Our intention here was only to point out the presuppositions of this calculus and its general significance.

We shall only mention Nelson's most recent paper on the topic "Non Null implication" in which he combines himself his realisability definitions with attempts to reach negationless logics. He considers them however only from the purely formal point of view, while we have had occasion to mention necessary amendments for each of them if a fruitfull synthesis is to be possible.

We end here our critical review of negationless logics, and will close this section of our paper by a critical review of the definitions of negation that are present in the litterature, in so far as they aim to an elimination of that concept.

First proposal

Let us consider p as a sentence, and "truth" as a possible predicate for sentences. Let us consider equally "false" as a predicate of sentences.

The proposal is then made to define negation by means of falsity. The negation -p of a sentence is either the disjunction of all sentences that are false when p is true, or any sentence that is false when the first sentence p is true.

This first proposal is unacceptable for the following reason:

a) the disjunction of all sentences false, when a given sentence is true, may be a infinite sentence and we reject the possibility to write down or to understand infinite sentences.

b) if to the contrary the negation of p is any sentence that is false when p is true then for one given sentence there exists in general an infinity of non equivalent negations. We did not impose the requirement that for any sentence only one negation exists. Yet the existence of an infinity of in principle independent negations could not be accepted.

c) a third argument against this definition we have to reject: it has been claimed that falsity cannot be defined except by means of negation itself. This is a false assertion and we shall show it by means of an example.

Let us consider a very simple formal system: the sequence of natural positive numbers and as sentences equalities among these numbers. The relation S: "successor of" and the operation "relative product" are included in the language, with suitable axioms.

1) We give a truth definition for this simplified language as follows: "$n = m$" if and only if "$n = S^r o$" and "$m = S^s o$", and $r = s$ (this last identity is not of the same type as the first type because it expresses simply that $(S/S)^n$ is equivalent to $(S/S)^r I$).

$n = m$ is false if and only if either n is an x-th successor of m or m is an y-th successor of n. It is then clearly possible at least for this language to define falsity without using negation.

It would be an important problem to solve, to discover for what types of formal systems the falsity definition can be given completely independently from the truth definition. We can not solve this important problem in general but we can at least state that it cannot be done for a calculus of unanalysed propositions. The possibility must however be frequent because of the fact that formal systems can be arithmetised and because of the fact that our arithmetical example shows at least by an illustration the plausibility of giving independent truth and falsity definitions.

It could be the case that for two natural numbers neither the sentence "$n — m = o$" nor the sentence "$n — m = e$" where "e is larger than o" can be proven. In this case the principle of excluded third would disappear. It is moreover possible that for given types of sentences, the positive definition of falsity cannot be constructed. In this case there are sentences without negation.

Moreover $Vp \rightarrow - Fp$, where V and F are both positive qualities does not in general imply $Fp \rightarrow - Vp$, (except if we abandon our present strategy and define V by means or — F, or F by — V). If we do not add additional postulates for V and F; both are positive predicates.

It is interesting to note that Immanuel Kant seems in agreement with our two first requirements that reject this definition. He asserts in the "Kritik der Reinen Vernunft" (1787, pp. 97-98) that the sentence "man is not mortal" has no definite meaning, because the only claim made is that the class of men is included in an unorganized infinite class. This is not, according to Kant a definite statement because it leaves open an infinite possibilities for the situation of the class of men.

Herman Weyl, far from being only an intuitionist states, as quoted by Valpola clearly: the negation of the statement "all numbers are even" would be as infinite and thus unusuable disjunction 1 is an odd number or 2 is an odd number or 3 is such asf. Weyl's example is however essentially linked with the infinity of the set of natural numbers. Our own reasons to reject the definition of negation by means of falsity, defined in a positive way, do not imply the existence of infinite sets.

c. Second proposal

A second proposal to introduce on the basis of a positive definition, a negative operator is the following one: "a is not red" means "a is blue or green or..." It is again unacceptable for similar reasons:

a) in general we cannot construct either a finite or an infinite set of alternatives that are true when the first is false (it is obvious that here the attention is more drawn upon the predicate than it was in the earlier case)

b) moreover it will in general not be provable that it is logically necessary that the subject in question must have one of the alternative predicates, and neither can it be proven in general that it is impossible for the alternatives to be simultaneously true (to take a very famous example: we cannot prove that every object must have one of a finite list of colours from logical considerations alone, and moreover we cannot prove that it is a logical truth (even though it is factually unthinkable) that the same object can at the same moment not have two different colours).

If we accept this proposal a) many sentences will not possess negations and b) the principles of non contradiction and of the excluded third will become synthetic probable propositions.

d. Third proposal

As a third proposal we can use the relations of identity and of difference. Negation can here be introduced in the following way:

$$— Pa = (x)\ (Px \rightarrow (x \neq a)) \text{ or } (Q)\ (Qa \rightarrow (Q \neq P)).$$

Either all elements posessing the predicate P are different from a or all predicates posessed by the element a are different from the predicate P.

The adequacy or inadequacy of this proposal depends completely upon the meaning of the relations "identical" and "different". The proposal reduces to nothing if we cannot give a positive definition for the relation of "difference" (if different "from" is simply "not identical with", we are turning in circles and we do not really give an interpretation of negation).

Unhappily enough neither the concept of identity nor the concept of difference are without serious problems. Certainly we can easily propose a few evident axioms for them:

Identity:

1. $(Ax)\ (x = x)$
2. $(Ax)\ (Ay)\ [(x = y) \leftrightarrows (y = x)]$
   $(Ax)\ (Ay)\ (Az)\ [\{(x = y) \wedge (y = z)\} \rightarrow (x = z)]$
3. $(Af)\ (x = y) \rightarrow [f(x) \leftrightarrows f(y)]$

Difference:

4. $— [(Ex)\ (x \neq x)]$ or $[(Az)\ (z = x) \rightarrow — (z \neq x)]$
5. $(Ax)\ (Ay)\ [(x \neq y) \leftrightarrows (y \neq x)]$
6. $(Ef)\ [(x \neq y) \rightarrow (f(x) \wedge — f(y))]$

This last axiom again presupposes negation.

It is not suficient to have the three first types of axioms as definitions of identity and difference too many, (as the reader can easily verify) relations satisfy these first three. To avoid the weakness the difference axioms must be strengthened. But identity itself offers already sufficient difficulties. The set of all functions or predicates is not a constructible set. The concept of identity implies either (for absolute identity) a quantifier ranging over all functions or properties (a non defined and in most formal systems antinomic set), or if one wants to avoid this danger, no absolute identity but only partial resemblance can be defined (and it is our conviction that in non antinomical systems we can reach no more than that). The relativation of identity implies the relativisation of difference.

If we want to overcome the difficulties of difference (relative difference at most, in function of our former remarks upon identity) then we must have recourse to a recursive procedure: all atomic sentences or objects are different if they have not the same form; all atoms are different from all non atoms: if two elements are different then added by concatenation or applied as operators to identical objects the results remain different.

The reader can see how in this way we can at least reach a satisfactory definition of a relative difference. If this is the case however the classical properties of negation are lost: a is partially different from itself even and the definition does not guarantee the validity of the principle of non contradiction.

Valpola's treatment of difference on p. 110 cannot be accepted 1.a is identical to be or a is different from b cannot be defended if we demand that either we show that all properties (or even very large sets of properties) of a are equally properties of b) or if, to the contrary, in this enormous set it is demanded that we find a property of a that is not a property of b (or that for a and b we can give a construction procedure that allows us out of the constructive axioms for difference, to show the difference). Both for the absolute or for the recursive definition of difference this property will be lost. The second axiom is a form of the "ex falso sequitur quodlibert" that is certainly false for any acceptable type of implication $((a \neq a) \rightarrow Pb)$. The third property must be rejected also.

The sentence

7. $(x)\ [((a \neq x) \rightarrow (b \neq x)) \rightarrow a = b)]$

is not a logical truth. (If $x = a$, or $x = b$, (the only crucial case), then only we get a logical truth). If it is weakened as not to be applied to a and b, then there is certainly no difficulty. If it keeps its strength then the antecedent of the implication can be true and the consequent false and become a case of ["a diff b" implies " diff a"], then $a = b$, this simply asserts that the identity of a and b is a logical truth. We cannot accept an axiom that turns all identities into logical necessities.

To come to a conclusion, concerning this definition of negation by means of the difference relation, proposal already made by Plato, in his Parmenides; we must draw upon a remark made by Wittgenstein. He refuses to use the identity relation in his language of the Tractatus (and doing this he must also refuse to use the difference relation). Indeed we can give to this identity relation either an ontological interpretation (object language) or quite to the contrary a semantical reinterpretation:

a) If we make the second choice, stating that the object having name n1 is also the object having name n2 then we can, with Hegel (sic) repeat the remark that the same object is looked at from two different points of view, and thus is not absolutely the same object. b) If we give to the identity relation an ontological interpretation, then we must modify our concept of relation in this sense that we introduce a relation about which we could not even imagine what it would mean to be false, valid for all objects, and yet in this respect different from all other logical truths that it claims to be a relation between objects and objects. Valpola's own objection against Plato's definition of the negation sign by means of the difference relation is the following one: in order to get the classical properties of negation (double negation equal to assertion, and the transposition principle) we need (either in 1st or 2nd order functional logic).

1) $(Pa \supset Qb) \supset ((x)\ (Qx \supset b \neq x) \supset (x)\ (Px \supset a \neq x))$
2) $Pa \supset [(x)\ ((y)\ (Py \supset x \neq y) \supset (a \neq x)]$
3) $(x)\ [(y)\ (Py \supset x \neq y) \supset a \neq x)] \supset Pa.$

We do not see any deep objections of a major nature against these three assertions. But, as we said already before, we cannot accept sentence 7. Only by means of the combination of the sentence 7 with the following two sentences:

a) If a is different from b, then for all x identical to b, a is different from x

b) if a is identical to b, then for all x different from b, a is different from x, can it be proved that x identical to y and x different from y are, in the sense of negation accepted in this context, negations of each other.

1) $-(x = y) \leftrightarrows (x)\ [(x = y) \rightarrow (y \neq y)]$
2) $(x = y) \leftrightarrows -[(x)\ (x = y) \rightarrow (y \neq y)]$

e. fourth proposal

The fourth proposal tries to define negation by means of implication. It is a very well known procedure:

— p equivalent to $p \rightarrow F$. The problem however is the definition of F. Either it is a metalogical symbol designating an infinite set of false propositions (and then the expression is meaningless) belonging both to logic and metalogic or it is an infinite disjunction or conjunction of false propositions (and then once more F cannot be constructed); or F is a definite false proposition. We give some examples for F: in protothetics (p)p; in arithmetic $o = 1$; in class logic (x) (y) (x included in y). Only one of these last proposals can be taken into account. But in that case we have an infinity of different negations (indeed we can even construct the conjunctions of some of these false propositions). If we have type theories we have even negations of different types. Naturally this difficulty has not seemed dangerous as long as the presupposition was made that all false propositions reciprocally imply each other. *If we accept however natural entailment we cannot make such a presupposition* and we come to the conclusion that we have an infinity of different negations, some of which imply or are implied by each other, some of which are completely independent from the others.

Valpola is one of the few authors who recognises that even without the presupposition of the equivalence of all falsehoods serious difficulties arise a) if the false sentence implied is a contradiction, it is a moot point to know if a contradiction can be entailed by any statement (*entailed, not implied*) b) if it is a factual statement then the number of sentences the falsehood of which can be defined by it, will be limited to those sentences that are content-relevant to the factual statement in question.

In general: if we do not concentrate absent mindedly on the negation but if we inquire simultaneously about entailment, then it must be the case for a real entailment to exist (in the system of Anderson-Belnap for instance) that there is a common term between antecedent and consequent. How then could we define $-$ p as $p \rightarrow F$?

We do not avoid, it seems to us the infinity of different types of negation. This could be considered a new discovery, but it certainly increases strongly the complexity of the underlying logic.

It could even be the case that the problem of the subjunctive conditional, as unsolved now as it ever was, plays a role it seems that the relation between p and F (whatever the F may be) should be read as follows "if it were the case that p were true then it would be the case that F would be true". This is a subjonctive conditional and so it seems that we do not only a) multiply enormously the number of different types of negations b) but moreover make the solution of the problem of negation dependent upon the solution of the problem of entailment and of the subjonctive conditional.

These arguments are not sufficient to reject completely the last proposal but we must add, starting from our positive view on reality the following conclusion: we must add to our view of reality entities like absurdities, falsities, contradictions and only then can we get our positive definition of negation. Doubtlessly these entities have psychological existence, but they have only psychological existence. A general logic not tightly linked with psychology would become impossible.

The discussion of the four proposals that has just been presented seems to show that in a language the aim of which is to describe the universe as it exists, no means by which negation could be introduced in an undisputable way could be said to exist.

The decision to eliminate negation could be made radically: no negation in the synthetic sciences and no negation in logic or mathematics either or could be made with more moderation: no negation in the synthetic sciences but negation allowed in the field of logic mathematics. No attempts have been made to carry through the moderate elimination. It seems that certain purely logical arguments, independent from the ontological ones presented in our introduction, could be used:

1. The problem of the excluded third disappears if we eliminate negation,
2. A certain number of paradoxes (though not all: Curry's paradox does not use negation) disappear,
3. It seems that the history of science, an accumulation of new information and not a deletion, is better represented by a purely positive logic.

B. A pragmatical justification of negation

1. In order to define what negation is, we should ask ourselves first what negation applies to, what it is operating upon. Our first problem is that we see so many different, though related objects that are denied. We enumerate them in order:

a) a statement can be denied. The statement is the totality of a speech act in a given situation, made by a given person at a certain time. The statements "No" are in this sense indexical statements that they refer to other statements that did occur before.

A statement, being a concrete unitary speech act, can be considered as member of a set of statements (type) or in its individuality (token).

b) a statement has an internal correspondent. We know since Wittgenstein that this internal correspondent can be very complex and can be a function of very many internal states earlier and later. For simplicity, in this first introduction, let us however call this internal state a belief set. Here again we can be interested in the platonistic or the nominalistic version the "type" of given beliefs or their "token" quality (the fact to occur at given times and in given persons).

c) a statements has an external product, that is relatively independent from the statement: the utterance (more specially, for indicative cases: the sentence). In written language, this relative autonomy of the written sentence is more clearly seen than in spoken language (the acoustic waves and visual impressions used in context dependent speech). But in both cases we are entitled to make the distinction. Here again we have sentence types and sentence tokens. Israel Scheffler is on record to have proposed, as a pupil of Quine and Goodman, a nominalistic semantics.

d) finally we can also deny propositions. Propositions seems to be hypothetical constructs that can be defined or utterances. Unhappily enough, this concept is unclear. Here again the type and token distinction can be applied (though we should have a beter analysis of proposition, than the one we have to see if a proposition is not by its very definition a type: a class of related sentences, or beliefs, or statements).

e) we can also deny sequences or set of entities of the four mentioned classes. (f.i.: when I say: "he is not poor", the operator "not" seems to apply to the predicate "poor", it is a moot question to know if "he is not poor" is synonymous to "I deny that he is poor" or "it is false that he is poor" or "it is not the case that he is poor" and so forth). What are the partition operators that have to be applied upon elements of our four levels and in what higher totalities can they be inserted as natural parts ? Presumably predicate calculus could give a suggestion towards the partitioning of propositions. But how does this construction stand with reference to the partitioning of statements, utterances, beliefs ? In logic, after Aristoteles brief attempt to study polysyllogisms, nothing that can be compared to Harris' "Discourse Analysis" is to be found (the study of the effect of operators upon sets of atoms is neglected). Only one article of Bar-Hillel on Bolzano's Wissenschaftslehre makes such an attempt.

2. All the entities mentioned under 1. are connected with the communication act. This communication act however can be studied from three points of view at least: the syntactic, the pragmatic and the semantic point of view. I assume the distinction known (even though it has its problem: see our article on the topic in the Pleiade volume on epistemology). The level closest to concrete reality seems to be the pragmatic level on which we study the relation between language, language users and denoted objects. It seems thus natural (though contrary to historical development) to begin with the study of attempts to define negation on the pragmatic level.

We now have at least 24 problems (8 different types of denied elements, to be studied each on 3 levels). To these 24 problems have to be added the study of negation on sets of entities or on parts of entities. This makes us 3 times the 24 problems: 72 problems.

In all these cases, negation seems to be definable by means of incompatibility; a concept for which we are now seeking a few definitions.

Let us say that p and q are incompatible, (I do not specify what entities they are) if they cannot be simultaneously asserted. Formally:

Incompatible (p,q) $=_{\text{Def}}$

Impossible [Simult (Ass $\times$ p, Ass $\times$ q)].

To expres this I need a) alethic modal logic: impossible b) tense logic: simultaneously c) assertion logic: the assertion operator.

The concept of "impossibility" is not in general a primitive concept of modal logic. It is usually defined by means of negation itself and thus could not be used to define negation. The question arises: is it possible to build some acceptable modal logic with impossible as a primitive term ? If so, this definition of the incompatibility is not circular for our purposes.

We write down a few laws for I (impossibility) to show that we can take it as a primitive:

1. If p is impossible, p is false: $Ip \rightarrow Fp$ (false is here, positively defined as was shown to be possible before in this article).
2. $I(p \vee q) \leftrightarrows [Ip \wedge Iq]$.
3. If p is necessary, it is impossible that it is impossible: $\Box p \rightarrow IIp$ (the converse is not in general true, but we can develop a calculus with $IIp \rightarrow \Box p$ added).
4. $IIp \rightarrow \Diamond p$: if it is impossible that p is impossible, p is possible.
5. $(Ip \vee Iq) \rightarrow I(p \wedge q)$ (the converse here is false: $I(p \wedge q)$ may be true without Ip or Iq being true).
6. $[I(p \rightarrow q)] \rightarrow \Diamond(p \wedge Iq)$: if "p implies q" is impossible, then it is possible that p is true and q impossible (the converse is false).
7. $I((Ex)\ (fx)) \leftrightarrows [If(a) \wedge If(b)' \wedge \ldots]$ (analogous to disjunction, but the series cannot be completed. Strictly speaking we can only write down a finite fragment of the conjunction, and then we have no equivalence, but only implication from left to right).
8. $I[(Ax)\ (fx)] \rightarrow \Diamond[(Ex)Ifx]$ If it is impossible that all x satisfy f, then it is possible that there is an x for which fx is impossible (we point again to the weak analogy with law 5).

We have by means of the here mentioned laws, without using negation, combined I with the usual logical constants and in doing this we have shown that incompatibility can be positively described (one has only to use $I(Axp \wedge Axq)$ where A means "asserts", or $(Axp) \rightarrow I(Axq)$, or $(Axq) \rightarrow I(Axp)$.

Instead of working in alethic modal logic, one could however also start out with deontic modal logic: "It is forbidden to assert simultaneously p and q" (and I have then again the task to look for an independent formalisation of "forbidden" that is not derived from the usual axiomatisations of "permitted" or "obliged"). This seems to be possible.

The rules are analogous to I:

1) $(\mathrm{Fr}p \vee \mathrm{Fr}q) \rightarrow \mathrm{Fr}(p \wedge q)$
2) $(\mathrm{Fr}p \wedge \mathrm{Fr}q) \rightarrow \mathrm{Fr}(p \vee q)$
3) $\mathrm{Fr}(p \rightarrow q) \rightarrow (p \rightarrow \mathrm{Fr}q)$
4) $\mathrm{Fr}\ \mathrm{Fr}p \rightarrow \mathrm{P}p$. a.s.f.

The present writer has a preference for the deontic version over the alethic version because for him logic is a normative discipline. This preference has however to be defended. Neither assertion nor deontic logic seem sufficient however to define the negation concept.

If it is asserted that John did not come, it seems to be implied that it is believed by the audience that he could come and that it was believed by the same audience that he would come. It thus seems the case that we need the presuppositions in the speaker or the audience, but that we could express these presuppositions by combinations of time and belief logic.

Now in general there will be many propositions $q_1 \ldots q_n$ incompatible either in the alethic or in the deontic sense with p. To take care of this situation, we can stipulate the two following postulates:

1. Deontically: there is at least one proposition r, and not more than one proposition r, such that for all q_i, when q_i is asserted then one is obliged to assert r (this postulate can also be formulated as follows: all propositions to which one is committed by the assertion of any incompatible with p are either materially, or necessarily equivalent, or have a common consequent r). Here the logic of committment and of implication have necessarily to be brought in.

2. For alethic modality: there is at least one proposition r and not more than one, we necessarily assert when we assert

one of the incompatibles with p. Various systems of modality have to be brought in.

In the beginning instead of considering the common implicate of the incompatibles, I was considering the strong negator, namely: a unique proposition r whose assertion either commits you to or necessitates the assertion of all incompatibles with p. This concept as Batens and Vandamme however pointed out is more or less useless, then it presupposes the compatibility of the incompatibles, a condition rarely if ever satisfied.

In general quoting Greco and Bresson, discussions with psychologists and psycholinguists about negation gave the result: *negation is a modality, in this sens that it expresses a propositional attitude of the subject towards the entity denied.*

This leads us to other definitions of incompatibility

a) p and q are incompatible if it is impossible or forbidden to believe simultaneously p and q (doxastic incompatibility)

b) p and q are incompatible if it is impossible or forbidden to know simultaneously p and q (epistemic incompatibility).

We still need

a) chronological logic for the definition of simultaneity

b) logic of belief and knowledge to define believing and knowing

c) and our positive definitions of "impossibility" or "being forbidden".

A. Prior will inform us about the logic of simultaneity (an equivalence relation on the field of the before or after relations), and J. Hintikka will do the same as to the belief of knowledge concepts. We ourselves have shown that I and fr can be used without negation. Again we shall have to postulate either the existenceof the unique common implicate, or the strong negator.

All of the cases considered untill now come however under the heading: *trying to find positive definitions or axioms for negative concepts using simultaneously the non classical logics also necessary to define "presupposition" and "context"*. (Cont (p) = the set of all q such that $B \times [(Er)\ (p \rightarrow r)] \rightarrow [B \times q]$ that what is needed to give a proposition a meaning. The same task is to be undertaken if on the pragmatic level we now undertake the definition of the action of denial as an operation on a belief.

Let us define the concept of an action aimed at the destruction of a belief. To define it positively we start from the intuitive concept: destroying a thing. Destroying a table means eliminating its specific type of organisation, for instance: dividing it physically into unrelated parts. The positive definition of this could be that the D operator (destruction operator) transforms a set with a given relation defined upon it into a number of subsets with relations defined upon them but whose relations among each other allow a much greater degree of liberty and variability than was the case before the destruction operation was applied. This operation D is not yet clearly defined, but I think we need to look for a precise definition of it, if we want to define negation.

$$D(a) = \{A_1 \ldots a_n\} \wedge (R)[(R \in S) \rightarrow \{\Diamond(R(a_1 \ldots a_n) \wedge \Diamond I(R(a_1 \ldots a_n)\}]$$

D applied to a, yields a set of objects $a_1 \ldots a_n$ and for all relations elements of given relation set S, it is possible that either these relations exist among the a_i, just as it is possible that these relations are impossible among the a_i. This definition does not use negation, and uses only known operators.

This destruction operator, if it can receive a positive definition (again the same difficulty shows itself), can also be applied to semantic nets, related structures of concepts. It can also be applied to beliefs. Saying "John is not poor" would then mean disconnecting some link in the semantic net of the hearer, relating "John" to "poor". It would thus be an interesting fact that in Markov's very important '$_i$Theory of Algorithms' there is present a annihilation operator that has some, though not all the properties of the D operator. The annihilation operator maps any word it is applied to upon the empty word. The difference between Markov's operator and our D shows up however under iteration:

DD is further fragmentation; AA is simply the identity mapping.

We think it necessary to continue studying Markov because we believe that we should define negation in the theory of algorithms (among other tasks this is also related to the problem of defining negation in combinatorial logic). We consider it very important that Frege in his study on negation explicitly considers the destruction operator. (^{5})

Frege states that the meaning of a question (he calls it a "thought") exists, even though the truth value of this meaning is unknown. Thus the existence of meanings is independent from their truth values. This also appears from conditional sentences with false antecedents. When a meaning of a sentence, a thought is denied, Frege asks if this can be construed as a destruction or a separation (id est: an action on the thought) He denies it: if, after the negation (or denial) there exists only a disconnected mass of concepts, there was never a thought that could be denied. The meaning of the denied sentence exists and is an ingredient part of any true denial. As second argument he gives that we can iterate denial: if denial were destruction, it would not be understandable that one could deny a denial. As third argument he asks (p. 368) what elements could be separated from each other by means of a denial. He tells us: it could not be parts of sentences, neither objects nor representations(then these representations are personal and would not be common to n persons understanding the same denial). These arguments are forcefull and yet we think we can refute them. It is possible to keep the thought denied, accompanied by the negative injunction, and yet separate, in memory a) this thought from the stock of information we preserve to act upon and b) to separate from each other in the permanent memory the predicates that were joined together in the denied proposition. These operations are akin to the ones Freges uses his weakest arguments against, namely the separation of representations. Frege's arguments are indeed not conclusive: indeed representations, are personal but it is possible to have common elements in them and to perform interpersonal actions of separation.

Frege claims that there is no essential difference between an affirmative and a negative judgement, giving examples showing that it is not possible to separate both purely syntactically. This however does not imply that this impossibility is intrinsic (we do not have only syntax). But he finds the very important problem that Klima has taken up again later: the various places (subject, verb, asf.) where the negative particle can be found. Frege cannot prove that it is impossible to distinguish an affirmative from a negative judgement but he claims rightly that we have still to look for such a criterium. The search for such a criterium is precisely one of the purposes of this series of papers. We must take note however of the fact that Frege did not give a disproof and that the problem he touches so lightly upon is now taken up again much more thoroughly. *It is clear on page 371 that Frege's Platonism, his belief in the fact that neither the truth of a thought nor the internal organisation of it is produced by the thinker in an action, is at the foundation of his belief in the fact that the asserted denial does not do anything to the thought.* His comparison of the thought with a mountain, and the judgement with motion through the mountain, is revealing. Even if the thought is produced its truth could be timeless (we should have to construe it as: were somebody to produce thought T at moment m, then it would be true).

The further argumentation of Frege is precisely proof of the fact that those who are no Platonists must construe negation as destruction. To deny is not to wipe out completely, to disconnect, utterly, but to wipe out relatively in certain functions and in certain relations. The same concept can occur n times in r ways. Completely false is what is written on p. 372; "die Verneinung aber als Bestandteil des Gedankens bedarf wie der Gedanke selbst kein Träger" (372).

Frege claims also by means of simplicity and economy arguments that it is more economical not to have a denying operation, but only a negationterm. If not, one needs a) assertion b) denial c) neg. terms. If one restricts oneself to the unique assertion operation one needs only a) neg. term b) assertion.

For Frege negation has a specific role by defining all thought complexes from the simple "A und B" (Gefuge erster Art) by means of negation. It seems to us that his analysis of denial is primarily intended to guarantee this reconstruction principle.

Let us after these brief and insufficient remarks go to semantics. Here we encounter the distinction "true" and "false". In general falsity is defined by means of a negation present in the semantic metalanguage. It is worthwhile task, still to be undertaken to do one of the two following things a) define a negationless semantics, in which there will be a positive independent definition of "truth" and "falsity".

The concept of incompatibility will then be redefined as follows: p and q are incompatible if it is false in every model that p and q are both true. If an independent definition of "falsity" can be given this defines incompatibility without presupposing negation.

We have shown, both when discussing Valpola and by giving a negationless falsity definition for the identity theory of natural numbers that on an object level language and on a semantic meta-level falsity can indeed be independently defined (without any negation).

And then, combining incompatibility with implication we can again make the same construction for r, the common implicate, now upon a purely semantical basis. A special case of this idea is the following one: suppose that we have the theorem "q0p10...q_n" in te language L. In that case not p can be represented by the finite disjunction "q10q2...0q_n". In general however the number of alternative cases is infinite (f.i.: any quantitative statement, involving real numbers) and either we have to use Carol Karp's logic of statements of infinite length, or we have to make the nonconstructive move to say that for this infinite and even in general non denumerable set, there exists in the language a unique name that will be the name of the negation. It is this reasoning that makes us sometimes think that negation is a fundamentally non constructive operation.

As we are not able to decide for the wild platonism of classical set theory, but are aware of the fact that constructive mathematics is certainly not sufficient for even the development of non contemporary classical mathematics, we want to look for both a constructive and a non constructive negation. For this reason the study of David Nelson's paper "Constructible Falsity" becomes so important in semantics and for this reason also the study of Karp's work is indispensable.

Finally we come to syntax. Haskell Curry in his "Theory of Formal Deducibility" gives various versions, all purely syntactical. It is significant that he only comes to negation in ch. IV, even after introducing quantifiers.

His versions of negation are the following ones: CI. Not p is proved if it is proved that a) either no proof yet exists for p (certainly not what Curry wanted: a pragmatico-syntactical concept) b) or no proof can exist in L for p. This presupposes again the concept of negation if we cannot give the proof of unprovability in the following way: a) all provable propositions have a property P and b) all properties of p are different from the property P (Curry himself says: the proof consists in showing that the unprovable proposition does not have the property P, but by expressing himself in this fashion he finally presupposes negation as given in syntax). We try to avoid this pitfall and the only way we can see is to introduce as a primitive and positive concept the concept of "Difference". Again we stand in front of the same task: the axiomatisation, as strong as possible of "different". This concept of negation does not satisfy Curry for another reason: it is not extensible. When we insert L in a richter L' having presumably more power of proof, then an negated proposition can become asserted.

Another way to reach the same result is the direct axiomatisation of "unprovability", as a positive concept.

1. $Fp \rightarrow Up$: the false is not provable (the converse is false as any incompleteness theorem will show)
2. $U(p \vee q) \leftrightarrows [U(p) \wedge U(p)]$
3. $[U(p) \vee U(p)] \rightarrow U(p \wedge q)$
4. $(Prp \wedge Uq) \rightarrow U(p \rightarrow q)$ (where Pr means provable)
5. $Prp \rightarrow UUp$: If p is provable, then it is unprovable that p is unprovable
6. $U[(Ex)\,(fx)] \rightarrow U(fa)$ (for arbitrary a)
7. $U[(Ax)\,(fx)] \rightarrow (Ex)U(fx)$ (it is essential here to note that $U[(Ex)\,(fx)]$ and $(Ex)\,[U(fx)]$ are fundamentally different expressions.

We once more see that, as in the similar cases of impossibility or being forbidden, unprovability can receive a positive definition by axioms.

C2. Looking for stronger proposals Curry comes then up with a very strong one "not p" means "p is absurd". And absurdity means: every proposition is derivable from p. We have several remarks a) reading Brouwer's doctors thesis "Over de Grondslagen van de wiskunde" I cannot accept this absurdity negation as an adequate expression of his intention. His intention is clear: a proposition is absurd if all constructions that can lead to the proposition, have failed and necessarily so. We need the concept of construction, of trying to construct, of failure, and of necessary failure to express Brouwer's intention

b) what does Curry mean ? Either that it is provable in L that p implies every other proposition ? Or that it is true in L, and provable in a semantic metalanguage of L, that p implies every proposition ? Again, we see that we cannot study negation without studying simultaneously implication. c) is it really necessary to go from the very weak proposal C1; "p is not implied by any theorem" to the very strong proposal in C2: "p implies every proposition". Are there no types of negations in between both ? Especially since a quantifier ranging over the set of all propositions is not extremely constructive. We could weaken, use non standard quantifiers and say: not p mens: "p implies members of those subsets". This is really what happens in the third proposal considered by Curry:

C3. Carnap, in his introduction to Semantics uses p. 163 direct refutation rules (this is the semantic counterpart of our attempts to transform the negative into the positive present everywhere in these pages). We are of the opinion that we should study these direct refutation rules, but that they should be put into relationship with the D operator. They should not be introduced ad hoc but justified. The proposal in C2 seems to us very inadequate, if we refer to the real process of reasoning. Absurdity in general is more often defined as p implying not p, or not (p implying p) but these two versions presuppose the concept of negation and thus do not lead to its definition.

It is an interesting fact that Curry distinguishes quite care-fully between the negation of atomic or simple propositions (containing no logical operators) and negation of complex propositions (containing logical operators, and more specially other negations). It is presumably to be recommended to give a type theory of negtions defining first negations of propositions of level 1, then negations of propositions of level 2 and so forth.

From a purely syntactical point of view all three proposals of Curry seem to define negation as a metalinguistical operator, belonging to general syntax (a concept Curry himself does not like). If this is true, then the level structure of metalanguages must be reflected by the level structure of negations. We remember here that Bergson's treatment of negation in "L'Evolution Créatrice", having simply the intention to get rid of the concept of nothingness also intimates in a non formal fashion that negation is a metalinguistic operator.

We now have mentioned some problems to be solved in a pragmatical, a semantical and a syntactical conception of negation.

Fundamentally we claim that negation is te be defined in pragmatics by the D-operator or some similar eliminator positively defined. Semantic and syntactic properties seem all to be derivable from this point of view: *the various Curry negations seem to be variously complete destructions.* We admit that we have more difficulty to relate the D with positively defined falsity (perhaps we could try to give a modal definition of falsity: Fp would mean "It is illicit to assert p" and "illicit" could be defined by a destruction operator on the relation of legitimity or by R).

We want to present two generalisations of the concept of negation, both suggested by the concept of the operator D.

1. Two processes can be called incompatible if they cannot occur simultaneously as processes affecting the same systems. If we have an exact process concept, the concept of incompatibility can perhaps be generalised for it. This seems important to us because we do not think that the study of negation can be executed without the study of contradiction (in a sense by

using fundamentally the incompatibility concept. We use contradiction to define negation).

2. If two processes can be called incompatible, there must exist a possibility to define an automaton interaction that has the properties of a negation. Such an interaction could be the following one: let a_0 be the initial state and let the word p (a sequence of inputs) applied to a_0 have produced the state e. Then we shall say that another automaton denies p if after p it produces a_n input in the first one, that maps e upon the initial state. It destroys the effect of p through the input i $(i(pa_0) = a_0)$. This is naturally a definition of negation through counteraction, through the inverse operation.

The two last ideas are only very inchoative but deserve I think, to be persued. We shall state why.

Negation as general biological action.

1. Why should negation be studied ? What is the scientific, philosophical or social importance of the study of the concept of negation ? Why should it be studied more than other words or word complexes ?

We think that the direction our development will take will only be clear if we give an answer to those questions.

Certainly the first motivation of most of the contributions to this collective undertaking are that they knew that important material on negation was present in linguistics, in logic and in psychology. There were convinced of the fact that this information should be brought together and they shared the conviction, by no means common to all philosopher that only by interdisciplinary approaches solutions could be found.

This certainly was the first motivation, and this motivation is common to all.

2. The writer of the present paper however, in this encouraged by suggestions coming from many members of the working group, had a deeper motivation. He shares with Jean Piaget the conviction that knowledge can only be understood as a form of life. If we now observe the behavior of living

systems then we find that living systems show behavior of the following type:

1) The system A stops one of its actions
2) The system A begins one of its actions
3) The system A erases the result of one of its actions
4) The system A avoids (flees) the system B
5) The system A destroys the system B.

Acts like taking distance from, are simply ways of avoiding contact with, similar is flight.

Fundamentally two organic systems can either incorporate the one in the other, join the one to the other, reproduce each other, eliminate the one from the other. Is agression only a form of incorporation or incorporation a form of agression ? It is difficult to say.

The writer of the present note however came to the following conclusion: a) this set of extremely primitive actions proper to all living systems can be separated in two classes (or at least ordered in function of a given dimension) b) this set of elementary actions can be defined in such a way that they can be performed on very abstract sets so that the same classification or ordering system can also be applied to very abstract sets of sets.

The classification principle we want to propose is the following one: For a sequence A of operations: one set of operations begins from a large variety of initial states and is defined essentially by its final state; the other type of operations is to the contrary starting from one definite initial state and ends in a large set of different final states (having some property in common). It is better to speak about an order than about a dichotomy because of the fact that a given operation can in various degrees be determined by its final and by its initial state. For instance: take agression, and at the limit incorporation. Agression has no uniquely final state, simply the destruction of the power of attack of the agressee. Even incorporation, if accompanied as usual by decomposition is not uniquely determined by its final state. Yet there is certainly a degree of determination. The clearest opposition is certainly the one between flight, and incorporation. One can flee in all directions, and only attack in the direction of the object attacked. Stopping an action does not determine at all what will happen next: beginning an action to the contrary does clearly do this.

We want to introduce a definition: let us consider a universe of sets. Let us consider mappings of sets or sequences of sets upon sets or sequences of sets, and let us introduce a measure for the domain and for the codomain of these mappings. Let us consider the degree of negativity or positivity of such mappings or operations as measured by the quotients of the measure of domain and codomain. An operation is maximally positive if it maps a very large domain on a very small codomain, maximally negative if it maps a very small domain on a very large codomain. Let us analyse by means of these definitions the different operations mentioned. The stop is a mapping of a definite element upon the whole universe: maximally negative. The destruction is the mapping of one system upon a set of systems. Again negative. Avoidance or flight is the mapping of one system upon another set of systems having a given element in common. *It thus seems possible at least approximately to measure the degree of negativity of an operation in general.*

Let us now consider a new universe, the set of trajects or sequences of beings in the first universe. Here again we can form sets and form degrees of negativity and positivity.

Operating in this way it seems possible to define degrees of negativity and of positivity upon any given level of abstraction.

If this is the case, then the question arises: can we find laws of combination of operations of given degrees of positivity and negativity, and are these laws the same on all these levels ?

This question is an intellectual question and is perhaps not yet a sufficient justification for an independent study of negation.

We need more motivation. But this motivation is forthcoming: if we are goal directed systems, what is the function of positive and negative operations (in the meaning given to them by our very general definitions) in our goal directed actions ?

If our aims are to fly, to attack, to avoid, or to the contrary, to approach, to incorporate, to seek contact is there an objective system of measurement for the comparative efficiency of these two action systems ?

And if we have an efficient combination of negative and positive actions on a given level of abstraction or concreteness can we derive from our level definition and from the definition of this efficient mixture the efficient mixture on the other level ?

It thus seems to be the case that all living systems in as far as they are self steering goal directed systems have positive goals (states they are aimed at) and negative goals (states they avoid). Degrees and combinations of both are possible.

In as far as we and our societies are also such selfsteering goal directed systems, it is an important problem to know how to combine aims of both types. We seem to have encountered a fundamental bipolarity of all goal directed systems. It is not simply the mapping of large on small, or small on large domains that is the specific characteristic but the fact that such a mapping is an aim of a goal directed system. We know from earlier work that such goal directed systems can be formally defined. It is important moreover to stress that such sophisticated forms of approach and flight as refusal and acceptance, belief and rejection can simply be seen as similar operations in more abstract spaces. The question however remains open to know in how far the combinatorial properties remain invariant on all levels.

Our general aim is thus to increase the efficiency of our action by a more efficient combination of two action types.

Our more specific aim is to increase the efficiency of the special type of action that is problem solving on the one hand, and talking or writing on the other hand by a more efficient combination of positive and negative searches, of positive and negative declarations.

It is extremely probable that a much more refined classification of actions, problems solving and communications will be needed before we can answer such a question.

We think that it will now be sufficiently clear why one wants to study negation. We could state it in even more pregnant ways: the study of negation is the study of conflict. Not perhaps of all types of conflict but of a specific type of conflict either in our thinking or in our communicating. How should we handle such conflicts and what is their specific characteristic ? We have asked ourselves what would be the case if we could only think and speak in an assertive and never in a negative manner. It is naturally very difficult to execute such thought experiments that should be quite to the contrary be handled by sophisticated experimental methods. We can however quite safely say this: if we could not use negations either in thinking or in speaking, then we could not compare the universe as it is to the universe as it should be or could be or as we wish it to be or fear it to be. The complete positive thinker could not make use of the concepts of possibility, necessity, obligation, value or utility. By means of the concept of negation we can say that the income distribution in society is not just, that we do not have the friends we desire, that the second world war could have had another outcome than it had in fact.

The acts of protest, refusal, rejection, rebellion, revolt, all necessary ingredients of the acts of modification and in a sense prerequisites of the concept of action itself are necessary. Heinemann in his article "The meaning of Negation" stresses very clearly the fundamental anthropological importance of the negation operator. He is aware however of the fact that the negation of an assertion is not the only possible type; a) one can dissolve b) separate c) forbid. Heinemann's stress on "separation" is perhaps not the best way to express the negative character of an action. We think that our definition of degrees of negativity is able to do so in a positive way and without using the hidden negation that is present in the concept of separation.

To describe what exists it is necessary for human beings to compare it with what is not. This can be done by permuting the elements that are, or by comparison with future or past or possible states. Various ways of negation can thus be separated "Man is not yet immortal" "Oranges are not red (as some apples are)" "France is not a kingdom (as it used to be)". The taking of a certain distance with reference to what is in order to extract the relations of its parts and in order to search for its explanation and for its future seems a prerequisite for the possibility to act; action is to produce what is not or to destroy what is.

These remarks of a non technical nature have the intention to show the importance of the study of negative action in general, of negative thinking in particular. Let us however now introduce the following questions:

1. Is negative action and thinking the basis and foundation of positive action or thinking or is this not the case ?
2. Is positive action or thinking the basis of negative action or thinking ?
3. Or are both independent
4. In case 1, 2 or 3, is there any hierarchical order from the positive to the negative ?

It is entirely possible that on different levels of action and of negation, the answers to these questions are different. It is also entirely possible that on different levels of abstraction the combinatorial properties of positive and negative actions are different.

We can now summarise what we have said in the former paragraphs:

1. Every goal directed systems seems to present two types of actions in various spaces of abstraction, the two types are negative and positive types. It is not necessarily (and not even natural) to consider these two types in a dichotomous manner but it is more natural to introduce gradual seriation.
2. Every set of goal directed systems in conflict presents a type of action that can be considered as a negative action
3. Every set of goal directed systems that represent in itself either its environment or its own operation has to introduce

an operation of separation or distanciation between the model and the image that is of an essentially negative nature.

All these reasons p lead for a specific study of the operation of negation. And so we see for the necessities of action the very high importance of the same negation we always wanted to avoid in the first part of this paper where we started from ontology and epistemology. This implies however that we describe essentially in a positive fashion even the negative action.

For these reasons, we think that the foundations of negative logic should be sought in anthropology.

It is in the work of Paul Lorenzen that we find the clearest awareness of this facet[6].

Lorenzen's "Operative Logik"

Negationless logic is based upon ontology, an approximation to negation has to restrain itself to anthropology. The two important features of man are, a) the fact that he performs actions and b) the fact that he is a social animal, constructing a culture c) the interaction of these two features. Now we observe the following fact a) Lorenzen has started out, under the influence of Hugo Dingler and of Herman Weyl (thus of the intuitionists and of Brouwer) to develop an "operative Logik". This relates the negation operator to one of the main features of man: *man as actor* b) later, he has developpped another starting point for logic: namely the concept of antagonistic or dialogical logic. Here he takes into account the other main feature, we mentioned, *man as collaborator.* It thus seems to us that he, more than any other logician we know of, takes care of the real function of logic as a human product c) moreover he is very influential in applying his logical conceptions to the theory of natural languages. In our interdisciplinary study this is another title to recommendation.

Lorenzen wants to start the exposition of logic and mathematics with a theory of calculi. A calculus is for him a system of rules to operate with "figuren" (structures, schematic objects, or even actual objects); p. 4 of his "Einfuhrung in die Operative Logik" he explicitly stipulates that these schemata can be ordered sets of objects, and need not be signs on paper or voice sounds. The only crucial concept is thus the action according to certain rules, upon arbitrary objects.

The study of such system of action rules is for him logic and mathematics. We shall perhaps have to ask ourselves a) what types of objects he allows b) what type of action concepts he presupposes c) what it does mean to act according to a rule. He himself will not allow these questions to arise at this moment of systematic thinking: he simply presupposes the existence in man of the practical skill to act upon sequences of objects according to certain rules. When asked how many skills he presupposes, he refuses radically to engage in such a discussion saying that the art of schematic operation is a fact and that he only wants to use, not to describe it, in the beginning of the construction.

Now the method is specified according to which calculi are studied. No predicates of calculi are to be considered except definite ones. A predicate is definite if a) either a method for the proof of its existence is given by these definition of the predicate or b) a method for the disproof of its existence is given by the definition. In the first case the predicate is proof definite. In the second case the predicate is disproof definite. Here in the definition itself of definiteness, a requirement imposed upon every usefull predicate by Lorenzen the duality between assertion and negation is introduced: it is suggested that a different set of rules defines proof and another set disproof. But we saw earlier that disproof can be positively defined.

For any given calculus, two basic predicates are important: a certain figure is derivable (id est can be constructed by means of the rules of action of the calculus out of its initial elements) or a certain figure is underivable: it cannot be constructed out of the initial elements of the calculus by means of its rules of action.

The predicate "derivability" is proof-definite: one can give a proof for the derivability by giving a derivation.

The predicate "underivable" is disproof-definite because a method of refutation of it is given: namely by giving again a derivation of the figure. One can disprove its underivalibity. Underivability is not proof-definite.

The two basic predicates of derivability and underivability are again in the incompatibility relation typical for negation. It is typical for Lorenzen's system that it is stronger than the theory of recursive predicates because of the fact that a recursive predicate has both proof definiteness and disproof definiteness.

The basic predicates for figures thus being indicated, we must indicate his basic predicate for rules. This is the predicate of "admissibility". A rule R is admissible with reference to a calculus K if in the calculus K' consisting of K with R added, there are no figures derivable that were not already derivable before the addition in K.

The predicate of admissibility is disproof definite then we can show a figure x having the following properties a) in K, x is underivable b) in K + R, x is derivable. This constitutes a refutation of the admissibility of R.

The basic foundation of Lorenzen's logic is his "Protologik", namely the study of five methods enabling one to prove the admissibility of a given rule R with reference to a calculus K. These five rules are so essential to his undertaking because in essence a rule is for him a logical rule when and only when it is universally admissible, with reference to all calculi. The predicate of universal admissibility can be disproof definite then it can be refuted by giving a calculus in which a given rule is not admissible.

It is problematic to us in how far the set of all calculi and a selection operation within this set can be defended as a definite concept. It is also doubtful to us if the definition of a logical rule by the property of admissibility is defensible. Perhaps we should introduce modifications here already. But what seems most interesting to us is the study of the Protologik. When this "Protologik" is analysed we shall then introduce Lorenzen's "Consequence" operator and his "negation" operator.

"Protologik" and Negation.

Let us give some examples of calculi. To construct them we need the following skills.

a) the skill to recognize a sequence of objects as identical or different

b) the skill to abstract the concept of variable (a sign that stands for any sequence of objects)

c) the skill to understand and obey production rules leading from certain configurations to other configurations.

K1. o; o+; ao produces ao+; a+ produces a + o. The first two positions are initial positions; the two last rules state that one can from given positions produce other positions.

Two types of liberty characterise the concept of calculus. From a given position different succeeding positions can be derived.

For instance the following calculus is allowed: +; a produces ao; a produces +a+.

On the other hand two different positions can be joined together. Rules of the form "a, b produces ab" are allowed. P. 15 of his "Operative Logik" we find other examples of calculi.

As we said before the basic problem for Lorenzen is to discover the procedures leading to the proof of admissibility of rules. He tells us however p. 23 "eine Ubersicht über die moglichen Elliminationsverfahren durfte kaum zu gewinnen sein". We must ask ourselves if this is true after having studied his methods.

The general form of a proof of admissibility is the following one: let the rule R to be proven admissible be of the form A1...An produces A. Then the rule can be eliminated if A can be obtained by means of the other rules if A1...An are added to the initial formule. This method is called the deduction principle. In logic indeed (and not in protologic) p has q as a consequence in a calculus K if, when p is added to K, q can be obtained. The name is thus well deserved.

The second protological principle is a generalisation of the induction principle. Let us introduce variables that have as values only expressions that can be derived in a given calculus. Let us then have A1...An as initial positions. Let us have the following rules: B11, B12...B1r produce B1 and a sequence of B21, B22,...B2r produce B2, untill Bs. Let then be f a function that is derivable from the initial positions; that is moreover derivable from all the consequents of the rules if it is derivable from the antecedents of the rules: then the second elimination principle is that this function is derivable from all the derivable expressions of the calculus.

Our expression of the rule mentioned on p. 28 is perhaps different from Lorenzen's but seems to us more in accordance with the operational intention of his work.

The third principle is the inversion principle. The principle is the following one: if a certain figure can only be obtained by means of the accepted rules from certain given antecedents, then it is admissible to add these antecedents if the figure in question has been obtained. The three methods mentioned before are positive in nature. The fourth method deserves our special attention because of the fact that it is a method to prove underivability.

To do this we have to define the concept of difference. As in many contexts before, when we started this study of negation, it is needed to look for a positive definiton of a negative concept. The proof of underivability is of the following form: one proves, where s is a variable covering all derivable figures, the proposition "s is different from t" where t is the underivable figure.

To do this we use naturally the second principle (induction) and moreover an axiom system for "different from" that has the following features:

a) for atomic elements, every atomic element is identical to itself and different from each other atomic element. This can be expressed by means of a finite matrix if there are only a finite number of atomic elements.

b) For non atomic figures the following rules hold
1. uX is different from v (where u and v are variables for atoms. X is a variable for arbitrary sequences)
2. u is different from xX

3. X different from Y produces uX different from vY.
4. u different from v produces uX different from vY.
5. u identical to v and X identical to Y produce uX identical to vY.

This axiom system for "different from" (even though Lorenzen would not like to call it an axiom system) has certainly the drawback that it mixes sometimes very different levels and metalevels. But it has the essential importance to define a negative concept (non-derivability) by means of a positively defined concept (being different from every derived figure). It is thus very important to ask if the properties of difference and identity mentioned here are sufficient ones or if they are too weak or too strong.

Page 37 the five protological principles are enumerated. They are the three we mentioned with moreover the following two added, dependent upon the concepts that have just been introduced.

The principle of equality: if A is equal to B, B can be produced from A.

The principle of underivability: if A is different in K from every derivable figure then adding the rule that from A any other figure can be obtained does not increase the set of derivable figures in K and thus is admissible.

Lorenzen, quite wrongly according to us, considers this to be (p. 37) a validation of the "ex falso sequitur quodlibet": this would only be the case *if the production relation in arbitrary calculi were identical to the consequence relation in logic.* As this should obviously not be true, we reject the interpretation he gives to his last rule. As to the rule itself, it certainly is admissible but can only be written outside K (as A is underivable in K). Again it seems to us that the neglect of the language-metalanguage distinction in Lorenzen should be stressed.

The questions should be asked a) if there are any systematic reasons to add or substract certain rules for the proof of admissibility or inadmissibility b) if there are any systematic relations between these rules c) if the admissibility concept really is a good definition for logic.

Paul Lorenzen has rightly built up his protologic on action-procedures (then his operating with sets of objects is in effect any type of general action). In this context the definition of "Logical rules" by "admissible rules" (an activistic translation of "analytical sentences"; analytical sentences do not add anything to the context of synthetic sentences they are conjoined to in exactly the same way as admissible rules do not make other figures derivable in any of the calculi they are added to) is not adequate. We would submit the following definition for "logical truth". Let us consider a set of calculi (not the set of all calculi). With reference to this set of calculi that we stipulate to be a recursive set, we consider the operations that extend given fragments of calculi into the complete calculus they are fragments of. We call such operations "expansion operators". Let us now consider "logical reles" the rules for universal expansion operators. As every calculus is an individual construction, the universal expansion operators must be schemata of operators. This will yield us a positive definition of the logical rules with reference to recursive sets of calculi. We know we lose a lot a) we do not preserve the uniqueness of logic (it becomes relativised to sets of calculi; we think this is unavoidable because the set of all calculi, the set of all possible actions, is not a constructive set that can be meaningfully used to describe actions) b) we must prove that there are expansion operators on a fairly high level of abstraction, while Lorenzen, inspired by the concept of "analytical sentence" can easily take over earlier types of reasoning. The burden of proof lies on us.

This remark being made the two first questions asking for the systematic unity of the protological rules becomes much less important. But we cannot avoid making the remark that there is some systematic unity in the set of protological rules for proof of admissibility, a type of unity that we can perhaps in the future use for the construction of schematic expansion operators.

a) when a figure is constructed from initial or intermediary positions and from this figure a second one is constructed, the two construction procedures can be added to each other and the last one can be reached from the first one (rule for expansion addition, in general called: rule for modus ponens).

b) if a figure can be constructed from a set of initial positions and if it can moreover be reached from all positions derived from these initial ones, it can be reached fom all positions (rule for expansion generalisation, otherwise called rule for complete induction).

c) the inversion rule: expansions can be inverted when a given position can only be reached from a given subset of other ones (this reversal rule must be the natural inversa of the expansion rule). These three rules starting from the point of view that there are expansions, that can be added, and can be reverted, and that can on a meat-level, be universally added are really rules for addition (specific or general) and reversal of expansions.

The two other rules of Lorenzen's five are distinct in nature: the fourth one adds the process of expansion to the process of reversal and states that in this case each of the results can still be obtained again.

But the last rule is so closely related to the concept of "admissibility" that we can not accept it in the calculus of expansions: it starts from an unconstructible position, and such a position can precisely not be used, by its definition itself. We thus can not mention it.

The general conception of protologic, the general conception of logic as a function of action according to rules, seems thus completely acceptable (indeed this conception is the very reason why we refer to Lorenzen as an example for logicians to follow). But we find a certain systematicity in a reformulation of his protological rules and this systematicity excludes the last rule, and compells us to reject the essentially classical conception of admissibility for rules, admissibility that is in Lorenzen's text essentially negatively defined (an admissible rule does *not* add to the set of derivable positions).

How could we then, in Lorenzen's framework define a more adequate negation ? We start from his own remark on p. 74, when he tells us that for predicates, the denial of a predicate may be considered as a primitive operation so that "die negative Aussage nicht als zusammengesetzt gegenüber der positiver erschiene" (p. 74 op. cit.). When it comes to operating with configurations, he does not preserve however the same point of view that seems to us fundamentally correct.

1) To refute the derivability of a sentence one has to show it to be different from every derivable sentence.

2) To refute the unadmissibility of a rule we have to prove the admissibility, to refute the admissibility we have to show at least one calculus in which the rule expands the calculus.

We want to make a radically different proposal. Its idea is simply the following one: we know that some attempts have been made to define falsehoods in propositional calculus, propositions derivable from certain axiomatic falsehoods. We want now to apply the same idea to the protological expansion rules, and then define negation by these abnormal protological rules.

Gerold Stahl in "Zeitschrift für mathematisch logik und Grundlagen der Mathematik" (Bd. 4 p. 244-247, 1958) has defined "An opposite and expanded system" (for classical propositional calculus). We follow his idea but apply it to Hilbert's positive logic. Its opposite system would be:

1. $p \rightarrow (p \wedge q)$
2. $q \rightarrow (p \wedge q)$
3. $\sim [(p \wedge q) \rightarrow p]$
4. $\sim [(p \wedge q) \rightarrow q]$
5. $(p \wedge q) \rightarrow \sim p$
6. $(p \wedge q) \rightarrow \sim q$
7. $(p \vee q) \rightarrow p$
8. $(p \vee q) \rightarrow q$
9. $\sim [\{(p \rightarrow r) \wedge (q \rightarrow r)\} \rightarrow \{(p \vee q) \rightarrow r\}]$
10. or $[\{(p \rightarrow r) \wedge (q \rightarrow r) \rightarrow \sim \{(p \vee q) \rightarrow r\}]$
11. or $[(p \rightarrow r) \rightarrow \{(p \vee q) \rightarrow r\}]$
12. $(p \rightarrow q) \rightarrow (p \leftrightarrows q)$
13. $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow \sim (p \rightarrow r)$

or 14. $\sim \{[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)\}$

or 15. $\sim [(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

The deduction rules would be:

16. "$p \rightarrow q$ and q, thus p"

17. Or: p → q and p, thus q
or 18. (p → q) and p, thus q.
The substitution rules remain the same.

We now define $\sim p$ as: p is *derivable from the opposite axiom system* just described.

This definition of negation is naturally not a definition if we leave in the axioms or deduction rules all the negations we introduced. For this reason we make one of the following two changes

a) either whenever a formula is preceded by a negation, replace the negation by (Ap)p.

b) or we replace the negation by the impossibility modality for which we can give a positive definition in the opposite system as easily as in the direct system (except that here now will be declared impossible all the propositions that were declared true before). Given the fact that such opposite systems can be constructed while negation can be eliminated within them, we can indeed come to a definition of negation by the counteraxioms.

Let us apply the same technique to protologic.

1) the rule of addition of extension operators becomes: either any extension operator can be combined with any arbitrary other one to yield the possibility to go from the initial position of the first to the final position of the last (or, closer to the opposed propositional calculus): if we have in order two extension operators, and if the second one can be applied to a position in a calculus K, consider then that it is possible to apply the first and second one taken in that order as applicable to that position).

2) The rule of generalised induction: if either a position can be reached from some of the successors of a position in K (even from all successors) consider then that it can be reached from all positions in K, or if it can be reached from the positions of K, and also from some sucessors, consider that it can be reached from all positions of K.

3) if a position can be reached from several other positions

of K, it has been reached, consider then that all of the predecessors can be reached from the initial position of K.

4) if either the direct extension of A to B or the inverse one from B to K is not possible in K, consider then that if either or A or B is reached, both are reached, from the same initial position. Our definition of negation will be: if a proposition can be reached in K only by using one of the four earlier rules of oppositional protologic, then we shall consider that the negation of this proposition is reached in K.

We did not have yet the occasion to investigate the properties of opposed protologic, but we think that the reader will observe that we are applying to our action calculus, built up for anthropological reasons, the idea that negation can be introduced, but must be introduced by fundamental rules of refutation (the opposed protologic). We submit that to remain close to Lorenzen's initial inspiration we must replace his negation, as he introduced it, by some such negation as we here propose (our proposal has the advantage that negative and positive are equally primitive by now, exactly as the ascription and rejection of predicates Lorenzen mentions at the start of his remarks on negation).

Lorenzen's own definition of negation is inadequate. His concept is introduced on p. 75 in a quite classical fashion: we call an F-figure for K a figure from which it is allowed to derive any other figures of the calculus K. The last rule of protologic makes any underivable figure an F figure. We then say that the negation of A means the fact that from A is derivable an F-figure. There are calculi in which all figures are F figures (an example is given p. 73; they are the calculus equivalents of the strongly connected automata of Huzino). The fact that that definition is not adequate is shown by these examples themselves: perfectly acceptable calculi contain then the negation of all their figures. Another argument against the Lorenzen definition is, that in general only the existenc of underivable figures guarantees by means of our last rule the existence of at least one F figure. Thus in general non A will mean: it is possible to derive from A by means of rules of K, a figure not derivable in K. This means if A is atomic, that A is not an initial statement of K and if A is not atomic that A itself cannot be derived in K. Now it seems insufficient, to say the least, to take as a definition of negation of A either the fact that it is not an initial position in A or the fact that it is not derivable in A. It is clear that we come very close to Curry's rejected definitions of negation and that the promising starting point from a general theory of operations leads to disappointing results.

C. *Dialogical negation*

We finally consider the dialogical logic of Lorenzen, with special reference to his negation concept.

Some generalisations of this logic are proposed a) the n person no zero sum dialogue b) modifications of the asymmetries of the dialogue c) modifications of the object of the dialogue: namely: dialogues about the protological operations and about Curry's combinatorial logic (as a preparation for a closer combination of operational and dialogical logic).

1. *Lorenzen's ideas*

Our exposition is a blend of "Metamathematik" and of "Remarks on the completeness of Logical systems relative to the validity concept of Paul Lorenzen and Kuno Lorenz" (Notre Dame Journal of Formal Logic, April 1964, pp. 81-112, by Wolfgang Stegmuller).

The starting point of Lorenzen is that John von Neumann's "Theory of Games" can be used as the foundation of symbolic logic. We share to a certain extent this conviction but we shall have to criticise the way in which it is explicited in Lorenzen's work that paradoxically never refers to game theory.

We shall say that a game has two players: the prononent (the person who starts the play of the game in which we are interested) and the opponent (the player who follows by initiating the next move).

"Als Partner des Dialogs wollen wir den zuerst redenden als den "Proponenten" vom Gegenspieler als dem "Opponenten" unterscheiden" (Logische Propadeutik, pp. 158).

The logical game of Lorenzen is a game on form, not on content (presupposing the distinction between form and content).

a) It is a two person game; proponent against opponent (these two terms are relative to an individual play of the game).

b) the game contains a set of possible positions: the set M and

c) on M a relation R is defined stating what positions can be followed by what other positions

d) the M is subdivided into two set MO (the set under the control of the opponent) and MP (the set under control of the proponent).

e) If xRy ,x and y are elements of the two disjoint subdivisions of M, (no player can play twice in succession); xRx is excluded.

f) a set of positions E is the set of end positions: they are not in the domain of R: no player is allowed to make a move when such an element of E is reached.

g) no play of the game is allowed to remain undecided: either P or O wins. Lorenzen accepts the following rule(that is not essential for the game theoretical approach to logic): P wins if either he can compell the opponent to make a move after an end position has ben reached (this is not allowed) or if, in an infinite play (plays may be infinite) he can prohibit the opponent to compell himself to enter into such a situation. It is important to understand that the proponent has here a strong advantage (so it is very important to know who starts a play: the advantage is his). Logically one might have come to other decisions: in an infinite play nobody could win, or both win, or the opponent wins.

h) "the meaning of a logical sign is determined by specifying how sentences containing it as its principal logical sign, after having been presented by one player (the proponent) may be attacked by the counter player (the opponent) and defended against this attack by the first player" (pp. 85, Stegmuller).

i) a proposition is a logical theorem if it can be defended by its proponent against every possible strategy of its opponent.

j) the moves of the game are the following ones: asserting closed formulas, challenging either the right or the left part of the formula put forward or challenging it for a given value of one of its variables.

k) we are now going to describe the relation R, defining the possible transformations from one position to another. It has three parts: the logical rule L, the basic rule B and the structural rule S.

A. The logical rule L. M is indicating in general a molecular statement containing other logical connectives. In the degenerate case M can be an atomic proposition. We consider different cases:

1) The proponent proposes "M1 and M2". The opponent has then the choice between two attacks: he may challenge either M1 and M2. Against these attacks there is only one possible defense: putting forth the challenged statement.

"Die Konjunktion: behauptet der Proponent "a und b", so hat der opponent das recht, eine der beiden Teilaussagen zu wahlen und anzuzweifeln. Kann der Proponent die gewahlte Aussage nicht verteidigen, so hat der Opponent den Dialog bereits gewonnen" (Propodeutik, p. 158).

2) The proponent proposes "not M1". The only possible attack is M1. There is no possible defense: one can naturally attack M1 if it is a conjunction or a disjunction but one cannot attack it in other case, and the attack is not specificially determined by the fact that M1 is an attack against a negation.

3) The proponent proposes "M1 implies M2". The opponent states M1. There are two possible replies by the proponent: attack M1, or state M2.

4) The proponent states "For every x, fx". Then the opponent may select one of an infinite number of attacks: fa, fb, etc": is the first true, or the second etc. The only defense is to state the doubted statement and assert it.

5) The proponent states (Ex) (fx). The attack is total challenge; f is doubted everywhere. There are an infinite number of possible defenses: either fa, or fb etc. This is the basic rule.

Comment: the two last rules, considering quantifiers consist either for the proponent or for the opponent in a selection within an infinite set. This move is not constructive: the rule of this selection is not given (the infinite set is the set of possible values of the variable). The unconstructivity would show clearly if one makes the graph of the possible plays of the game: at that point an infinite bifurcation would be necessary and the rule of development for this infinite bifurcation is not given. Another non constructive feature of the dialogic logic of Lorenzen would be: a theorem has to be defended against every possible strategy of an opponent, The set of possible strategies is again not a constructive set. It is infinite and we do not know its rule of construction. For a constructivist these two concepts have no meaning. If we are to preserve them, we have to replace them by weaker concepts, where either finite sets or constructible sets are considered.

The other criticisms are more basic however: the rule for the defense of the implication, countenaces the "ex falso sequitur quodlibet". We should think that the following moves are equally allowed "Mn implies M2, if M1 implies M2" (showing that its antecedent is false). The discussion about this rule could be repeated with reference to every other rule and shows only one thing: Lorenzen does not add anything to the classical meaning of the logical constant by giving his dialogical formulation.

As our last comment we want to point out that dialogical logic should be related to other types of non classical logic. Let us for instance take only the rule for defense and attack of conjunctions. Assertion logic is certainly used when Lorenzen puts forward the assertion by an x of a conjunction. We had already in this article the occasion to point towards the use of assertion logic in Van Dantzig's intuitive foundation for his negationless logic. Erotetic logic is used when it is stated that the opponent asks if one of the terms of the conjunction holds. Deontic logic is used when it is stated that the opponent has the right to (is permitted to) interrogate about any of the two terms of the conjunction (in "Metamathematik" actually the question sign is used to indicate the move the opponent makes). And finally both the activities of attack and defense have to be formally defined as follows: a player attacks a statement if he asserts statements with the aim to cause the denial by the person who asserts some statement of that very statement. A person defends a statement when he asserts other statements (or the same) with the aim to continue in the future to have the right (to be permitted to) assert this statement or (otherwise put) with the aim to cause other persons (or eventually himself in the future) to assert this statement. The concept of denial has been positively defined in our second article in this series on "Negation" and the concept of "doing an action with the aim of producing a state of affairs" has been axiomatically defined in Roderick Chisholm's article: "Some Puzzles about agency" (p. 199-218) in "The logical way of doing things" (editor Karel Lambert Yale University Press 1969).

Lorenzen's dialogical logic should thus be developed as a combination of erotetic, deontic, assertion and action logic. His terms themselves show that it can not be left in the isolation in which it finds itself here.

We cannot in this article dedicated to negation rewrite on these new foundations. Lorenzen's dialogical logic. We had to point out however this dependence because it shows still deeper the anthropological foundations of logic.

We are now going to study the other rules. Paradoxically enough, to us the most contested rules: the basic and the structural rule are the most interesting ones.

B. *The basic Rule B*

The proponent has not the right to introduce atomic formula. He has only the right to introduce them if they have before been introduced by the opponent.

What is the meaning of this ? It expresses the distinction between form and content, between logical truth and factual truth, and gives to the prononent the function of defending formal truth. The very fact that in the dialogue atomic formula can never be attacked is another expression of the same distinction.

We have to make the following comment: if we want a dialogical logic that is in any sense close to actual dialogue then we must allows to all participants the right to introduce atomic formulae, and we have the duty not to impose upon our dialogical rules an a priori strong separation between form and content. We can even encounter dialogues in which hypothetical permissions as to the introduction of atomic formulae are introduced (i.e.: under certain conditions a given type of player has or has not the right to introduce atomic formulae).

C. *Structural rule S.* Let us define a round as a triple, the first element of which is a statement, the second element of which is an attack and the last element of which is the defense against the attack. The round is closed of all three elements of the round are present. It is opened if the first is present. The round opening with a negation statement can never be closed (there is no defense against an attack on negation). The structure specifies the number of times given attacks may be made i) the opponent O may attack once and only once by means of an attack move in a round k a formula of the proponent occuring in an earlier round ii) the prononent may attack an arbitrary number of times earlier formulae of the opponent iii) both players may defend themselves against attacks in a given round only if all later rounds are closed (a later round is a round started later).

We could try to defend the structural rule in the following fashion: the proponent, who has the charge of proof must have maximal possibility of proof.

One could also reject it and say: a fair game is a symmetric game: either both P and O must have an infinite number of possibilities of attacks and defenses, or they must have an equal finite number (at the limit: both may defend and attack a given formula once). Kuno Lorenz however claims that this restriction on the attack repetitions will not increase or decrease the number of theses that can be defended. If it is at all impossible to have complete symmetry in the structural rule, one should at least counterbalance the definition of "winning" that gives a strong dominance to the proponent (he wins already if he does not lose), by not again favouring him in the intuitionistic structural rule (this counterbalancing could occur again in two different ways: either have a symmetrical winning rule, or favour in the winning rule that assymetrical party that is disfavoured in the structural rule).

Our own opinion is that only a symmetric game with an unlimited number of possibilities of attack and of defense is a fair representation of reality on the one hand; if we take the normative view on the other hand, we might limit the number of attacks and defenses allowed but we should still preserve the symmetry. The symmetry in the rules of attack and defence should be mirrored normatively by the symmetry in the winning or losing rules; realistically this symmetry does not always exist (but we can think of situations in which the proponent has major chances and about situations in which the opponent has major chances).

Many different combinations of winning rules, basic rules and structural rules can be thought out, that would be different from Lorenzen's. Are these topics relevant for the analysis of negation ?

According to us they are. The whole set up of the game theorethic approach to logic as Lorenzen handles it, is based upon the positive-negative opposition: 1. The opposition opponent-proponent 2. The opposition winning-losing (here radically enforced by the impossibility of the draw) 3. The opposition defence-attack, as we said before.

These roles could be expressed in assertion logic (see Rescher op. cit.): 1. The proponent is the person who asserts a statement (ix) (Ep)Asxp), and the opponent is the person who asserts a statement a) only under the condition that somebody else has asserted a statement and b) to reach the goal of making this person abandon his statement in the future. If the means-ends relationship can be expressed then this distinction can be formalised. If "abandoning a statement" can be construed as "no longer making the statement in situations in which it should normally have to be made" then "abandoning a statement" can be distinguished from "asserting its negation". We think this should be done. Finally the distinction between "defense and attack" can also be expressed in assertion logic: a statement is a defense of another statement if it is asserted with the aim to enable one to continue in future occasions to assert the first. It is an attack if it is asserted with the aim to make it impossible to assert on future occasions an earlier one.

The logical operators should not be introduced because historically they happen to exist, but because the general theory of attack and defense of assertions makes them necessary.

The generalisations of Lorenzen's approach we are going to suggest in the following paragraph have the intention to free ourselves somewhat from the positive-negative distinction that seems to be dominant in the initial attempt.

If the proponent has the right simultaneously to repeat attacks and to repeat deenses, than classical logic is obtained. If the proponent has the right to repeat defenses, but has no longer the right to repeat attacks, then a so called anti-intuitionistic logic is obtained.

If complete symmetry is imposed on proponent and opponent as to the rules of repetition of defenses and attacks, then a common sublogic of intuitionism and anti intuitionism is obtained (with the desirable property of being close to strict implication).

Our aim being the study of negation, let us then procede to generalisations in which the strong opposition between attack and defense, winning and losing is weakened.

2. Generalisation of Lorenzen's dialogic logic.

Von Neumann's "Theory of Games" has been mainly useful in the study of a) two person b) zero sum games (the total gain being always zero, the one player losing what the other wins). There unique maxim in strategies that are optimal strategies for all players have been developed. It is our conviction that Lorenzen's dialog game is a two person zero sum game for the following reasons a) it certainly has only two players b) it is always the case that one loses and the other wins (by the rules of the game). The utility seems to be: holding a proposition or abandoning a proposition. If this is true than in all plays either the proponent abandones his statement and the opponent can keep his first attack as dominant, or the proponent can keep his statement and the opponent loses his attack-statement. The total gain is zero: one statement kept, one statement lost. This idea however has been attacked in discussion by Mr. F. Van Dun who claims these games not to be zero sum games. If we represent correctly the argument, than he thinks so because the same propositions can be held at the final stage by both participants. We would then still believe that this apparent common possessions, is implying the abandoning of one statement and the keeping of another.

The discussion hinges upon the definition of "gain" and "loss" for these games (the utility function). In what follows, we shall continue to claim that Lorenzen games are zero sum two person. This is not a realistic representation of real discussion that is n-person non-zero sum (with the possibility of forming coalitions). Through bargaining processes the total group, even when constituted by partially antagonistic subgroups can have a non zero total utility for the outcome of the game. It seems fundamental to us to introduce dialog games a) with n players b) with possibility of coalitions c) with possibility of common gain.

This problem has been set up by us and some first attempts towards its solution have been made by Mr. F. Van Dun. He introduced two situations: I. One proponent, n opponents. In the way of coalition formation he allowed the opponents (he did not specify if this was to happen under certain conditions or unconditionally) to make free use of each other's statements in their common fight against the proponent. He did not state how this would modify the logic obtained.

2. N proponents, one opponent. Here the same situation could obtain 3. N proponents, N opponents. The game becomes only meaningful if some relationship is introduced between the proposals of the various involved parties.

4. We think it would be interesting to introduce the concept of "partial opponent" and "partial proponent". But we see

but dimly their meaning. They seem to be needed to give sense to the idea of coalition in discussion. Is this relevant for the study of negation ? We think it to be very relevant if we consider that negations, in whatever syntactical form they may appear, have as pragmatic function to eliminate certain speakers from the coalition to which one belongs and by doing so to trace the frontiers of the coalitions. If this is true then it would be useful to combine various structural and basic rules with various coalition structures.

But before getting into that we want to show some other possible generalisation of Lorenzen's dialogical procedure. As we said, we felt rather disappointed seeing the old and well known logical constants once more with their usual introduction and elimination rules. The idea then arose: *why did Lorenzen so completely forget his own operational logic when building up his dialogical logic ?*

Why formalise conflict about negations and disjunctions (in a way presupposing what is to be analysed) and why not fight over calculi (in Lorenzen's sense of the word). The proponent will then be a builder of certain constructions and the opponent will have to destroy them. It is not clear how we have to work, but as a first attempt one could use this: Curry's combinatorial logic is well known. We mention some of his combinators: Kabc = ab; Wabc = abcc; Sabc = a(bc); Pabc = acb. We can organise plays with combinators. A proponent holds a initial position. An opponent can attack it. How can he attack it and by means of which combinators ? If we say that the proponent has won if he can hold his initial position, then we can always make the opponent win by attacking by means of the combinator K. Indeed after the deletion of c, there is no combinator that can reintroduce it. If the opponent attacks by means of W a counterattack by means of K will restore the initial position. If he attacks by means of S, there is again no possible countermove. We could only introduce possible defenses by means of introducing new combinators: for instance a deletor, or a recreator. But in that case the game is again trivial, because of the fact that the defense always wins. According to us the following problem is not trivial how to define a game with combinators that is not trivial in the sense that either necessarily the defense or the attack wins. We have thought (without exploring to the end) about the following: let P and O both put forward initial positions and let threats be possible: if the one attacks by means of K the other can attack by means of K and both have lost. But if this is not the case (and they can stipulate, not using K except in certain circumstances) then only certain other combinators are allowed. A second idea that came to us was the following one: we could introduce equivalence relations between positions and say that a player wins if he can restore a position equivalent to the initial one.

Our introduction of combinators is only a first attempt to introduce the theory of algorithms and to see what are the problems of combining algorithms with theory of games (let us also point towards the affinity of the annihilator and the K operator).

We should now after having understood the relation between game theory and combinators in general return to Lorenzen's "Operative Logik" and study the relation there (always with the object in mind to see what type of logical laws could be founded upon "Protologik"). The fundamental question would be: if a logical rule is universally admissible in operational logic, and is on the other hand a position that can be upheld against any strategy of any opponent, what is the exact relationship between universal admissibility and universal defensibility ? I cannot solve this question, and yet it is the fundamental anthropological question of logic.

If we replace, as we have proposed the concept of "admissibility" by the concept of "universal extension schema" then the same fundamental question becomes "is a universal extension schema in some sense related to a position that can be defended against any attack by any opponent ?"

Harsanyi (Contributions to the theory of Games vol IV: "A Bargaining Model for the cooperative n-person Game" (pp. 325-356) considers a set of N players, and the set S of all its subsets. Let us characterise these players by the fact that each of them defends a certain number of propositions in the initial move and attacks another series. This is already more general than Lorenzen's initial situation. A player can commit himself to defend equally the positions of other players and not to attack theirs, under the condition of a certain payoff, namely under the condition that they commit themselves to the defense of certain of his own positions. The player, in Harsanyi's general case makes contracts with many different subsets (syndicates as he calls them) and his bargaining position (namely the probability that other players will increase their defense of his own position) will become better. A meta-play is possible: one may bargain about the structural and basic rule.

In a sense this means a disjunction of commitments as second move: namely each player enters under certain payoff conditions into a syndicate (id est: accepts to defend the positions of the member of the syndicate). The dividends that the member of the syndicate receives from the other members are the propositions they commonly accept to defend. Dividends may be negative: namely sets of players may withdraw their support of certain propositions. For every bargaining situation there is also a threat strategy: namely the positions the member of the syndicate announce to attack if there is no agreement reached between the members and non members of S. Obviously we should exclude such agreements that would yield at the end for some member of some syndicate less than the number of positions he would gain if no agreements were reached anywhere and if every player was attacked by every other. Contradictions are presumably such agreements that would give less than the pure conflict payoff.

It is allowed after having announced all forms of threat and cooperation to redistribute at least once. Then the game is played. It is important to examine the aplication of Harsanyi's axioms to the dialogue game:

Axiom 1. If u is a solution of the game (being a final imputation to every player of a certain number of assertions that continue to be defended and others that continue to be rejected) then it may not be the case that it is possible that there is anoher solution yielding for all players a larger part of their original assertion and rejection set.

Axiom 2. Every game that has the situation of its players symmetrical must have at least one symmetrical solution.

Axiom 3. If we change for one player the zero point and/or the unit of measurement (this meaning: if we analyse the assertions in smaller units) the solution remains the same.

Axiom 4. If we have a solution for a given game, and if we restrict the possibilities of motion for some or all players in the game but still in such a fashion that the solution remains a possibility (reachable within the game) then the solution remains a solution (so restrictions on the rules of proof leaving open the possibility to get to certain results have no influence). A detailed rewriting of Harsanyi's text for logical games would be necessary. We simply want to stress that the main difficulties are a) the modeling of the utility function (depends upon the number and type of propositions defended and attacked in the end, compared to number and type defended and attacked in the beginning but cannot easily be uniquely determined) b) the meaning of transformations upon the utility function (analysing the propositions in predetermined ways-difficult to see in which ones).

The work has to be continued in the following directions: a) the application of Lorenzen's theories upon the grammar of natural languages (a task already undertaken by himself) and eventual reference to Kraak and Klima (see elsewhere in this number). b) the real execution of the programmatic remarks made in this paper. c) the development of a theory of the relation between the negation and implication (in the framework of an adequate theory of entailment embedded both in operational and in dialogical logic).

Conclusion

In the first part of this paper, dedicated to negationless logics and starting both from ontological and epistemological presuppositions we have pointed out that the negationless logics developed untill the present moment need a) to be brought together and b) to be founded on assertion logic, strong existence logic, truth modality logic, asf. Our aim was to point out that our ontological framework engages us into a serious undertaking of reconstructing our deduction rules. By trying to eliminate negation, we think we got some insight into its nature and function.

In the second part of this paper, we essentially stressed first the pragmatic function of negation (introducing the D operator and the general biological definition of the operator) and we tried to unify, generalise and develop the attempt towards the study of logic that seems to us to take into account in the clearest manner the pragmatic functions, both operational and dialogical, of logical constants (and thus also of negation). We hope that the many questions we had to ask without being able to answering them will encourage others to continue. If we may finish with a philosophical remark: without having in the least any intension to reach such a result we felt ourselves compelled to stress a negationless ontology and an anthropology, where negation is present but, as we did show many times, where it can be positively described. There is some analogy, though no identity of this result with an ontology similar to J. P. Sartre's "L'Etre et le Néant". (with the following strong difference: we do our best to describe even the anthropological negativity in a positive way). The reader may rest assure that we are ourselves the first to be astonished, when looking at these results. We hope that it will be possible to see some relationship between this logico-philosophical inquiry and our other contribution, stressing the linguistic and psychological aspect of the problem.

NOTES

(1) David Nelson in his "Negation and Separation of Concepts" p. 208 states: "the absence of a property P if it may be established at all must be established by the observation of another property N; which is taken as a token for the absence P. The way N is chosen in general seems to be a complicated matter". (p. 208, Constructivity in Mathematics).

(2) J. N. Findlay, "Meinong's Theory of Objects and Values" Oxford University Press.

(3) This opinion is shared by a number of investigators. Moritz Schlick: "In der That haben negative Urteile nur praktischer, psychologischen nicht theoretisch logischen Wert. Die Gebaude unserer Wissenschaften bestehen ausschliesslich aus positiven Aussagen" (Allgemeine Erkenntnisslehre, 1925, pp. 59). And Bertrand Russell "The world can be described without the use of the word "not" " (1948, Human knowledge, p. 520).

(4) We agree with both H. Freudenthal's "Zur intuitionistischer Deutung logischer formula" (1937, pp. 112-116 — Compositio Mathematica, and with G. F. C. Griss, "Similar refection of negation (see bibliography under Griss, G.) unconvinced by A. Heyting's replies (Compositio Mathematica 1936, pp. 117-118).

(5) The following articles analyse negationless logic (even though this term has not everywhere the same content):

Hugo Freudenthal, Zur Intuitionistischer Deutung logischer Formeln, Compositio Mathematica, vol. 4.

G. F. C. Griss, 1944, Negatieloze intuitionische wiskunde, Koninklijke Akademie der Wetenschappen, afdeling natuurkunde, 53, pp. 26, 1-268.

—, —Negationless intuitionistic mathematics I-IV, Ibidem, 49, pp. 1127-1133, 53 — pp. 456-463, A54, pp. 193-452-462, 473-471.

—, — Logic of negationless mathematics, Ibidem, A54, pp. 41-49.

P. C. Gilmore, The effect of Griss' criticism of the intuitionistic logic on deductive theories formalised within intuitionistic logic, Kon. Acad. der Wet., Proc. A56, pp. 162-174, pp. 175-186.

—, Veli Valpola Ein System der Negationlosen Logik mit ausschliesslich realisierbarer prädikaten (Acta Philosophica Fennica, fasc. IX, 1955).

P. G. J. Vredenduin, The Logic of negationless mathematics, Compositio Mathematica, vol. II, 1953, pp. 204-277.

David Nelson, Non-null implication, JSL, vol, 31 — nr. 4, Dec. 1966.

—, — Constructible Falsity, JSL, nr. 1, March 1949.

—, — Negation and Separation of concepts in constructive systems David Nelson, Constructivity in Mathematics, pp. 208-225.

A. A. Markov, A Constructive Logic, Uspehi matematiceskinauk (NS), 5, 1950, pp. 187-188.

N. V. Vorobev, a constructive propositional calculus with strong negation, Doklady Akademii Nauk SSSR 85, 1952, pp. 465-468.

—, — The problem of deduction in the constructive propositional calculus with strong negation, Doklady Akademii Nauk SSSR, 85, pp. 689-692.

D. Van Dantzig, On the Principles of intuitionistic and affirmative mathematics, Kon. Ned Akad. Wet., Section of Sciences, 50, pp. 918-929.

—, — Indagationes mathematicae, 1947 (pp. 429-440, 506-517).

H. Rasiowa, Algebraische Characterisierung der Intuitionistischen Logik mit starker Negation (Constructivity in mathematics, Amsterdam, 1959), pp. 234-240.

(6) Frege: Kleinere Schriften (ed. I Angelelli, 1967, Hildesheim): "Die Verneinung" (p. 362-377).

(7) P. Lorenzen's views are expressed in "Einführung in die Operative logik und Mathematik" (Grundlehren der Mathematischen Wissenschaften, Band LXXVIII, 1955, Springer, Berlin) for his operational logic and in "Metamathematik" (Manheim 1962 par 2 "Effektive Logik der funktoren und Quantoren" pp. 18-33, "Logische Propädeutik, Vorschule des Vernunftigen Rendens" (Kap. V and VI) Manheim 1967 (by N. Kamlah and P. Lorenzen), "Dialogspiele als semantische Grundlagen von Logikkalkülen; Archiv für Math. Logik v. Grundlagenforschung, 11, 1968 by K. Lorenz and "Ein dialogisches Konstruktive tätskriterium" (Infinitistic Methods, Pergamon Press, 1961, (by P. Lorenzen).