CUT ELIMINATION, CONSISTENCY AND COMPLETENESS IN CLASSICAL LOGIC

Luis E. SANCHIS

In this note we discuss several notions in Gentzen formalization of classical logic. The fundamental property of the system — the elimination of the cut rule — is related with other important notions, namely consistency and completeness. The relation that connects these notions is not clear in the original proof given by Gentzen. On the other hand it is well known that the elimination of the cut rule is an easy consequence of the consistency and completeness properties. We shall show that consistency is enough to prove the theorem and that Gentzen's proof involves essentially the use of such property. The argument is carried out for very general system formulated with Gentzen-type rules. Only the sentential calculus is considered.

1. We shall consider a system K of sentential logic in which there are sentential variables: $e_1, e_2, \ldots, e_n, \ldots$, and one or more connectives, each one with a given number of arguments. The symbol # will denote any such connective with n arguments. The set of formulas is the smallest set that contains the variables and contains $\# A_1 \ldots A_n$ whenever it contains $A_1, \ldots, A_n$ for any connective #.

Letters A, B, ... will denote formulas. Finite sequences of formulas (eventually empty) will be denoted with letters M, N, ... The formulas in the sequence M are called components of M. If every component of M is also a component of N, we say that N is an *expansion* of M and write $M < N$. If every component of M is a variable we say that M is *atomic.*

Expressions of the form $M \vdash N$ are called *sequents*. The components of M(N) are called *left* (*right*) *components* of the sequent $M \vdash N$. If both M and N are atomic we say that $M \vdash N$

is atomic. If P is an expansion of M and Q is an expansion of N we say that $P \vdash Q$ is an *expansion* of $M \vdash N$. If both M and N are empty, $M \vdash N$ is the *empty sequent*. A sequent $M \vdash N$ is *closed* if there is some formula which is both a right and a left component. The *union* of the sequents $M \vdash N$ and $P \vdash Q$ is the sequent $M, P \vdash N, Q$.

A *partition* of the variables $e_1, \ldots, e_n$ is a non closed atomic sequent $M \vdash N$ such that every component is one of the variables. The partition is *complete* if every variables in the list is a component of the sequent. The empty sequent is called the *empty partition*.

We assume that for every connective # with n arguments two finite sets $R_{\#}$ and $L_{\#}$ have been defined. The elements of both sets are non empty partitions of the variables $e_1, \ldots, e_n$. Any set may be empty. The elements of $R_{\#}$ are called *right axioms* of #, and the elements of $L_{\#}$ are called *left axioms* of #. An *axiom* of # is either a right axiom or a left axiom of #.

Let $\# A_1 \ldots A_n$ be some formula and $P \vdash Q$ be some right (left) axiom of #. The sequent obtained by simultaneous substitution of $A_1, \ldots, A_n$ for $e_1, \ldots, e_n$ in $P \vdash Q$ is called a *right (left) axiom* of $\# A_1 \ldots A_n$.

We introduce now the derivation rules for the system K. The rules operate on sequents. The first Axiom Rule give a set of *initial sequents*. There is an *Expansion Rule* that allows permutations, cancellations and the introduction of new components. For each connective there is a right rule and a left rule. Any sequent that can be obtained using these rules is said to be *derivable in* K.

Axiom Rule: All sequents $A \vdash A$ were A is any variable.

Expansion Rule:

$$\frac{M \vdash N}{P \vdash Q}$$

provided $M < P$ and $N < Q$.

Right Rule for $\#$:

$$\frac{P_1, M \vdash N, Q_1, \ldots, P_k, M \vdash N, Q_k}{M \vdash N, \# A_1 \ldots A_n}$$

where $\# A_1 \ldots A_n$ is any formula, M and N are arbitrary sequences and $P_1 \vdash Q_1, \ldots, P_k \vdash Q_k$ are all right axioms of $\# A_1 \ldots A_n$.

Left Rule for $\#$:

$$\frac{R_1, M \vdash N, S_1, \ldots, R_m, M \vdash N, S_m}{\# A_1 \ldots A_m, M \vdash N}$$

where $\# A_1 \ldots A_n$ is any formula, M and N are arbitrary sequences and $R_1 \vdash S_1, \ldots, R_m \vdash S_m$ are all left axioms of $\# A_1 \ldots A_n$.

In the right and left rule for $\#$ the components of M and N are called *parameters*, the formula $\# A_1 \ldots A_n$ is called the *main formula* of the rule and all other components are called *secondary formulas*.

On these very general assumptions we can prove several results for the system K.

Proposition 1. If $M \vdash N, \# A_1 \ldots A_n$ is derivable in K and $P \vdash Q$ is some right axiom of $\# A_1 \ldots A_n$ then $P,M \vdash N,Q$ is also derivable in K.

Proposition 2. If $\# A_1 \ldots A_n, M \vdash N$ is derivable in K and $R \vdash S$ is a left axiom of $\# A_1 \ldots A_n$ the $R,M \vdash N,S$ is also derivable in K.

Proposition 3. If $M \vdash N,A$ and $A,T \vdash W$ are both derivable in K and A is some variable, the $M,T \vdash N,W$ is also derivable in K.

Proofs of these properties can be easily supplied by the reader.

The *cut rule* — which is not of course a primitive rule in the system K — has the following form:

$$\frac{M \vdash N,A \qquad A,T \vdash W}{M,T \vdash N,W}$$

We say that the cut rule *holds* in the system K if whenever $M \vdash N,A$ and $A,T \vdash W$ are both derivable then $M,T \vdash N,W$ is also derivable.

A connective # in the system K is *saturated* if it is possible to obtain the empty sequent from the set of all axioms of # using the cut rule and the expansion rule.

Theorem 1. If every connective in the system K is saturated then the cut rule holds.

We prove that whenever $M \vdash N,A$ and $A,T \vdash W$ are derivable then $M,T \vdash N,W$ is also derivable by induction on the number of connectives in the formula A. The case in which A is a variable is Proposition 3. Assume that A is the formula $\# A_1 \ldots A_n$. Then by Propositions 1 and 2 we know that for every right axiom $P \vdash Q$ of A the sequent $P,M \vdash N,Q$ is derivable and for every left axiom $R \vdash S$ of A sequent $R,T \vdash W,S$ is derivable. Since # is saturated we can obtain the empty sequent from the axioms of A using only cut rule and expansion rule. Note that every cut in this derivation eliminates formulas with less connective than A. In this derivation of the empty sequent we can add to every sequent M,T in the left and N,W in the right and we get at the end the sequent $M,T \vdash N,W$. In this derivation the initial sequents are derivable in K and each application of the expansion rule or the cut rule produce derivable sequents by the induction hypothesis. This proves the theorem.

An *assignment* s is a mapping that associates with every variable e_i a truth value T (true) or F (false). We define the *truth value* of a formula A and of a sequent $M \vdash N$ under the assignment s, by the following rules:

The truth value of a variable under s is the value given by s to the variable.

The truth value of a sequent $M \vdash N$ is T under s if and only

if component of N is T under s or some component of M is F under s.
The truth value of a formula $\#A_1 \ldots A_n$ is T under s if and only if all right axioms of the formula are T under s.

In the case the truth value of $M \vdash N$ is F under s we shall say that s *refutes* $M \vdash N$. Clearly the formula $\#A_1 \ldots A_n$ is T under s if and only if no right axiom of the formula is refuted by s.

Theorem 2. The following conditions are equivalent for any connective $\#$ with n arguments:

i) The connective $\#$ is saturated.
ii) Every assignment refutes at least one axiom of $\#$.
iii) Every complete partition of the variable $e_1, \ldots, e_n$ is the expansion of some axiom of $\#$.

Assume i) holds. The cut rule and the expansion rule preserve the property of being T under some assignment. Since the empty sequent is refuted by all assignments it follows that ii) holds.

Assume ii) holds. Every complete partition of the variables $e_1, \ldots, e_n$ is refuted by some assignment s; since s refutes at least one axiom of $\#$ it follows that every complete partition must be the expansion of some axiom of $\#$. Hence iii) holds.

Assume iii) holds. To prove i) we show that every partition $M \vdash N$ of the variables $e_1, \ldots, e_n$ can be obtained from the set of all axioms using only the cut rule and the expansion rule. This is proved by induction on the number k of variables of the list which are not in $M \vdash N$. If $k=0$ we use ii). Suppose now e_i is a variable of the list that does not appear in the partition $M \vdash N$. By the induction hypothesis both $M \vdash N, e_i$ and $e_i, M \vdash N$ can be obtained from the set of all axioms using cut rule and expansion rule. With an application of the cut rule we get then $M \vdash N$.

A sequent $M \vdash N$ is *valid* if no assignment refutes it. The system K is *consistent* if every derivable sequent is valid.

Theorem 3. The system K is consistent if and only if every assignment refutes at least one axiom of every connective # in the system.

Assume the system is consistent and suppose some assignment s does not refute any axiom of a connective # with n arguments. Let $M \vdash N$ be the complete partition of the variables $e_1, \dots, e_n$ in which every component of M is give value T by s and every component of N is given value F by s. If $R \vdash S$ is some left axiom of # then $R, M \vdash N, S$ must be closed, hence derivable. It follows that the sequent $\# A_1 \dots A_n, M \vdash N$ is derivable and this is a contradiction since this sequent is refuted by s.

Assuming that every assignment refutes some axiom of every connective # it is possible to prove that every derivable sequent is valid by induction in the derivation rules. The assumption must be used only to check the left rules. The details are left to the reader.

Theorem 4. If the cut rule holds in the system K then the system is consistent.

We assume the cut rule holds in K. To get a contradiction assume there is an assignment s and some connective # such that no axiom is refuted by s. Again let $M \vdash N$ be the complete partition of the variables $e_1, \dots, e_n$ which is refuted by s. It follows that the sequents $\# e_1 \dots e_n, M \vdash N$ and $M \vdash N, \# e_1 \dots e_n$ are derivable. Since the cut rule holds $M \vdash N$ is derivable but this is impossible because it is an atomic sequent which is not closed.

Corollary. The cut rule holds in the system K if and only if the system is consistent.

2. In this section we shall consider the completeness of the system K. We say that the system is *complete* if every valid sequent is derivable.

Theorem 5. The following conditions are equivalent:

i) for every formula A the sequent $A \vdash A$ is derivable.
ii) For every connective # the union of a right axiom and a left axiom is a closed sequent.
iii) The system K is complete.

Assume i) holds and # is some connective with n arguments. Since $\#e_1 \ldots e_n \vdash \#e_1 \ldots e_n$ is derivable it follows from Propositions 1 and 2 that the union of a right axiom and a left axiom of # must be derivable. Since such union is an atomic sequent it must be closed, so ii) holds.

Assuming ii) we prove that every valid sequent $M \vdash N$ is derivable, by induction on the number of connective in $M \vdash N$. If there is no connective so $M \vdash N$ is atomic it must be a closed sequent so it is derivable. Assume $M \vdash N$ is of the form $M \vdash N_1, \#A_1 \ldots A_n$ and it is valid. It follows from the definition that for every right axiom $P \vdash Q$ of $\#A_1 \ldots A_n$ the sequent $P,M \vdash N_1,Q$ is valid, hence by the induction hypothesis is derivable. It follows that $M \vdash N$ is derivable. Assume now that $M \vdash N$ is of the form $\#A_1 \ldots A_n, M_1 \vdash N$. Let $R \vdash S$ be some left axiom of $\#A_1 \ldots A_n$ and let s be some assignment. If $\#A_1 \ldots A_n$ is T under s it follows that $M_1 \vdash N$ is T under s, hence $R,M \vdash N,T$ is also T under s. If $\#A_1 \ldots A_n$ is F under s, some right axiom $P \vdash Q$ is F under s, and since $R,P \vdash S,Q$ is closed it follows that $R \vdash S$ is T under s, hence $R,M_1 \vdash N,T$ is T under s. This shows that the sequent $R,M_1 \vdash N,T$ is valid for every left axiom, hence derivable by the induction hypothesis. From this we have that $M \vdash N$ is derivable.

That iii) implies i) is trivial.

The results we have obtained give a formal characterization of Gentzen type systems which are consistent and complete. For any truth function it is possible to introduce a connective that represents such function. Suppose H is one such truth function with n arguments. We can express the function as a conjunction of terms, each of which is a disjunction and the terms in each disjunction are either variables or negation of variables. For each such disjunction introduce a right axiom of a connective representing H just by taking the negated variables as left components and the unnegated variables as right components. A connective with such right axioms represents clearly the given truth function. We need now to obtain left axioms in such a way that the following conditions are satisfied: every assignment refutes at least one axiom and the union of a left axiom with a right axiom is a closed sequent. Such set of left axioms can be obtained as follows: from each right axiom select one variable and if the variable was a right component take it now as a left component; if the variable was a left component take it now as a right component. All the non closed sequents obtained in this was are taken as left axioms. By construction it is clear that the required conditions are satisfied.

3. To conclude this note we want to discuss the situation in the predicate calculus. We assume for this a system containing predicate letters, function letters, free and bound variables, and some connectives. Formulas are defined as usual. We take as axioms all sequents $A \vdash A$ with A any atomic formula. The extension rule and the rules for the connectives are the same as in the system K. For each quantifier two rules are introduced; for details see [2] or [3].

Propositions 1, 2 and 3 hold exactly as formulated in section 2 (for Proposition 3 the formula A is now an atomic formula). Some kind of inversion rules hold also for the quantifier; for example the right rule of the universal quantifier is invertible but the left rule is not.

Theorem 2 is true in the new calculus; in fact is the theorem proved by Gentzen. The proof now becomes more involved for the quantifiers rules are not completely invertible; a secondary induction is required now to complete the argument. The other theorems of section 2 and the corrollary are also true.

For the completeness Theorem 5 the situation is the same. The equivalence holds in the predicate calculus but the proof of the implication from ii) to iii) requires a completely different approach (in fact the proof is now non constructive). The argument given by Beth in [1], p. 196 can be applied here.

REFERENCES

[1] BETH, E. W.: *The foundations of mathematics*, North Holland Pub. Comp., 1965.

[2] CURRY, H. B.: *Foundations of mathematical logic*, McGrawHill, 1963.

[3] GENTZEN, G.: Untersuchungen über das logische Schliessen, *Math. Z.*, 39:176-210,405-431 (1934).